17 February 2000

ELSEVIER

PHYSICS LETTERS B

Physics Letters B 474 (2000) 309–313

Schwarz–Sen duality made fully local

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Received 10 December 1999; accepted 6 January 2000
Editor: L. Alvarez-Gaumé

Abstract

Duality symmetric electromagnetic action à la Schwarz–Sen is shown to appear naturally in a chain of equivalent actions which interchange equations of motion with Bianchi identities. Full symmetry of the electromagnetic stress tensor is exploited by generalizing this duality symmetric action to allow for a space-time dependent mixing angle between electric and magnetic fields. The rotated fields are shown to satisfy Maxwell-like equations which involve the mixing angle as a parameter, and a generalized gauge invariance of the new action is established.

1. Introduction

Current interest in duality in string theory has brought about a wealth of studies on similar symmetries present in other contexts, such as abelian p-forms theories. The simplest among the latter, namely free electromagnetism, has long been known to remain invariant under the interchange of equations of motion with Bianchi identities. The first attempt to implement this symmetry at the level of the basic fields of the action [1] involved non local transformation among the A and E fields, in the Hamiltonian (first order) version of the electromagnetic action. Later on, a non covariant action was proposed by Schwarz and Sen [2] where the transformation is made local at the expense of doubling the number of gauge fields. When the equations of motion for some of the fields are used, the usual Maxwell action is recovered.

As it is well known, duality symmetry is actually more general than the discrete interchange of electric with magnetic fields. It is a continuous symmetry, which gets reduced to a $U(1)$ group when invariance of the symmetric stress tensor [2] is also imposed [3]. The conserved momentum associated with this continuous symmetry is found to be the integral of Chern–Simons terms, and full equivalence between Maxwell and Schwarz–Sen actions has been shown to remain valid at the quantum level [4]. Moreover, covariant generalizations of the Schwarz–Sen action have been found by introducing either an infinite number of auxiliary gauge fields [5] or finite additional fields in a non polynomial way [6].

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[2] All future references to the stress tensor will be to the symmetric gauge-invariant one.

0370-2693/00/$ - see front matter
PII: S0370-2693(00)00023-X

On the other hand, when considered as a symmetry of the stress tensor, this duality symmetry is not the most general invariance, because the stress tensor is also invariant under rotations with a different angle at each spacetime point. We call this symmetry of the stress tensor fully-local duality, to distinguish it from that of the Schwarz–Sen action, which is usually called local duality (in opposition to the non-local transformations in [1]). In fact, from our perspective, the duality symmetry of the Schwarz–Sen action could be called global. Our work is reminiscent of the passage from global to local gauge invariance of complex matter fields, wherein gauge fields are introduced in the covariant derivative.

The purpose of this work is twofold. We first show how the Schwarz–Sen action emerges from a chain of equivalent actions which interchange equations of motion with Bianchi identities. Then we rewrite the Schwarz–Sen action in terms of a complex gauge field. We proceed afterwards to probe the effects of such space-time dependent rotations on Maxwell equations and their implementation in a generalized version of the Schwarz–Sen action. We find that gauge invariance remains valid when suitable generalized. At the end we present the conclusions.

2. The Schwarz–Sen action

Free electromagnetism equations of motion [3],

$$\partial_\mu F^{\mu\nu}[A_\mu] = 0\,, \tag{1}$$

with

$$F_{\mu\nu}[A_\mu] = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{2}$$

can be obtained from the action

$$S[A_\mu] = -\tfrac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu}\,, \tag{3}$$

and Bianchi identities

$$\partial_\mu \tilde{F}^{\mu\nu}[A_\mu] = 0\,, \tag{4}$$

hold automatically, with

$$\tilde{F}^{\mu\nu}[A_\mu] = \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}[A_\mu]\,. \tag{5}$$

Alternatively, we can vary

$$S[Z_\mu] = -\tfrac{1}{4}\int d^4x \tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \tag{6}$$

with

$$\tilde{F}_{\mu\nu}[Z_\mu] = \partial_\mu Z_\nu - \partial_\nu Z_\mu \tag{7}$$

obtaining

$$\partial_\mu \tilde{F}^{\mu\nu}[Z_\mu] = 0, \tag{8}$$

as equations of motion and

$$\partial_\mu F^{\mu\nu}[Z_\mu] = 0 \tag{9}$$

as Bianchi identities, with $F^{\mu\nu}[Z_\mu]$ defined by a relation analogous to Eq. (5).

If we define $E^i = F^{i0}$ and $B^i = -\tfrac{1}{2}\epsilon^{ijk}F_{jk}$, regardless of whether $F_{\mu\nu}$ depends on A_μ or Z_μ, we see that the effect of passing from $S[A_\mu]$ to $S[Z_\mu]$ is to interchange equations of motion with Bianchi identities.

Now, $S[A_\mu]$ is equivalent to the first order action

$$\begin{aligned} S[A_\mu,F_{\mu\nu}] &= \int d^4x\left[\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu)\right] \\ &= \int d^4x\left[\tfrac{1}{2}(\mathbf{B}^2 - \mathbf{E}^2) - \mathbf{E}\cdot\dot{\mathbf{A}} - \mathbf{E}\cdot\nabla A_0 \right. \\ &\quad \left. -\mathbf{B}\cdot\nabla\times\mathbf{A}\right], \end{aligned} \tag{10}$$

where $A^\mu = (A^0,\mathbf{A})$. As it is well known, A_μ and $F_{\mu\nu}$ are independent fields in this approach, but varying with respect to $F_{\mu\nu}$ definition (2) is recovered, and replacing it in $S[A_\mu,F_{\mu\nu}]$, we get back $S[A_\mu]$. Now, varying $S[A_\mu,F_{\mu\nu}]$ with respect to B^i and A_0, yields $\mathbf{B} = \nabla\times\mathbf{A}$ and $\nabla\cdot\mathbf{E} = 0$, respectively. Replacing these last equations in (10) we get

$$S[A_i,E_T^i] = \int d^4x\left\{E_T^i\dot{A}_i - \tfrac{1}{2}\left[\mathbf{E}_\mathbf{T}^2 + (\nabla\times\mathbf{A})^2\right]\right\}, \tag{11}$$

[3] We use the metric $g_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$.

where $\mathbf{E_T}$ indicates that only the transversal part of $\mathbf{E}$ survives. Hence, another vector potential can be introduced through $\mathbf{E_T} = \nabla \times \mathbf{Z}$. As we shall see, $\mathbf{Z}$ will be later identified with the spatial components of the tetravector Z_μ, already introduced. Then $S[A_i, E_T^i]$ can be written as

$$S[A_i, Z_i] = -\tfrac{1}{2}\int d^4x \Big[\nabla \times \mathbf{Z} \cdot \dot{\mathbf{A}} - \nabla \times \mathbf{A} \cdot \dot{\mathbf{Z}} + (\nabla \times \mathbf{A})^2 + (\nabla \times \mathbf{Z})^2 \Big], \tag{12}$$

where an integration by parts has been performed, thus exhibiting the symmetry between $\mathbf{Z}$ and $\mathbf{A}$. The equations of motion are now

$$\dot{\mathbf{E}} = \nabla \times \dot{\mathbf{Z}} = \nabla \times \nabla \times \mathbf{A} = \nabla \times \mathbf{B},$$
$$\dot{\mathbf{B}} = \nabla \times \dot{\mathbf{A}} = -\nabla \times \nabla \times \mathbf{Z} = -\nabla \times \mathbf{E} \tag{13}$$

Some comments are in order. First note that the assignment in $S[A_i, Z_i]$ of (13) as equations of motion and $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$ as "Bianchi identities", corresponds neither to $S[A_\mu]$ nor to $S[Z_\mu]$. It is a mixture between the assignments in both actions. Secondly, we could go through steps similar to those that led us from $S[A_\mu]$ to $S[A_i, Z_i]$, but now in reverse order and with Z_μ taking the place of A_μ. The antisymmetric disposition of $\mathbf{Z}$ and $\mathbf{A}$ in the first terms of $S[A_i, Z_i]$ necessitates the definition $Z_\mu = (Z_0, \mathbf{Z})$, which should be compared to $A^\mu = (A^0, \mathbf{A})$. We would end up with an action which can be identified with $S[Z_\mu]$ as defined in (6).

At this point, Schwarz–Sen action is obtained from $S[A_i, Z_i]$ noting that for any functions A_0 and Z_0,

$$\int d^4x \nabla A_0 \cdot \nabla \times \mathbf{Z} = \int d^4x \nabla Z_0 \cdot \nabla \times \mathbf{A} = 0,$$

so that

$$S[A_i, Z_i] = S[A_\mu, Z_\mu] = = -\tfrac{1}{2}\int d^4x \Big[\nabla \times \mathbf{Z} \cdot \big(\dot{\mathbf{A}} + \nabla A_0\big) - \nabla \times \mathbf{A} \cdot \big(\dot{\mathbf{Z}} + \nabla Z_0\big) + (\nabla \times \mathbf{A})^2 + (\nabla \times \mathbf{Z})^2 \Big], \tag{14}$$

which is easily recognized as the Schwarz–Sen action. Eq. (14)summarizes one of the main results of this work: we see that $S[A_\mu, Z_\mu]$ emerges in the middle point of a chain of equivalent actions that lead from $S[A_\mu]$ to $S[Z_\mu]$ and backwards, interchanging equations of motion with Bianchi identities.

It should be stressed that the A_0 and Z_0 fields of the Schwarz–Sen action bear no relation to those which appear in $S[A_\mu]$ and $S[Z_\mu]$. In the former case, advantage is taken of relation (14) to make the integral nicely dependent on two four-potentials, while in the latter case they are used to impose the constraints $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$.

3. Complex field formulation

If we define $\mathbf{\Phi} = \mathbf{A} + i\mathbf{Z}$, $\Phi_0 = A^0 + iZ_0$, $\Phi_\mu = (\Phi_0, \mathbf{\Phi})$ [4], Schwarz–Sen action can be written as

$$S[\Phi_\mu, \Phi_{\mu *}] = -\tfrac{1}{2}\int d^4x \left\{ \frac{i}{2}\Big[\big(\dot{\mathbf{\Phi}} + \nabla\Phi_0\big) \cdot \nabla \times \mathbf{\Phi}^* - \big(\dot{\mathbf{\Phi}}^* + \nabla\Phi_0^*\big) \cdot \nabla \times \mathbf{\Phi} \Big] + \nabla \times \mathbf{\Phi} \cdot \nabla \times \mathbf{\Phi}^* \right\}. \tag{15}$$

Varying with respect to $\Phi^{\mu *}$ yields

$$\nabla \times \dot{\mathbf{\Phi}} = i \nabla \times \nabla \times \mathbf{\Phi} \tag{16}$$

which is the same as Eq. (13), while variation with respect to Φ^μ yields the complex conjugate equation.

$S[\Phi^\mu, \Phi^{\mu *}]$ is separately invariant under the local gauge transformations

$$\mathbf{\Phi} \to \mathbf{\Phi} + \nabla\Psi_1, \tag{17}$$

$$\mathbf{\Phi}^* \to \mathbf{\Phi}^* + \nabla\Psi_2, \tag{18}$$

$$\Phi_0 \to \Phi_0 + \Xi_1, \tag{19}$$

$$\Phi_0^* \to \Phi_0^* + \Xi_2, \tag{20}$$

where Ψ_1, Ψ_2, Ξ_1 and Ξ_2 are arbitrary gauge functions satisfying apropriate boundary conditions. In case we want the surface term picked by the Lagrangian to be real, conditions $\Psi_1 = \Psi_2^*$ and $\Xi_1 = \Xi_2^*$ should be further imposed. The La-

[4] With our choice Φ_μ is not a tetravector since in its definition covariant and contravariant components are summed. Other choices would render it a tetravector.

grangian is also invariant under the simultaneous global U(1) duality rotations

$$\Phi_\mu \to e^{i\alpha}\Phi_\mu, \tag{21}$$

$$\Phi_\mu^* \to e^{-i\alpha}\Phi_\mu^*. \tag{22}$$

Under rotation (21),

$$\mathbf{F} \equiv \nabla \times \mathbf{\Phi} = \mathbf{B} + i\mathbf{E} \tag{23}$$

is transformed to

$$e^{i\alpha}\mathbf{F} = (\cos\alpha\,\mathbf{B} - \sin\alpha\,\mathbf{E}) + i(\cos\alpha\,\mathbf{E} + \sin\alpha\,\mathbf{B}), \tag{24}$$

which gets reduced to the known discrete duality transformation for $\alpha = \pi/2$. This continuous $U(1)$ symmetry has associated the conserved real current $j^\mu = (j^0, \mathbf{j})$ [5], where

$$j^0 = \tfrac{1}{4}(\nabla \times \mathbf{\Phi}^* \cdot \mathbf{\Phi} + \nabla \times \mathbf{\Phi} \cdot \mathbf{\Phi}^*),$$

$$\mathbf{j} = \tfrac{1}{4}\nabla \times (\Phi_0 \mathbf{\Phi}^* + \Phi_0^* \mathbf{\Phi}) + \tfrac{1}{4}(\dot{\mathbf{\Phi}}^* \times \mathbf{\Phi} + \dot{\mathbf{\Phi}} \times \mathbf{\Phi}^*) + \frac{i}{2}[(\nabla \times \mathbf{\Phi}^*) \times \mathbf{\Phi} - (\nabla \times \mathbf{\Phi}) \times \mathbf{\Phi}^*]. \tag{25}$$

In terms of the original **A** and **Z** fields, the conserved momentum reads

$$\int d^3x\, \nabla \times \mathbf{A} \cdot \mathbf{A} + \nabla \times \mathbf{Z} \cdot \mathbf{Z}, \tag{26}$$

which is of the usual Chern–Simons type.

4. Fully-local duality

Given a configuration of electromagnetic fields

$$\mathbf{F}' = \mathbf{B}' + i\mathbf{E}', \tag{27}$$

which satisfies Maxwell equations

$$\nabla \cdot \mathbf{F}' = 0, \quad \dot{\mathbf{F}}' = i\nabla \times \mathbf{F}', \tag{28}$$

if we rotate them through

$$\mathbf{F} = e^{i\alpha(x)}\mathbf{F}', \tag{29}$$

with a space-time dependent angle $\alpha(x)$, the rotated fields satisfy

$$\mathbf{D} \cdot \mathbf{F} = 0, \tag{30}$$

$$D_t\mathbf{F} = i\mathbf{D} \times \mathbf{F}, \tag{31}$$

where

$$D_\mu = (\partial_\mu - i\partial_\mu \alpha) \tag{32}$$

is a kind covariant derivative. Now, taking the real and imaginary parts of Eqs. (30) and (31) we have a total of eight equations. Were one to solve the four $\partial_\mu \alpha$ therefrom, they would be overdetermined. The four compatibility conditions turn out to be

$$\tfrac{1}{2}\partial_t(\mathbf{E}^2 + \mathbf{B}^2) + \nabla \cdot (\mathbf{E} \times \mathbf{B}) = 0,$$

$$\partial_t(\mathbf{E} \times \mathbf{B}) + \mathbf{E} \times \nabla \times \mathbf{E} + \mathbf{B} \times \nabla \times \mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{B}) - \mathbf{E}(\nabla \cdot \mathbf{E}) = 0, \tag{33}$$

which are just the conservation equations of the stress tensor, but now for the rotated fields. This is actually not a piece of news, since the stress tensor is invariant under (29). Of course, not every configuration satisfying the four Eq. (33) comes from a U(1)-local rotation of another configuration which satisfies Maxwell equations, since the $\partial_\mu \alpha$ field solved from Eqs. (30) and (31) must satisfy the integrability conditions $\partial_\mu \partial_\nu \alpha = \partial_\nu \partial_\mu \alpha$.

Now, since the original fields $\mathbf{F}'$ were derived from

$$\mathbf{F}' = \nabla \times \mathbf{\Phi}', \tag{34}$$

the rotated fields can be derived from a rotated potential using a covariant rotor

$$\mathbf{F} = e^{i\alpha(x)}\mathbf{F}' = \mathbf{D} \times \mathbf{\Phi}, \quad \mathbf{\Phi} = e^{i\alpha(x)}\mathbf{\Phi}' \tag{35}$$

Eq. (31) reads now

$$D_t\mathbf{D} \times \mathbf{\Phi} = i\mathbf{D} \times \mathbf{D} \times \mathbf{\Phi}. \tag{36}$$

This equation can be obtained from the following generalization of the Schwarz–Sen action,

$$S[\Phi_\mu, \Phi_{\mu *}, \alpha] = -\tfrac{1}{2}\int d^4x \left\{ \frac{i}{2}[(D_t\mathbf{\Phi} + \mathbf{D}\Phi_0) \cdot \mathbf{D} \times \mathbf{\Phi}^* - (D_t\mathbf{\Phi}^* + \mathbf{D}\Phi_0^*) \cdot \mathbf{D} \times \mathbf{\Phi}] + \mathbf{D} \times \mathbf{\Phi} \cdot \mathbf{D} \times \mathbf{\Phi}^* \right\}. \tag{37}$$

Variation with respect to Φ_μ yields Eq. (36), and with respect to Φ_μ^* yields it complex conjugate. Varying with respect to α yields no new equations.

[5] Note that j^μ is not a tetravector, since the action $S[\Phi_\mu, \Phi_\mu^*]$ is not a scalar.

$S[\Phi^{\mu},\Phi^{\mu *},\alpha]$ is now separately invariant under the local transformations

$$\mathbf{\Phi} \to \mathbf{\Phi} + \mathbf{D}\Psi_1 \tag{38}$$

$$\mathbf{\Phi}^* \to \mathbf{\Phi}^* + \mathbf{D}^*\Psi_2 \tag{39}$$

$$\Phi_0 \to \Phi_0 + \Xi_1 \tag{40}$$

$$\Phi_0^* \to \Phi_0^* + \Xi_2 \tag{41}$$

where the conditions $\Psi_1 = \Psi_2^*$ and $\Xi_1 = \Xi_2^*$ should again be imposed in case we want the surface term picked by the Lagrangian to be real. The Lagrangian is also invariant under the simultaneous local U(1) rotations

$$\Phi_\mu \to e^{i\beta(x)}\Phi_\mu , \tag{42}$$

$$\Phi_\mu^* \to e^{-i\beta(x)}\Phi_\mu^* , \tag{43}$$

together with

$$\alpha \to \alpha + \beta . \tag{44}$$

Under the transformations (42)–(44), the fields

$$\mathbf{F} = \mathbf{D} \times \mathbf{\Phi} = \mathbf{B} + i\mathbf{E} \tag{45}$$

are rotated into

$$e^{i\beta(x)}\mathbf{F} = (\cos\beta(x)\mathbf{B} - \sin\beta(x)\mathbf{E}) + i(\cos\beta(x)\mathbf{E} + \sin\beta(x)\mathbf{B}) . \tag{46}$$

We see that the action $S[\Phi_\mu,\Phi_{\mu *}]$ is a particular case of $S[\Phi_\mu,\Phi_{\mu *},\alpha]$, which is obtained from the latter through the transformations (42-44) with $\beta = -\alpha$.

The conserved current associated to the $U(1)$ symmetry in the generalized action is now

$$\begin{aligned} j^0 &= \tfrac{1}{4}(\mathbf{D} \times \mathbf{\Phi}^* \cdot \mathbf{\Phi} + \mathbf{D} \times \mathbf{\Phi} \cdot \mathbf{\Phi}^*) , \\ \mathbf{j} &= \tfrac{1}{4}\nabla \times (\Phi_0\mathbf{\Phi}^* + \Phi_0^*\mathbf{\Phi}) \\ &\quad + \tfrac{1}{4}(\mathbf{D_t}\mathbf{\Phi}^* \times \mathbf{\Phi} + \mathbf{D_t}\mathbf{\Phi} \times \mathbf{\Phi}^*) \\ &\quad + \frac{i}{2}[(\mathbf{D} \times \mathbf{\Phi}^*) \times \mathbf{\Phi} - (\mathbf{D} \times \mathbf{\Phi}) \times \mathbf{\Phi}^*] . \end{aligned} \tag{47}$$

5. Conclusions

We have shown that the duality symmetric Schwarz–Sen action is the middle point of a chain of equivalent actions, which interchange equations of motion with Bianchi identities. Space-time dependent duality rotations were studied and the equations obeyed by the rotated fields were obtained, along with a generalized action from which these equations are derived. Remarkably, the gauge symmetries of this action are a natural extension of those of the Schwarz–Sen action.

An important property of the new Eqs. (30) and (31) is that for any $\alpha(x)$, every solution of them is mapped one-to-one to a solution of Maxwell equations. Among further developments of this work would be to show how this equivalence holds at the quantum level. This involves an analysis of the constrain structure of $S[\Phi_\mu,\Phi_{\mu *},\alpha]$. Moreover, coupling to external currents in a fully-local-duality-preserving way is also worth studying. We hope to deal with these topics in a next work.

Acknowledgements

We are indebted to Nicolás Grandi and Enrique Moreno for useful comments. We also wish to thank Fidel Schaposnik for constant encouragement during this work.

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