Relativistic treatment of Verlinde's emergent force in Tsallis' statistics

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Abstract

Following Chakrabarti,Chandrasekhar, and Naina [Physica A **389** (2010) 1571], we attempt a classical relativistic treatment of Verlinde's emergent entropic force conjecture by appealing to a relativistic Hamiltonian in the context of Tsalli's statistics. The ensuing partition function becomes the classical one for small velocities. We show that Tsallis' relativistic (classical) free particle distribution at temperature T can generate Newton's gravitational force's r^{-2} *distance's dependence*. If we want to repeat the concomitant argument by appealing to Renyi's distribution, the attempt fails and one needs to modify the conjecture.
Keywords: Tsallis' and Renyi's relativistic distributions, classical partition function, entropic force.
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Contents

1 Introduction

In 2011, Verlinde [1] put forward a conjecture that connects gravity to an entropic force. Gravity would then arise out of information regarding the positions of material bodies (it from bit). This idea links a thermal gravity-treatment to 't Hooft's holographic principle. As a consequence, gravitation ought to be be regarded as an emergent phenomenon. Verlinde's conjecture attained considerable reception (just as an example, see [2]). For a superb overview on the statistical mechanics of gravitation, we recommend Padmanabhan's work [3], and references therein.

Verlinde's initiative originated works on cosmology, the dark energy hypothesis, cosmological acceleration, cosmological inflation, and loop quantum gravity. The literature is immense [4]. A relevant contribution to information theory is that of Guseo [5], who proved that the local entropy function, related to a logistic distribution, is a catenary and vice versa. Such invariance may be explained, at a deeper level, through the Verlindes conjecture on the origin of gravity, as an effect of the entropic force. Guseo puts forward a new interpretation of the local entropy in a system, as quantifying a hypothetical attraction force that the system would exert [5].

The present effort does not deal with any of these issues. What we will do is to show that a simple classical reasoning centered on Tsallis' relativistic probability distributions proves Varlinde's conjecture. For Renyi's relativistic instance, one needs to modify the conjecture to achieve a similar result.
Our point of departure is Ref. [6], in which their authors studied a canonical ensemble of N particles for a classical relativistic ideal gas, and found its specific heat in the Tsallis-Mendes-Plastino (TMP) scenario [7]. We will not use here the TMP scenario. Inspired by [6], we appeal as well to our previous effort [8] for non-relativistic results and deal with Tsallis' statistics with linear constraints as a priori information [7]. In addition to finding, for the first time ever, relativistic Verlinde-results in a Tsallis'context, we will, for the sake of completeness, register some advances regarding the relativistic Tsallis scenario with linear constraints for the ideal gas.

2 Tsallis' relativistic partition function for the free particle

The celebrated and well-known Tsalis entropy is a generalization of Shanon's one, that depends on a free real parameter q [7].

The $q<1$ instance

We consider first the case $q<1$. This case is not relevant to our Verlinde's endeavor [8], but is a logical addition to the results of [6].

Tsallis' relativistic q-partition function for $N-$free particles of mass m reads [6]

$$\mathcal{Z}=\frac{V}{N!h^{3N}}\int\left[1+(1-q)\beta(\sqrt{m^2c^4+p^2c^2}-mc^2)\right]_+^{\frac{1}{q-1}}d^4p. \tag{2.1}$$

Using spherical coordinates and integrating over the angles the precedent integral we have

$$\mathcal{Z}=\frac{4\pi V}{N!h^{3N}}\int\limits_0^{\infty}\left[1+(1-q)\beta(\sqrt{m^2c^4+p^2c^2}-mc^2)\right]^{\frac{1}{q-1}}p^2dp. \tag{2.2}$$

With the change of variables $y^2=p^2+m^2c^2$ one now has

$$\mathcal{Z}=\frac{4\pi V}{N!h^{3N}}\int\limits_{mc}^{\infty}y\sqrt{y^2-m^2c^2}\,[1+(1-q)\beta c(y-mc)]^{\frac{1}{q-1}}dy. \tag{2.3}$$

Let x be given by $y=mcx$. We have then

$$\mathcal{Z}=\frac{4\pi Vm^3c^3}{N!h^{3N}}\int\limits_1^{\infty}x\sqrt{x^2-1}\left[1+(1-q)\beta mc^2(x-1)\right]^{\frac{1}{q-1}}dx. \tag{2.4}$$

With s defined as $x=s+1$ we obtain:

$$\mathcal{Z}=\frac{4\pi Vm^3c^3}{N!h^{3N}}\int\limits_0^{\infty}\left(s^{\frac{3}{2}}+s^{\frac{1}{2}}\right)(s+2)^{\frac{1}{2}}\left[1+(1-q)\beta mc^2s\right]^{\frac{1}{q-1}}ds, \tag{2.5}$$

or

$$\mathcal{Z} = \frac{4\pi V m^3 c^3}{N! h^{3N}} [(1-q)\beta mc^2]^{\frac{1}{q-1}} \int\limits_0^\infty s^{\frac{3}{2}} (s+2)^{\frac{1}{2}} \left[s + \frac{1}{(1-q)\beta mc^2} \right]^{\frac{1}{q-1}} ds +$$

$$\frac{4\pi V m^3 c^3}{N! h^{3N}} [(1-q)\beta mc^2]^{\frac{1}{q-1}} \int\limits_0^\infty s^{\frac{1}{2}} (s+2)^{\frac{1}{2}} \left[s + \frac{1}{(1-q)\beta mc^2} \right]^{\frac{1}{q-1}} ds. \quad (2.6)$$

Appealing to reference [9] we have now a result in terms of Hyper-geometric functions F and Beta functions B, namely,

$$\mathcal{Z} = \frac{4\pi V m^3 c^3}{N! h^{3N}} [(1-q)\beta mc^2]^{-\frac{3}{2}} \left[\frac{B\left(\frac{5}{2}, \frac{1}{1-q} - 3\right)}{\beta mc^2(1-q)} \times \right.$$
$$F\left(-\frac{1}{2}, \frac{5}{2}, \frac{1}{1-q} - \frac{1}{2}; 1 - \frac{1}{2\beta mc^2(1-q)}\right) +$$
$$\left. B\left(\frac{3}{2}, \frac{1}{1-q} - 2\right) F\left(-\frac{1}{2}, \frac{3}{2}, \frac{1}{1-q} - \frac{1}{2}; 1 - \frac{1}{2\beta mc^2(1-q)}\right)\right]. \quad (2.7)$$

For $\beta mc^2 >> 1$, $mc^2 >> k_B T$, we are in the non-relativistic case and have

$$\mathcal{Z} = \frac{2\pi V}{N! h^{3N}} \left[\frac{2m}{\beta(1-q)}\right]^{\frac{3}{2}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{1-q} - \frac{3}{2}\right)}{\Gamma\left(\frac{1}{1-q}\right)}. \quad (2.8)$$

The case $q > 1$

Let is now consider gravitationally relevant [8] case $q > 1$. We have for the partition function

$$\mathcal{Z} = \frac{4\pi V m^3 c^3}{N! h^{3N}} \int\limits_0^\infty \left(s^{\frac{3}{2}} + s^{\frac{1}{2}}\right) (s+2)^{\frac{1}{2}} \left[1 - (q-1)\beta mc^2 s\right]_+^{\frac{1}{q-1}} ds. \quad (2.9)$$

Integrating on the angles we have again

$$\mathcal{Z} = \frac{4\pi V m^3 c^3}{N! h^{3N}} \int\limits_0^{\frac{1}{\beta mc^2(q-1)}} \left(s^{\frac{3}{2}} + s^{\frac{1}{2}}\right) (s+2)^{\frac{1}{2}} \left[1 - (q-1)\beta mc^2 s\right]^{\frac{1}{q-1}} ds, \quad (2.10)$$

or

$$\mathcal{Z} = \frac{4\pi V m^3 c^3}{N! h^{3N}} [(q-1)\beta mc^2]^{\frac{1}{q-1}} \int\limits_0^{\frac{1}{\beta mc^2(q-1)}} s^{\frac{3}{2}} (s+2)^{\frac{1}{2}} \left[\frac{1}{(q-1)\beta mc^2} - s\right]^{\frac{1}{q-1}} ds +$$

$$\frac{4\pi V m^3 c^3}{N! h^{3N}} [(q-1)\beta mc^2]^{\frac{1}{q-1}} \int\limits_0^{\frac{1}{\beta mc^2(q-1)}} s^{\frac{1}{2}} (s+2)^{\frac{1}{2}} \left[\frac{1}{(q-1)\beta mc^2} - s\right]^{\frac{1}{q-1}} ds. \tag{2.11}$$

By recourse to [9] we now obtain

$$\mathcal{Z} = \frac{2\pi V}{N! h^{3N}} \left[\frac{2m}{\beta m(q-1)}\right]^{\frac{3}{2}} \left[\frac{B\left(\frac{5}{2}, \frac{1}{q-1}+1\right)}{\beta mc^2(q-1)} \times \right.$$

$$F\left(-\frac{1}{2}, \frac{5}{2}, \frac{7}{2} + \frac{1}{q-1}; -\frac{1}{2\beta mc^2(q-1)}\right) +$$

$$\left. B\left(\frac{3}{2}, \frac{1}{q-1}+1\right) F\left(-\frac{1}{2}, \frac{3}{2}, \frac{5}{2} + \frac{1}{q-1}; -\frac{1}{2\beta mc^2(q-1)}\right)\right]. \tag{2.12}$$

For $\beta mc^2 >> 1$, the classic case, the partition function reads

$$\mathcal{Z} = \frac{2\pi V}{N! h^{3N}} \left[\frac{2m}{\beta(q-1)}\right]^{\frac{3}{2}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{q-1}+1\right)}{\Gamma\left(\frac{1}{1-q}+\frac{5}{2}\right)}, \tag{2.13}$$

which is the usual non relativistic Tsalli's partition function for $q > 1$ already obtained in [8]. Figure 1 displays the graph of the function $H(T)$ given by

$$\mathcal{Z} = \frac{2\pi V}{N! h^{3N}} \left[\frac{2m}{\beta(q-1)}\right]^{\frac{3}{2}} H(T), \tag{2.14}$$

for $q = \frac{4}{3}$, the specific $q-$value needed for gravitaional considerations [8]. It tells us that $\mathcal{Z}$ is always positive, as it should be.

3 Tsallis' relativistic mean energy of the free particle

Case $q < 1$

Let us now calculate the average energy corresponding, firstly in the case $q < 1$. For it we have

$$\mathcal{Z} < \mathcal{U} >= \frac{V}{N!h^{3N}} \int [\sqrt{m^2c^4 + p^2c^2} - mc^2] \times$$

$$\left[1 + (1-q)\beta(\sqrt{m^2c^4 + p^2c^2} - mc^2)\right]_+^{\frac{1}{q-1}} d^4p, \tag{3.15}$$

or

$$\mathcal{Z} < \mathcal{U} >= \frac{V}{N!h^{3N}} \int [\sqrt{m^2c^4 + p^2c^2}] \times$$

$$\left[1 + (1-q)\beta(\sqrt{m^2c^4 + p^2c^2} - mc^2)\right]_+^{\frac{1}{q-1}} d^4p - mc^2\mathcal{Z}. \tag{3.16}$$

With changes in the variables similar to those made for the partition function, we obtain here

$$\mathcal{Z} < \mathcal{U} >= \frac{4\pi V m^4 C^5}{N!h^{3N}} \int\limits_0^\infty x^{\frac{3}{2}}(x+1)(\sqrt{x+2} \times$$

$$\left[1 + (1-q)\beta mc^2 x\right]^{\frac{1}{q-1}} dx. \tag{3.17}$$

This last equation can be rewritten as

$$\mathcal{Z} < \mathcal{U} >= \frac{4\pi V m^4 C^5}{N!h^{3N}} [\beta mc^2(1-q)]^{\frac{1}{q-1}} \int\limits_0^\infty x^{\frac{3}{2}}(x+1)(\sqrt{x+2} \times$$

$$\left[x + \frac{1}{(1-q)\beta mc^2}\right]^{\frac{1}{q-1}} dx. \tag{3.18}$$

Returning again to reference [9], we obtain for $< \mathcal{U} >$

$$< \mathcal{U} >= \frac{\sqrt{2}\, 4\pi V m^4 c^5}{N!h^{3N}\mathcal{Z}} \left[\frac{1}{\beta mc^2(1-q)}\right]^{\frac{5}{2}+\frac{1}{q-1}} \left[\frac{B\left(\frac{7}{2}, \frac{1}{1-q} - 4\right)}{\beta mc^2(1-q)} \times\right.$$

$$F\left(-\frac{1}{2},\frac{7}{2},\frac{1}{1-q}-\frac{1}{2};1-\frac{1}{2\beta mc^2(1-q)}\right)+$$

$$B\left(\frac{5}{2},\frac{1}{1-q}-3\right)F\left(-\frac{1}{2},\frac{5}{2},\frac{1}{1-q}-\frac{1}{2};1-\frac{1}{2\beta mc^2(1-q)}\right)\Bigg]. \tag{3.19}$$

From this last equation we obtain the mean energy expression for the non-relativistic case

$$<\mathcal{U}>=\frac{3}{\beta[2-5(1-q)]}. \tag{3.20}$$

Case q larger than one

When $q>1$ we have

$$\mathcal{Z}<\mathcal{U}>=\frac{4\pi Vm^4C^5}{N!h^{3N}}\int\limits_0^\infty x^{\frac{3}{2}}(x+1)(\sqrt{x+2}\times$$

$$\left[1-(q-1)\beta mc^2x\right]_+^{\frac{1}{q-1}}dx-mc^2\mathcal{Z}. \tag{3.21}$$

Making a similar reasoning as for the case $q<1$ we obtain

$$<\mathcal{U}>=\frac{\sqrt{2}\,4\pi Vm^4c^5}{N!h^{3N}\mathcal{Z}}\left[\frac{1}{\beta mc^2(q-1)}\right]^{\frac{1}{q-1}}\left[\frac{B\left(\frac{7}{2},\frac{1}{q-1}+1\right)}{\beta mc^2(q-1)}\times\right.$$

$$F\left(-\frac{1}{2},\frac{7}{2},\frac{1}{q-1}+\frac{9}{2};\frac{1}{2\beta mc^2(q-1)}\right)+$$

$$B\left(\frac{5}{2},\frac{1}{q-1}+1\right)F\left(-\frac{1}{2},\frac{5}{2},\frac{1}{q-1}+\frac{7}{2}-\frac{1}{2\beta mc^2(q-1)}\right)\Bigg]. \tag{3.22}$$

For $\beta mc^2>>1$ (the non-relativistic case) we obtain the result of [8], i.e.,

$$<\mathcal{U}>=\frac{3}{\beta[2+5(q-1)]}. \tag{3.23}$$

4 Specific heat in the linear constraints Tsallis' scenario

Let is now calculate the specific heat for the case $q = \frac{4}{3}$, relevant for Verlinde-endeavors [8]. This was not done in [6]. We should first note, with respect to Hyper-geometric functions, that

$$\frac{d}{dz}F(\alpha,\beta,\gamma;z) = -\alpha\beta F(\alpha+1,\beta+1,\gamma+1;z). \tag{4.24}$$

We now use the notation

$$F_1 = F\left(-\frac{1}{2},\frac{7}{2},\frac{9}{2}+3;-\frac{3k_BT}{2mc^2}\right), \tag{4.25}$$

$$F_2 = F\left(-\frac{1}{2},\frac{5}{2},\frac{7}{2}+3;-\frac{3k_BT}{2mc^2}\right), \tag{4.26}$$

$$F_3 = F\left(-\frac{1}{2},\frac{3}{2},\frac{5}{2}+3;-\frac{3k_BT}{2mc^2}\right), \tag{4.27}$$

$$F_4 = F\left(\frac{1}{2},\frac{9}{2},\frac{9}{2}+4;-\frac{3k_BT}{2mc^2}\right), \tag{4.28}$$

$$F_5 = F\left(\frac{1}{2},\frac{7}{2},\frac{7}{2}+4;-\frac{3k_BT}{2mc^2}\right), \tag{4.29}$$

$$F_6 = F\left(\frac{1}{2},\frac{5}{2},\frac{5}{2}+4;-\frac{3k_BT}{2mc^2}\right). \tag{4.30}$$

Thus, we can write

$$<\mathcal{U}>= 3k_BT\frac{\frac{3k_BT}{mc^2}B\left(\frac{7}{2},4\right)F_1 + B\left(\frac{5}{2},4\right)F_2}{\frac{3k_BT}{mc^2}B\left(\frac{5}{2},4\right)F_2 + B\left(\frac{3}{2},4\right)F_3}, \tag{4.31}$$

and, for the specific heat we have then

$$C = \frac{\partial<\mathcal{U}>}{\partial T} = \frac{<\mathcal{U}>}{T} + \frac{9k_B^2T}{mc^2}\frac{B\left(\frac{7}{2},4\right)F_1 - \frac{21k_BT}{3mc^2}B\left(\frac{7}{2},4\right)F_4 - \frac{5}{8}B\left(\frac{5}{2},4\right)F_5}{\frac{3k_BT}{mc^2}B\left(\frac{5}{2},4\right)F_2 + B\left(\frac{3}{2},4\right)F_3} -$$

$$\frac{3k_B<\mathcal{U}>}{mc^2}\frac{B\left(\frac{5}{2},4\right)F_2 - \frac{15k_BT}{8mc^2}B\left(\frac{5}{2},4\right)F_5 - \frac{3}{8}B\left(\frac{3}{2},4\right)F_6}{\frac{3k_BT}{mc^2}B\left(\frac{5}{2},4\right)F_2 + B\left(\frac{3}{2},4\right)F_3}. \tag{4.32}$$

This expression is plotted in Figure 2. We see that the specific heat is always positive, as it happens in the non-relativistic case [8].

5 The relativistic, Tsallis entropic force

We arrive now at our main present goal. We specialize things now to $q = \frac{4}{3}$. Why do we select this special value $q = \frac{4}{3}$? There is a solid reason. This is because

$$\mathcal{S} = \ln_q \mathcal{Z} + \mathcal{Z}^{1-q} \beta < \mathcal{U} > .$$

Since the entropic force is to be defined as proportional to the gradient of $\mathcal{S}$, there is a unique q-value for which the dependence on r of the entropic force is $\sim r^{-2}$ when $\nu = 3$. Thus we obtain, for $q = 4/3$,

$$\mathcal{S} = 3 - (3 - \beta < \mathcal{U} >)\mathcal{Z}^{-\frac{1}{3}}. \tag{5.1}$$

From (2.12) we can write

$$< \mathcal{Z} >= ar^3, \tag{5.2}$$

from which it is obtained that

$$\mathcal{S} = 3 - \frac{3 - \beta < \mathcal{U} >}{a^{\frac{1}{3}} r}. \tag{5.3}$$

Following Verlinde [1] we define the entropic force as

$$\vec{\mathcal{F}}_e = -\frac{\lambda}{\beta} \vec{\nabla} \mathcal{S}, \tag{5.4}$$

where $\vec{\nabla}$ indicates the four-gradient in Minkowskian space.

$$\vec{\mathcal{F}}_e = -\frac{\lambda}{\beta} \frac{3 - \beta < \mathcal{U} >}{a^{\frac{1}{3}} r^2} \vec{e}_r, \tag{5.5}$$

where $\vec{e}_r$ is the radial unit vector. We see that F_e acquires an appearance quite similar to that of Newton's gravitational one, as conjectured by Verlinde en [1]. In Figures 3 and 4 the function $L = 3 - \beta < \mathcal{U} >$ is plotted. We see that L is always positive. This entails that the relativistic entropic force is purely gravitational.

6 The relativistic, Renyi's entropic force

In Renyi's approach to our problem [8] the entropy is

$$\mathcal{S} = \ln \mathcal{Z} + \ln[1 + (1-\alpha)\beta < \mathcal{U} >]_+^{\frac{1}{1-\alpha}}. \tag{6.1}$$

For $\alpha = \frac{4}{3}$, the expression for the entropy is

$$\mathcal{S} = \ln \mathcal{Z} + \ln \left[1 - \frac{\beta < \mathcal{U} >}{3}\right]_+^{-3}. \tag{6.2}$$

The second term on the right hand of (6.2) is independent of r. Additionally, from (5.2) we obtain

$$\ln \mathcal{Z} = 3 \ln r + \ln a. \tag{6.3}$$

Here we need to derive the entropy with respect to the area, thus changing Verlindes conjecture. As in the non-relativistic case [8], we have then

$$\vec{F}_e = -\frac{\lambda}{\beta}\frac{\partial \mathcal{S}}{\partial A}\vec{e}_R = -\frac{\lambda}{\beta}\frac{3}{8\pi r^2}\vec{e}_r. \tag{6.4}$$

This is again a gravitational expression for the entropic force.

7 Conclusions

We obtained here the relativistic partition function $\mathcal{Z}$ of Tsalli's theory with linear constraints, that adequately reduces itself to its non-relativistic counterpart for small velocities.
We do the same for the mean value of the energy$< \mathcal{U} >$ for the relativistic Hamiltonian of the ideal gas.
We obtain the associated specific heat that turns out to be positive, as befits an ideal gas.
From $\mathcal{Z}$ and $< \mathcal{U} >$ we obtained the relativistic entropy $\mathcal{S}$
We have presented two very simple relativistic classical realizations of Verlinde's conjecture. The Tsallis treatment, for $q = 4/3$, seems to be neater, as the entropic force is directly associated to the gradient of Tsallis' entropy S_q, which acts as a "potential", as Verlinde prescribes. This is not so in the Renyi instance, in which one has to modify Verlinde's F_e definition and derive S with respect to the area.

Strictly speaking, Verlinde's conjecture can be unambiguously proved for the Tsallis entropy with $q = 4/3$. The Renyi demonstration correspond to a modified version of Verlinde's conjecture.
Of course, ours is a very preliminary, if significant, effort. A much more elaborate treatment would be desirable.

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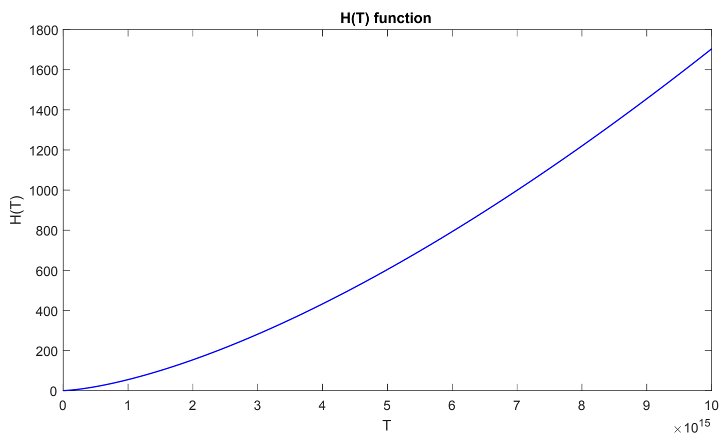

Figure 1: H function

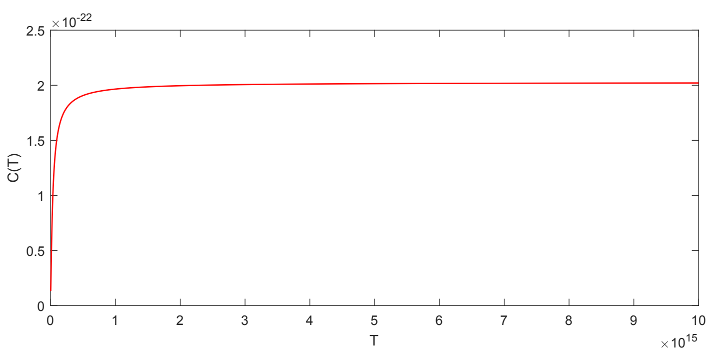

Figure 2: Specific heat

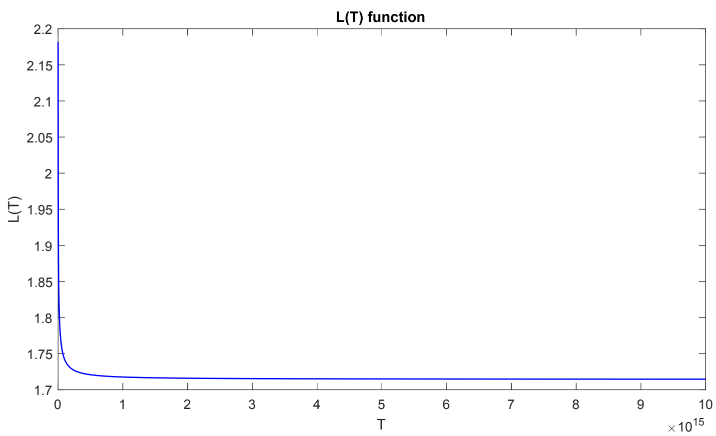

Figure 3: $L(T) = 3 - \beta < \mathcal{U} >$

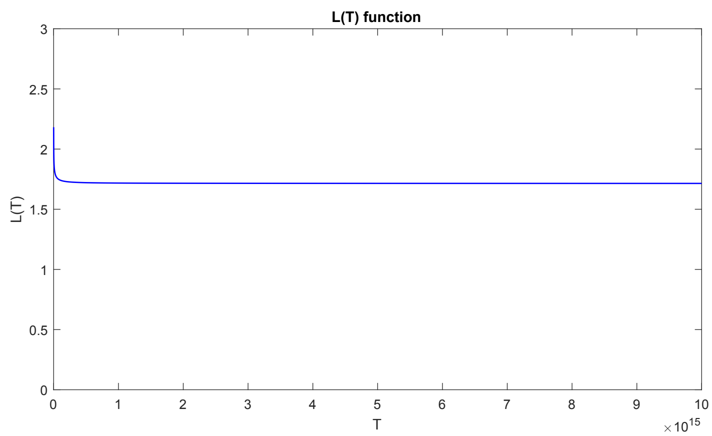

Figure 4: Centered $L(T) = 3 - \beta < \mathcal{U} >$