On Coordinate Transformations in Planar Noncommutative Theories

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Abstract

We consider planar noncommutative theories such that the coordinates verify a space-dependent commutation relation. We show that, in some special cases, new coordinates may be introduced that have a constant commutator, and as a consequence the construction of Field Theory models may be carried out by an application of the standard Moyal approach in terms of the new coordinates.

We apply these ideas to the concrete example of a noncommutative plane with a curved interface. We also show how to extend this method to more general situations.

1 Introduction

Noncommutativity has been intensively studied in past years, in part due to its role in String Theory ([1] and references therein) and its application to Condensed Matter Physics (see, for example, [2] and references therein). It also offers interesting alternatives for gauge theories. From the mathematical point of view, the development of Noncommutative Geometry [3] and the Formality Theorem in Deformation Quantization [4], have been fundamental to the understanding of noncommutativity.

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The case of a constant noncommutativity parameter, which leads to the familiar Moyal product, is well understood (see, for example, [5]). However, space-dependent noncommutativity is much more involved, since there is no general procedure to find an explicit representation for the resulting infinite-dimensional algebra, except when it comes from a Poisson structure [4]. The whole physical picture corresponding to this situation is, consequently, not completely known, although a lot of progress has been made due to recent efforts. For example, the connection between curved D-branes and space-dependent noncommutativity has been established in [6], and the construction of gauge theories in curved noncommutative spaces has been presented in [7]. Besides, an illuminating field-theory interpretation of Kontsevich´s construction in terms of a sigma model has been provided in [8].

The aim of this paper is to show how certain cases of coordinates satisfying a space-dependent commutation relation may be analyzed with the help of a change of variables. The main idea is that it may be possible to define new variables in terms of which one obtains a constant commutator. In terms of those coordinates, we can work with the usual Moyal representation of the noncommutative product and then easily construct a calculus and study the geometry of the space. If the relevant objects (product, derivatives, integrals,...) may at the end be written in terms of the original variables, we can obtain explicit representations for the original algebra, as well as for the derivatives and integrals. Interestingly enough, cases in which this can be achieved are not artificial but arise in a variety of problems, for example related to the construction of noncommutative solitons and instantons [9]-[10].

This paper is organized as follows: in section 2, we discuss the relation between the metric and a space-dependent commutation relation for the coordinates of a two-dimensional space, based on general considerations. In section 2.1, we consider the particular case of a commutation relation that depends on only one of the coordinates. We show that, for this particular case, it is always possible to perform a change of variables to new coordinates verifying a constant (i.e., space-independent) commutation relation so that the Moyal product representation can be used with the resulting simplifications it implies. We then extend all the previous results to the example of a curved interface dividing two regions that have (different) constant values for θ in section 3, where we present the concrete example of a scalar field theory. The possibility of generalizing the method is discussed in section 4. In section 5 we present our conclusions.

2 Noncommutative products and two dimensional metrics

Coordinate commutators in a non-flat background spatial metric, necessarily imply the introduction of a coordinate-dependent noncommutativity parameter θ^{ij}:

$$[x^i, x^j]_\star = i\theta^{ij}(x) \ . \tag{1}$$

Now, as it is well known, there are severe constraints that have to be satisfied for a consistent definition of a noncommutative associative $\star$-product. In general, even when those constraints are satisfied, it is not possible to give a closed and explicit formula for the $\star$-product between two arbitrary functions, as it can be done for the constant-θ case (where one has the standard Moyal product). The general conditions under which such a product can be defined are derived in [4].

A sufficient condition to have an associative noncommutative product may be stated as follows [9]:

$$\nabla_i \theta^{jk}(x) = 0 \ , \tag{2}$$

where ∇_i is the covariant derivative corresponding to a metric connection. Although up to this point we have not introduced any metric into the game, a natural one will appear precisely when solving (2) in 2 dimensions. That metric will be determined (up to a constant scalar factor) by θ^{jk} itself; see (10) and (11) below.

Condition (2) may be derived by starting from the definition of the Poisson bracket $\{f, h\}$ for two functions f and h:

$$\{f, h\} \ = \ \partial_i f \, \theta^{ij} \, \partial_j h \tag{3}$$

and then noting that a noncommutative product, up to order θ, takes the form [1]:

$$f \star h \equiv fh + \frac{i}{2}\{f, h\} + \mathcal{O}(\theta^2) \ . \tag{4}$$

The associativity of the $\star$-product (4) requires the Jacobi identity for the Poisson bracket (3) to hold true, which in turns results in the equation:

$$\theta^{ij}\partial_j\theta^{kl} + \theta^{lj}\partial_j\theta^{ik} + \theta^{kj}\partial_j\theta^{li} = 0 \ , \tag{5}$$

i.e., θ^{ij} is a Poisson structure. In fact, this implies [6] that the product will be associative to all orders.

[1]Higher orders in an expansion in powers of θ are presented in [4].

The previous condition can be written covariantly in the form:

$$\theta^{ij}\nabla_j\theta^{kl} + \theta^{lj}\nabla_j\theta^{ik} + \theta^{kj}\nabla_j\theta^{li} = 0 \ , \tag{6}$$

since the terms in ∇_j containing the (symmetric) connection cancel out. Condition (2) is a sufficient condition for (6) to be true, namely, for the associativity to be valid to all orders in θ.

Moreover, (2) is very easy to deal with on a two dimensional space. Indeed, the most general θ^{ij} can be written in $d = 2$ in the form

$$\theta^{ij}(x^1, x^2) = \frac{\varepsilon^{ij}}{\sqrt{g(x)}}\theta_0(x^1, x^2) \tag{7}$$

where $\theta_0(x^1, x^2)$ is a scalar and $g(x)$ is the determinant of the two dimensional metric $g_{jk}(x)$. But then, condition (2) reduces to

$$\nabla_i \left(\frac{\varepsilon^{jk}}{\sqrt{g(x)}}\theta_0(x^1, x^2) \right) = \frac{\varepsilon^{jk}}{\sqrt{g(x)}}\nabla_i\theta_0(x^1, x^2) = \frac{\varepsilon^{jk}}{\sqrt{g(x)}}\partial_i\theta_0(x^1, x^2) = 0 \tag{8}$$

and hence θ_0 must be a constant.

Then, the sufficient condition (2) is equivalent, in $d = 2$, to the equation:

$$\theta^{ij}(x^1, x^2) = \frac{\varepsilon^{ij}}{\sqrt{g(x)}}\,\theta_0 \ , \tag{9}$$

where θ_0 is a real (non-vanishing) constant. This expression relates the space-dependent associative noncommutativity to a non-trivial background metric g_{ij}. In fact, it is believed [11] that quantum gravity is at the origin of noncommutative effects, so that kind of relation is not entirely unexpected. The g_{ij} tensor can be explicitly written by taking into account that in 2 dimensions every metric is conformally flat:

$$g_{ij}(x) \ = \ e^{\sigma(x)}\ \delta_{ij} \ , \tag{10}$$

and that the conformal factor $e^{\sigma(x)}$ is determined by g:

$$e^{\sigma(x)} \ = \ \sqrt{g(x)} \ . \tag{11}$$

Let us also note that, in the presence of a metric, the natural integration measure, $d\mu$, should be

$$d\mu \ = \ d^2x\sqrt{g(x)} \tag{12}$$

which is, as we shall see, consistent with the definition of a noncommutative product in which the integral acts as a trace.

2.1 The case $\theta = \theta(x^1)$

Let us specialize the previous discussion in two dimensional space to the case in which θ depends on only one coordinate, x^1 say. We know that θ can then be written as follows:

$$\theta \;=\; \theta_0 \, t(x^1) \;, \tag{13}$$

where $t(x^1) \;=\; 1/\sqrt{g(x)}$ is a positive definite function.

Our aim is to present an explicit formula for a noncommutative associative $\star$-product such that

$$[x^1, x^2]_\star \;=\; x^1 \star x^2 - x^2 \star x^1 \;=\; i\,\theta_0 \, t(x^1) \;. \tag{14}$$

An expression for $\star$ in powers of θ_0 was already found by Kontsevich [4], where the first terms are given by (4). However, the procedure to construct it, based on the Formality Theorem, can be quite involved in practice. Rather than following that approach, we shall show how to arrive to an associative noncommutative product by a different path. Inspired by the change of variables found in [9], while studying noncommutative vortices in a curved space, we left and right multiply (14) by $1/\sqrt{t(x^1)}$ where the square root is the noncommutative one, i.e., $\sqrt{a} \star \sqrt{a} = a$. Then:

$$x^1 \star \frac{1}{\sqrt{t(x^1)}} \star x^2 \star \frac{1}{\sqrt{t(x^1)}} - \frac{1}{\sqrt{t(x^1)}} \star x^2 \star \frac{1}{\sqrt{t(x^1)}} \star x^1 = i\theta_0 \;. \tag{15}$$

Here we have used the associativity of the $\star$-product, which is valid for any positive function $t(x^1)$.

We then change variables, from (x^1, x^2) to new ones (y^1, y^2) defined by:

$$\begin{cases} y^1 \;=\; x^1 \\ y^2 \;=\; \frac{1}{\sqrt{t(x^1)}} \star x^2 \star \frac{1}{\sqrt{t(x^1)}} \end{cases} \tag{16}$$

obtaining:

$$y^1 \star y^2 \;-\; y^2 \star y^1 \;=\; i\,\theta_0 \;. \tag{17}$$

Note that (16) involves the (well-defined) square root of a positive element, and besides, the definition of the new variables is not sensitive to the sign ambiguity of the square root, since that square root appears quadratically. Also, the variables y^1 and y^2, as defined in (16), are Hermitian and, because of (17), the noncommutative $\star$-product can then be realized as an ordinary Moyal product. We shall use '$*$' to denote the constant θ $(= \theta_0)$ Moyal product in terms of variables (y^1, y^2):

$$f(y) \star g(y) \;=\; f(y) * g(y) \;=\; \exp\Big(\frac{i}{2}\theta_0 \varepsilon^{jk} \, \frac{\partial}{\partial y^j}\frac{\partial}{\partial \tilde{y}^k}\Big) \, f(y) g(\tilde{y}) \Big|_{\tilde{y}=y} \;. \tag{18}$$

For functions of y^1 and y^2, working with the $*$-product, we can use all the standard (flat-space) noncommutative geometry tools [5].

Besides, we can rewrite the Moyal formula in terms of the original variables, where it leads to a concrete expression for the $\star$-product. To that end, we first note that from (16) one has

$$\left\{ \begin{array}{lcl} x^1 & = & y^1 \\ x^2 & = & \sqrt{t(y^1)} * y^2 * \sqrt{t(y^1)} \, . \end{array} \right. \tag{19}$$

Using (18), we obtain

$$\sqrt{t(y^1)} * y^2 * \sqrt{t(y^1)} \; = \; t(y^1) \, y^2 \quad . \tag{20}$$

where all the noncommutative artifacts have disappeared. Then:

$$\left\{ \begin{array}{lcl} x^1 & = & y^1 \\ x^2 & = & t(y^1) \, y^2 \, , \end{array} \right. \tag{21}$$

is an (exact) expression for the change of variables, that will allow us to derive explicit expressions for $\star$ and for the integration measure in the 'physical' variables x^1, x^2.

As an example, consider the $\star$-product between two functions of the original variables x^1, x^2. It can be defined in the form

$$\begin{aligned} f(x^1,x^2) \star g(x^1,x^2) &\equiv f(y^1,t(y^1)y^2) * g(y^1,t(y^1)y^2) \\ &= \exp\Big(\frac{i}{2}\theta_0 \varepsilon^{jk} \, \frac{\partial}{\partial y^j} \frac{\partial}{\partial \tilde{y}^k}\Big) \, f(y^1,t(y^1)y^2) \, g(\tilde{y}^1,t(\tilde{y}^1)\tilde{y}^2) \, \Big|_{\tilde{y}=y} . \end{aligned} \tag{22}$$

It is easy to verify that (22) provides a consistent associative realization of the algebra (14).

The first line in eq.(22) may be applied to verify *a posteriori* the associativity of the $\star$-product to all orders. Indeed, that property is, in this light, simply *inherited* from the (well-known) associativity of the $*$ product:

$$\big(f(x) \star g(x) \big) \star h(x) \; = \; \big(f(x(y)) * g(x(y)) \big) * h(x(y))$$

$$= \; f(x(y)) * \big(g(x(y)) * h(x(y)) \big) \; = \; f(x(y)) \star \big(g(x(y)) \star h(x(y)) \big) \, . \tag{23}$$

Of course, since we have at our disposal the formula for the change of variables (21), we may write everything in terms of x^1 and x^2 on the right hand side of the previous relation. One has to take into account the formulæ

$$
\begin{aligned}
\left.\frac{\partial}{\partial y^1}\right|_{y^2} &= \left.\frac{\partial}{\partial x^1}\right|_{x^2} + x^2 \frac{t'(x^1)}{t(x^1)} \left.\frac{\partial}{\partial x^2}\right|_{x^1} \\
\left.\frac{\partial}{\partial y^2}\right|_{y^1} &= t(x^1) \left.\frac{\partial}{\partial x^2}\right|_{x^1}
\end{aligned}
\tag{24}
$$

Using (24), the $\star$-product of functions of the original variables can be finally written, to all orders in θ_0, as

$$
\begin{aligned}
f(x)\star g(x) \;=\; \exp\frac{i\theta_0}{2}\Bigg[& t(\tilde{x}^1)\frac{\partial}{\partial x^1}\frac{\partial}{\partial \tilde{x}^2} - t(x^1)\frac{\partial}{\partial x^2}\frac{\partial}{\partial \tilde{x}^1} \\
& + \left(x^2 \frac{t'(x^1)}{t(x^1)} t(\tilde{x}^1) - \tilde{x}^2 \frac{t'(\tilde{x}^1)}{t(\tilde{x}^1)} t(x^1) \right) \frac{\partial}{\partial x^2}\frac{\partial}{\partial \tilde{x}^2} \Bigg] f(x) g(\tilde{x}) \Big|_{\tilde{x}=x}
\end{aligned}
\tag{25}
$$

By explicit computation order to order in θ_0 one can verify the *inherited* associativity. For example, up to order θ_0^2, the product (25) takes the form

$$
\begin{aligned}
f(x) \star g(x) \;=\;& f(x)g(x) + \frac{i\theta_0 t(x^1)}{2} \left(\partial_1 f \partial_2 g - \partial_2 f \partial_1 g\right) \\
& - \frac{\theta_0^2 t(x^1)^2}{8} \left(\partial_1^2 f \partial_2^2 g + \partial_2^2 f \partial_1^2 g - 2\partial_1\partial_2 f \partial_1\partial_2 g\right) \\
& - \frac{\theta_0^2}{8} \left(x^2 t(x^1) t(x^1)'' \partial_2 \left(\partial_2 f \partial_2 g\right) - 2t'(x^1)^2 \partial_2 \left(x^2 \partial_2 f \partial_2 g \right) \right) \\
& + \frac{\theta_0^2}{4} \left(t(x^1) t'(x^1) \partial_1 \left(\partial_2 f \partial_2 g\right) \right) + \mathcal{O}(\theta_0^3)
\end{aligned}
\tag{26}
$$

where ∂_1 and ∂_2 are partial derivatives with respect to x^1 and x^2. One can explicitly verify that this formula satisfies associativity.

Note that Kontsevich construction [4] leads to equivalence classes (one for each particular function θ) of associative products. The product we have defined is not the representant chosen in [4] where the procedure to change the representant is given.

We are ready now to consider integrals of products of fields and derivatives, starting from the known expressions in terms of the new variables. Taking into account that:

$$
dy^1\, dy^2 \;=\; dx^1\, dx^2\, t^{-1}(x^1)\ . \tag{27}
$$

and starting from an integral I_2 involving the $*$-product of two functions, we can write an equivalent expression in the new variables:

$$
I_2 = \int dy^1 dy^2 \tilde{f}(y^1, y^2) * \tilde{g}(y^1, y^2) \;=\; \int dx^1 dx^2 \frac{1}{t(x^1)} f(x^1, x^2) \star g(x^1, x^2)\ , \tag{28}
$$

where $\tilde{f}(y) \equiv f\big(x(y)\big)$. This expression guarantees that integrals of quadratic terms coincide with ordinary ones (when fields satisfy appropriate boundary conditions). Up to order θ^2 this can be verified using eq.(26).

Hence we see that also in this approach, the natural integration measure, in terms of variables (x^1, x^2) is that given in (12),

$$d\mu(x) = dx^1 dx^2 t^{-1}(x^1) \; . \tag{29}$$

This agrees with the result obtained in [12], based on an operatorial description of noncommutativity.

Let us end the previous discussion by noting that the machinery developed in this section is very helpful in the construction of solitons (vortices and monopoles) and instantons in noncommutative gauge theories. Indeed, one can connect an axially symmetric (in time t) ansatz for instantons in 4-dimensional noncommutative Yang-Mills theory with noncommutative vortices in two dimensional curved space with variables (r, t) . Now, the metric in which this connection is realized takes the form $g^{ij} = r^2\delta^{ij}$. Since it depends on just one variable, one can proceed, as explained above, to change variables in order to work with a standard Moyal product. In this way, passing as usual to the Fock space framework, explicit vortex, monopole and instanton solutions can be found in a very simple way [9]-[10].

3 Example: interface between regions with different θ.

As an application (and extension) of the previous results, we shall study now the case in which $\theta(x^1, x^2)$ is a function whose variation is concentrated along a curve $\mathcal{C}$, and is approximately constant elsewhere. In particular, we are interested in considering cases where θ has different (constant) values on two different regions which have $\mathcal{C}$ as a common interface.

A simple, and yet non-trivial example of this situation corresponds to a $\theta(x^1, x^2)$ with the following structure:

$$\theta(x^1, x^2) \; = \; \theta_+ \, H(x^1 - \varphi(x^2)) \, + \, \theta_- \, H(\varphi(x^2) - x^1) \tag{30}$$

where θ_+ and θ_- are two different real constants, and H denotes the Heaviside's step function, with integral representation

$$H(x) \; = \; \int_{-\infty}^{+\infty} \frac{d\nu}{2\pi i} \, \frac{e^{i\nu x}}{\nu - i0^+} \; . \tag{31}$$

The shape of the interface is determined by the function φ, which we assume to be smooth. The zeroes in the argument of H correspond to

$$x^1 - \varphi(x^2) \;=\; 0 \;, \tag{32}$$

which defines a smooth curve $\mathcal{C}$, dividing the plane into two regions with different values of θ, denoted θ_+ and θ_-; see Figure 1.

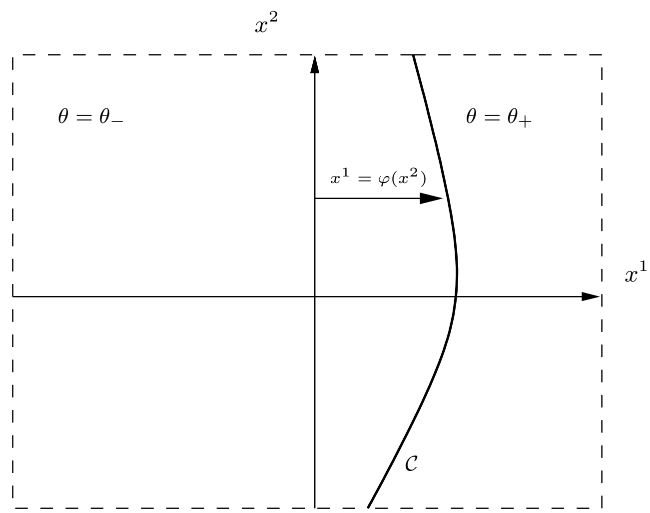

Figure 1: Example of a θ that takes the values θ_- and θ_+ in two regions which are separated by the curve $\mathcal{C}$

Our first step in constructing a noncommutative theory based on (30) consists in smoothing out the shape of the function $\theta(x^1, x^2)$, to avoid introducing singularities due to the existence of a finite jump on the interface [2]. That effect may be achieved by replacing $H(x)$ by a smooth approximation to it, $h_\epsilon(x)$, such that $h_\epsilon(x) \to H(x)$ for $\epsilon \to 0$. For example,

$$h_\epsilon(x) \;=\; \frac{1}{2}\,[1 + \tanh(\frac{x}{\epsilon})] \;. \tag{33}$$

With this particular approximant, which we shall adopt, $\theta(x^1, x^2)$ may be written more explicitly as follows:

$$\theta(x^1, x^2) \;=\; \theta_0 \,+\, \delta\, \tanh(\frac{x^1 - \varphi(x^2)}{\epsilon}) \;, \tag{34}$$

[2] The finite jump may be recovered at the end (if necessary) by considering the appropriate limit.

where we defined $\theta_0 = \frac{\theta_+ + \theta_-}{2}$ and $\delta = \frac{\theta_+ - \theta_-}{2}$.

We now proceed to change variables from the original coordinates (x^1, x^2) to new ones (Q^1, Q^2) which are defined by:

$$\left\{ \begin{array}{lcl} Q^1 & = & x^1 - \varphi(x^2) \\ Q^2 & = & x^2 \quad , \end{array} \right. \tag{35}$$

so that the new variables verify the commutation relation:

$$[Q^1 \, , \, Q^2]_\star \; = \; i \left(\theta_0 \, + \, \delta \, \tanh(\frac{Q^1}{\epsilon}) \right) \; . \tag{36}$$

In what follows we shall assume that the constants θ_+ and θ_- have the same sign (positive, say), since the opposite situation, which necessarily involves a passage of θ through zero, is qualitatively different, as it will become clear below. With this in mind, we may write (36) in the form:

$$[Q^1 \, , \, Q^2]_\star \; = \; i \, \theta_0 \, t(Q^1) \; , \tag{37}$$

where:

$$t(Q^1) \; \equiv \; 1 \, + \, \frac{\delta}{\theta_0} \, \tanh(\frac{Q^1}{\epsilon}) \; . \tag{38}$$

Assuming that both θ_+ and θ_- are positive, the function t shall always be positive, and we are of course in the same situation we considered in 2.1. Based on those results, we introduce a second set of variables, (y^1, y^2), such that:

$$\left\{ \begin{array}{lcl} y^1 & = & Q^1 \\ y^2 & = & \big[t(Q^1)\big]^{-1/2} \, \star \, Q^2 \, \star \, \big[t(Q^1)\big]^{-1/2} \; , \end{array} \right. \tag{39}$$

which verify the constant-θ algebra

$$[y^1 \, , \, y^2]_\star \; = \; i \, \theta_0 \; , \tag{40}$$

with $\theta_0 = \frac{\theta_+ + \theta_-}{2}$ as the noncommutativity parameter.

Then, in terms of the original variables, we have:

$$\left\{ \begin{array}{lcl} y^1 & = & x^1 - \varphi(x^2) \\ y^2 & = & \big[t(x^1 - \varphi(x^2))\big]^{-1/2} \, \star \, x^2 \, \star \, \big[t(x^1 - \varphi(x^2))\big]^{-1/2} \quad . \end{array} \right. \tag{41}$$

Of course, by a similar procedure to the one explained in 2.1, we may write everything in terms of the variables y^1 and y^2, which have a much simpler commutation relation than the original coordinates x^1 and x^2.

The inverse transformation is easy to find,

$$\left\{ \begin{array}{lcl} x^1 & = & y^1 + \varphi\big(t(y^1) \, y^2\big) \\ x^2 & = & t(y^1) \, y^2 \; , \end{array} \right. \tag{42}$$

where all the products are commutative. Therefore, we also have the direct transformation:

$$\begin{cases} y^1 &= x^1 - \varphi(x^2) \\ y^2 &= x^2/t(x^1 - \varphi(x^2)) \ . \end{cases} \tag{43}$$

An important tool in the construction of noncommutative field theories is the definition of derivatives. Since we know two variables y^1 and y^2, which verify a 'canonical' (i.e., constant-θ) commutation relation, it is natural to construct derivative operators in terms of them:

$$\begin{aligned} D_1 &\equiv \frac{\partial}{\partial y^1} = \frac{\partial}{\partial x^1} + x^2 \, \frac{\partial t(x^1 - \varphi(x^2))/\partial x^1}{t(x^1 - \varphi(x^2))} \left[\varphi'(x^2)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\right] \\ D_2 &\equiv \frac{\partial}{\partial y^2} = t(x^1 - \varphi(x^2)) \left[\varphi'(x^2)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}\right] . \end{aligned} \tag{44}$$

We can then obtain an explicit expression for the $\star$-product between f and g: from (22) and (43) using the derivatives above defined,

$$(f \star g)(x^1, x^2) \; = \; \exp\frac{i}{2} \left(\theta(\tilde{x}^1, \tilde{x}^2) D_1 \tilde{D}_2 - \theta(x^1, x^2) D_2 \tilde{D}_1\right) f(x) g(\tilde{x}) \Big|_{\tilde{x}=x} \; , \tag{45}$$

where:

$$\theta(x^1, x^2) \; = \; \theta_0 \, + \, \delta \, \tanh\left(\frac{x^1 - \varphi(x^2)}{\epsilon}\right) . \tag{46}$$

This expression shows explicitly how the interface enters in the star product.

It may be hard to attempt a direct proof of the fact that the operators defined in this way do satisfy Leibniz rule (with the star product (45)). However, this can be easily achieved if we note that D_i are in fact *inner* derivations. Indeed,

$$\frac{\partial}{\partial y^1} f(y) \; = \; i\theta_0 [y^2, f(y)]_* \; = \; i\theta_0 [x^2 \, t^{-1}(x^1 - \varphi(x^2)), f(y(x))]_\star \; , \tag{47}$$

and a similar expression for D_2. See, for example, [12] for a discussion of inner derivations in a general setting.

3.1 Noncommutative Field Theory

One immediate outcome of (43) is the expression for the integration measure, assumed to be defined for y_1 and y_2, in terms of x^1 and x^2:

$$d\mu \; \equiv \; dy^1 \, dy^2 \; = \; dx^1 \, dx^2 \, \left|\frac{\partial(y^1, y^2)}{\partial(x^1, x^2)}\right| \; \equiv \; dx^1 \, dx^2 \, |J| \; . \tag{48}$$

where

$$J \;=\; \left[t(x^1 - \varphi(x^2)\right]^{-1} . \tag{49}$$

It is now evident that the case in which θ changes its sign through the interface is qualitatively different: the change of variables becomes singular on $\mathcal{C}$, when $x^1 \;=\; \varphi(x^2)$, and as a consequence one has to introduce different changes of variables on different charts.

We have at this stage the complete set of tools to define a noncommutative field theory. For a scalar field in Euclidean time $\sigma(\tau, x^1, x^2)$, satisfying a reality condition which we will determine in a moment, we may define the action:

$$S[\sigma] \;=\; \int \frac{d\tau dx^1 dx^2}{|t(x^1 - \varphi(x^2))|} \left[\frac{1}{2}\big(\partial_\tau\sigma\star\partial_\tau\sigma + D_j\sigma\star D_j\sigma + m^2\sigma\star\sigma\big) + V_\star(\sigma)\right] \tag{50}$$

where the D_j's ($j \;=\; 1,2$) have been defined in (44), and the $\star$-product is the one of (45). This expression may of course be converted into its equivalent version in the variables y^j, a procedure that yields the much simpler expression:

$$S[\tilde\sigma] \;=\; \int d\tau dy^1 dy^2 \left[\frac{1}{2}\big(\partial_\tau\tilde\sigma * \partial_\tau\tilde\sigma + \partial_j\tilde\sigma * \partial_j\tilde\sigma + m^2\tilde\sigma * \tilde\sigma\big) + V_*(\tilde\sigma)\right] \tag{51}$$

with $\tilde\sigma(\tau, y) = \sigma(\tau, x(y))$, and $\partial_j \equiv \partial/\partial y^j$. Here it becomes evident that a positive action requires $\tilde\sigma$ to be a real function of (τ, y); then, noting that the transformation (43) is real, $\sigma(\tau, x)$ has to be real.

Working in terms of the variables (y^1, y^2) provides another important simplification: from the Moyal product (18), we see that

$$\int dy^1 dy^2\, f(y) * f(y) \;=\; \int dy^1 dy^2\, f(y)^2 \quad \forall\, f(y) \,. \tag{52}$$

Then the quadratic part of the action coincides with its commutative counterpart. Noncommutative effects appear in $V_\star$ and, when we return to the original coordinates (x^1, x^2), they show up in the existence of nontrivial metric and derivatives.

This procedure of 'pulling back' the action to the coordinates y^j may be used, for example, to derive the free propagator in terms of the 'physical' coordinates x^j. Considering for simplicity the massless case, we see that:

$$\begin{aligned}\langle\sigma(\tau, x)\sigma(\tau', x')\rangle \;&=\; \frac{1}{4\pi}\Big[(\tau - \tau')^2 + (x^1 - x'^{\,1} - \varphi(x^2) + \varphi(x'^2))^2 \\ &+\; \big(\frac{x^2}{t(x^1 \,-\, \varphi(x^2))} - \frac{x'^{\,2}}{t(x'^{\,1} \,-\, \varphi(x'^{\,2}))}\big)^2\Big]^{-1/2} .\end{aligned} \tag{53}$$

This expression allows one to derive, in particular, the propagator for points on the interface. Indeed, considering the propagator for the two points $x = (x^1, x^2)$ and $x' = (x'^{\,1}, x'^{\,2})$, where $x^1 = \varphi(x^2)$ and $x'^{\,1} = \varphi(x'^{\,2})$, we see that:

$$\langle \sigma(\tau, x)\sigma(\tau', x')\rangle|_{\mathcal{C}} \;=\; \frac{1}{4\pi}\Big[(\tau-\tau')^2 + (x^2 - x'^{\,2})^2\Big]^{-1/2} \;, \tag{54}$$

where we used the fact that $t = 1$ on $\mathcal{C}$. The result is different for pairs of arguments that are entirely inside each one of the two constant-θ regions. For example, for the case $x^1 >> \varphi(x^2)$, $x'^{\,1} >> \varphi(x'^{\,2})$, we have:

$$\langle \sigma(\tau, x)\sigma(\tau', x')\rangle \;\sim\; \frac{1}{4\pi}\Big[(\tau-\tau')^2 + (x^1 - x'^{\,1})^2 + (\frac{\theta_0}{\theta_+})^2\,(x^2 - x'^{\,2})^2\Big]^{-1/2} \;. \tag{55}$$

and a similar expression (with $\theta_+ \leftrightarrow \theta_-$) for the other region. Thus the presence of the interface introduces a fundamental change in the propagator, which never becomes equal to the free one, except for points on the interface.

We have mentioned the difficulties that arise when θ vanishes; indeed, it is clear now that this implies a singularity in the change of variables. Assuming that θ_- and θ_+ are such that θ vanishes on the interface, we know that t will vanish (with a non-zero normal derivative) at the interface. One can, however, still define an action for the region where $\theta = \theta_+$, say. It is easy to write that action in terms of the variables y^j, and then pass to the physical variables whenever necessary:

$$S[\tilde\sigma] \;=\; \int_{y_1>0} d\tau dy^1 dy^2 \,\Big[\frac{1}{2}\big(\partial_\tau\tilde\sigma * \partial_\tau\tilde\sigma + \partial_j\tilde\sigma * \partial_j\tilde\sigma + m^2\tilde\sigma * \tilde\sigma\big) + V_*(\tilde\sigma)\Big] \;, \tag{56}$$

since $y^1 > 0$ amounts to $x^1 > \varphi(x^2)$.

It is obvious that the existence of a border at $y^1 = 0$ requires the introduction of a boundary condition for $\tilde\sigma$. A non-trivial consequence of the non-linear relation between the y's and the physical variables is that the Neumann condition: $\partial_1\tilde\sigma(0, y_2) = 0$ becomes too strong in terms of the new variables:

$$\Big[\frac{\partial\sigma}{\partial x^1} \;+\; x^2\,\frac{\partial t(x^1 - \varphi(x^2))/\partial x^1}{t(x^1 - \varphi(x^2))}\,\big(\varphi'(x^2)\frac{\partial\sigma}{\partial x^1} + \frac{\partial\sigma}{\partial x^2}\big)\Big]|_{x^1\to\varphi(x^2)} \;=\; 0 \;. \tag{57}$$

Thus we should have not only $\frac{\partial\sigma}{\partial x^1} = 0$, but also

$$\varphi'(x^2)\frac{\partial\sigma}{\partial x^1} + \frac{\partial\sigma}{\partial x^2} = 0 \tag{58}$$

at the interface, since t vanishes with a non-zero derivative there. The last condition means that the derivative of σin the direction of the curve $\mathcal{C}$ should vanish, since $(\varphi'(x^2), 1)$ is tangent to that curve.

4 Generalization of the method

We now turn to generalizations of the change of variables approach to deal with the general situation [3]

$$[x_1, x_2]_\star = i\,\theta(x_1, x_2) \,. \tag{59}$$

One may wonder to what extent the method and results of sections 2 and 3 depend on choosing the Moyal product to represent relation (17), which is equivalent to the Weyl ordering prescription to define the map between operators and functions. To show that the change of variables is not restricted to this case, we will now work in a normal order, which arises naturally from the holomorphic representation of two-dimensional systems. This framework is particularly appropriate to construct scalar multisolitons, as described in [13] and extended in [14], where the generalization of Berezin approach to deformation quantization is applied to the case of arbitrary Khäler manifolds.

4.1 The holomorphic representation

Planar systems are characterized by creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ that satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \,, \tag{60}$$

and we shall use that representation to discuss different changes of variables. For future convenience, we symbolize them as

$$\hat{a} \to \hat{\bar{z}} \quad , \quad \hat{a}^\dagger \to \hat{z} \,. \tag{61}$$

We represent the algebra over the Hilbert space $\mathcal{H}$ with basis $\{|z\rangle\}$ of eigenstates of $\hat{z}$ (coherent states). On this basis, the action of the operators is

$$\hat{z} \to z \quad , \quad \hat{\bar{z}} \to \frac{\partial}{\partial z} \,. \tag{62}$$

The scalar product in $\mathcal{H}$ is defined by

$$\langle f|g\rangle \equiv \int \frac{dz\, d\bar{z}}{2\pi i}\, \mathrm{e}^{-\bar{z}z}\, \bar{f}(\bar{z})\, g(z) \,, \tag{63}$$

with

$$g(z) \equiv \langle z|g\rangle \quad , \quad \bar{f}(\bar{z}) \equiv \langle f|\bar{z}\rangle \,. \tag{64}$$

[3] In this section we will not continue to interpret $\theta(x)$ as arising from a nontrivial metric. Moreover, we shall not distinguish between sub and super-indices.

With this scalar product, z and $\partial/\partial z$ are relatively adjoint, and the eigenstates

$$\langle z|n\rangle \;=\; \frac{z^n}{\sqrt{n!}} \tag{65}$$

of the Hermitean operator

$$z\partial_z \;=\; a^\dagger\, a \tag{66}$$

form an orthonormal basis. Besides, we can obtain the integral representation of the identity operator:

$$f(z) \;=\; \sum_n \langle z|n\rangle\,\langle n|f\rangle \;=\; \int \frac{dz\,d\bar z}{2\pi i}\;\mathrm{e}^{-\bar z' z'}\,\mathrm{e}^{-\bar z' z}\,f(z')\;. \tag{67}$$

Now we represent the algebraic structure of $\mathcal{A}$ in the space of functions $f(z,\bar z)$ by defining a $\star$-product. To each operator $A(\hat z,\hat{\bar z})$ we associate the function

$$A(z,\bar z) \equiv \mathcal{S}[A(\hat z,\hat{\bar z})] \equiv \frac{\langle z|A(\hat z,\hat{\bar z})|\bar z\rangle}{\langle z|\bar z\rangle}\;. \tag{68}$$

This corresponds to the normal-ordering prescription in which all the $\hat z$ are to the left of all the $\hat{\bar z}$. We define the $\star$-product as

$$(A\star B)(z,\bar z) \;=\; \mathcal{S}[\mathcal{S}^{-1}(\hat A)\mathcal{S}^{-1}(\hat B)]\;. \tag{69}$$

Then

$$(A\star B)(z,\bar z) \;=\; \int \frac{dz'\,d\bar z'}{2\pi i}\;\mathrm{e}^{-(z-z')(\bar z-\bar z')}\;A(z,\bar z')\,B(z,\bar z')\;. \tag{70}$$

It is immediate to verify that $\star$-product reproduces the algebraic structure of $\mathcal{A}$:

$$[\bar z, z]_\star \;=\; 1\;. \tag{71}$$

Some useful relations are:

$$A(z)\star B(z,\bar z) \;=\; A(z)B(z,\bar z)\;, \tag{72}$$

$$A(z,\bar z)\star B(\bar z) \;=\; A(z,\bar z)B(\bar z)\;, \tag{73}$$

$$A(\bar z)\star B(z) \;=\; A(\bar z+\partial_z)B(z)\;, \tag{74}$$

where $\partial_z \equiv \partial/\partial z$ is the usual derivative.

The integral is defined as

$$\int d\mu(z,\bar z)\,A(z,\bar z) \equiv \mathrm{Tr}A(\hat z,\hat{\bar z}) \;=\; \int \frac{dz\,d\bar z}{2\pi i}\;A(z,\bar z)\;. \tag{75}$$

The usual derivatives ∂_z, $\partial_{\bar z}$ are realized as $\star$-product commutators,

$$\partial_z \;=\; [\bar z,\;]_\star \quad,\quad \partial_{\bar z} \;=\; -[z,\;]_\star\;. \tag{76}$$

4.2 General change of variables

Returning to Eq. (59), it is useful to move to the complex plane $(w, \hat{w})$:

$$w = \frac{1}{\sqrt{2}}(x_1 + ix_2) \quad , \quad \bar{w} = \frac{1}{\sqrt{2}}(x_1 - ix_2) \; . \tag{77}$$

Thus we have to study the commutation relation

$$[w, \bar{w}]_\star = \theta\Big(\frac{1}{\sqrt{2}}(w + \bar{w}), -\frac{i}{\sqrt{2}}(w - \bar{w})\Big) \; . \tag{78}$$

We want to define new coordinates $(z, \bar{z})$:

$$w = g(z, \bar{z}) \quad , \quad \bar{w} = \bar{g}(\bar{z}, z) \quad , \tag{79}$$

such that $[\bar{z}, z]_\star = 1$; i.e., the unknown function g must satisfy

$$g(z, \bar{z}) \star \bar{g}(\bar{z}, z) - \bar{g}(\bar{z}, z) \star g(z, \bar{z}) = \tilde{\theta}(z, \bar{z}) \; , \tag{80}$$

where $\tilde{\theta}$ is the function θ expressed in terms of $(z, \bar{z})$.

If this is the case, then we can still apply the formulae from the previous section. For instance

$$(A \star B)(x_1, x_2) = (\tilde{A} \star \tilde{B})(z, \bar{z}) = \int \frac{dz' \, d\bar{z}'}{2\pi i} \, \mathrm{e}^{-(z-z')(\bar{z}-\bar{z}')} \, \tilde{A}(z, \bar{z}') \, \tilde{B}(z', \bar{z}) \; , \tag{81}$$

where

$$\tilde{A}(z, \bar{z}) \equiv A\big(x_j(z, \bar{z})\big)$$

The integrals will be of the form

$$I = \int d\mu(x) \, A(x_1, x_2) \tag{82}$$

and comparing with Eq.(75), we obtain the integration measure

$$d\mu(x) = \frac{1}{2\pi i} J(x_1, x_2) \, dx_1 \, dx_2 \tag{83}$$

where J denotes the Jacobian

$$J(x_1, x_2) \equiv \Big| \frac{\partial(z)}{\partial(w)} \Big| (x_1, x_2) \; . \tag{84}$$

Finally, from (76), two possible inner derivations are

$$D_1 A(x) \equiv [\bar{f}(\bar{w}, w) \, , \, A(x)]_\star = \partial_z \tilde{A}(z, \bar{z}) \; , \tag{85}$$

$$D_2 A(x) \equiv -[f(w, \bar{w}) \, , \, A(x)]_\star = \partial_{\bar{z}} \tilde{A}(z, \bar{z}) \; , \tag{86}$$

being $f(w, \bar{w})$ the inverse function of $g(z, \bar{z})$. They satisfy Leibniz rule and

$$\int d\mu(x) D_j A(x) = 0 \; . \tag{87}$$

4.3 Conformal Transformations

Let us consider the special and important case of conformal transformations

$$w \;=\; g(z) \quad , \quad \bar{w} \;=\; \bar{g}(\bar{z}) \; , \tag{88}$$

with g an analytic function. Applying Eqs.(72)-(74) to our present situation, we see that g must satisfy the condition

$$g(z)\bar{g}(\bar{z}) - \bar{g}(\bar{z}+\partial_z)g(z) \;=\; \tilde{\theta}(z,\bar{z}) \; . \tag{89}$$

This is of course much simpler than the general condition (80), but it will only be useful to deal with some special cases.

For instance, we use this to study the rotation-invariant case

$$[x_1, x_2]_\star \;=\; i\frac{\theta_0}{2}(x_1^2 + x_2^2) \; . \tag{90}$$

In the complex plane,

$$[w, \bar{w}]_\star \;=\; \theta_0 \, w\bar{w} \; . \tag{91}$$

From the condition (89), $w = g(z)$ has to satisfy

$$\bar{g}(\bar{z}+\partial_z)g(z) \;=\; (1-\theta_0)\,\bar{g}(\bar{z})g(z) \; . \tag{92}$$

This suggests to take

$$w \;=\; g(z) \;=\; l\,\mathrm{e}^{az} \tag{93}$$

where l is a parameter with units of length and a is a dimensionless constant. Indeed,

$$\bar{g}(\bar{z}+\partial_z)g(z) \;=\; \mathrm{e}^{a^2}\,\bar{g}(\bar{z})g(z) \; ; \tag{94}$$

so, a must satisfy

$$\mathrm{e}^{a^2} \;=\; 1-\theta_0 \; . \tag{95}$$

According to (83), the integrals will be of the form

$$I \;=\; \int dx_1\,dx_2\,(x_1^2+x_2^2)^{-1}\,A(x_1,x_2) \; . \tag{96}$$

Since

$$z \;=\; \ln^{-1/2}(1-\theta_0)\,\ln\frac{w}{l} \tag{97}$$

the derivations are, using the chain rule,

$$D_1 A(x) \;=\; \partial_z \tilde{A}(z,\bar{z}) \;=\; \ln^{1/2}(1-\theta_0)\,(x_1+ix_2)\Big(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\Big)A(x) \tag{98}$$

$$D_2 A(x) \;=\; \partial_{\bar{z}} \tilde{A}(z,\bar{z}) \;=\; \ln^{1/2}(1-\theta_0)\,(x_1 - i x_2)\Big(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\Big) A(x) \quad (99)$$

From

$$x_1 \;=\; \frac{l}{\sqrt{2}}\,(\mathrm{e}^{az} + \mathrm{e}^{a\bar{z}}) \quad , \quad x_2 \;=\; -\frac{l}{\sqrt{2}}\, i(\mathrm{e}^{az} - \mathrm{e}^{a\bar{z}}) \quad (100)$$

and Eqs.(72)-(74), the fundamental $\star$-products are

$$x_j \star x_j \;=\; x_j^2 - \frac{\theta_0}{4}(x_1^2 + x_2^2) \quad (101)$$

(no summation over repeated indices) and

$$x_j \star x_k \;=\; x_j x_k + i\frac{\theta_0}{4}\epsilon_{jk}(x_1^2 + x_2^2) \quad (102)$$

for $j \neq k$. It is worth noting that with this star product,

$$x_1 \star x_1 + x_2 \star x_2 \;=\; \big(1 - \frac{1}{2}\theta_0\big)(x_1^2 + x_2^2) \quad (103)$$

is also rotation-invariant.

5 Summary and discussion

We have shown how, in some special cases, planar noncommutative theories with a space-dependent θ can be equivalently described in terms of a constant θ, by performing a suitable change of variables. In this way, we have been able to find an explicit representation of the noncommutative product, and to deal with its associated calculus (i.e., derivatives and integration).

We applied that procedure in section 2 to the concrete example of a $\theta(x)$ depending on only one coordinate. We have also unravelled the connection between space-dependent noncommutative parameter $\theta(x)$ and associative noncommutativity in a non-trivial background metric g_{ij}. As explained at the end of this section, apart from its interest *per se*, this connection can be useful in the solution of selfduality noncommutative soliton equations which can be derived, one from the other if one introduces an appropriate background metric.

The resulting $\star$-product was later on (section 3) adapted to the example of a smooth interface $\mathcal{C}$ dividing two regions with different constant θ-values.

The resulting noncommutative field theory obtained in this way could be used, for example, to study the physics of a reduced (relativistic) lowest Landau level description, when the magnetic field is not homogeneous, but rather has two different constant values in two respective regions (which have $\mathcal{C}$ as a boundary).

Finally, in section 4, we sketched the extension of the change of variables to more general situations, by taking advantage of the holomorphic representation for planar systems. The simple and yet important case in which the redefinition of variables is realized by a conformal transformation, has been analyzed in 4.3, applying the results to the particular case of a rotation-invariant θ.

A crucial point in the method is, of course, to find an adequate change of variables. This, in turn, depends on the explicit form of $\theta(x)$, in particular on its symmetries. That is the reason why it does not seem possible to construct an *explicit* coordinate transformation to address the general case. Nevertheless, we believe that the present results can contribute to an understanding of the properties of theories with a space-dependent noncommutativity in a simpler way, by mapping them to systems equipped with the standard Moyal product.

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References

[1] A. Connes, M. Douglas and A. Schwarz, JHEP **9802** 003 (1998). V. Schomerus, JHEP **9906** 030 (1999). N. Seiberg and E. Witten, JHEP **9909** 032 (1999).

[2] M. Ciccolini, C.D. Fosco and A. Lopez. J.Phys.A **35**, 10077 (2002). C. D. Fosco, A. Lopez. J. Phys. **A37**, 4123 (2004).

[3] A. Connes, *Noncommutative Geometry*, Academic Press, 1994 .

[4] M. Kontsevich, Lett. Math. Phys. **66**, 157 (2003).

[5] M. Douglas and N. Nekrasov, Rev. Mod. Phys. 73: 977-1029 (2001).

[6] L. Cornalba and R. Schiappa. Commun. Math. Phys. **225**,33 (2002).

[7] J. Madore, S. Schraml, P. Schupp and J. Wess. Eur. Phys. J. C **16**, 161 (2000). W. Behr and A. Sykora, hep-th/0309145. A. Das and J. Frenkel. Phys. Rev. D **69**, 065017 (2004). X. Calmet, hep-ph/0305027.

[8] A. Cattaneo and G. Felder. Commun.Math.Phys . **212**, 591 (2000).

[9] D. H. Correa, E. F. Moreno and F. A. Schaposnik, Phys. Lett. B **543**, 235 (2002).

[10] D. H. Correa, P. Forgacs, E. F. Moreno, F. A. Schaposnik and G. A. Silva, "Noncommutative 3 dimensional soliton from multi-instantons," arXiv:hep-th/0404015, JHEP in press.

[11] J. Madore gr-qc/9906059.

[12] C. D. Fosco and G. Torroba. Planar Field Theories with space-dependent Noncommutativity. hep-th/0405090.

[13] R. Gopakumar, S. Minwalla and A. Strominger, JHEP **0005**, 020 (2000).

[14] M. Spradlin and A. Volovich, JHEP **0203**, 011 (2002).