# Spectral shorted operators.

Jorge Antezana, Gustavo Corach and Demetrio Stojanoff

To the memory of Gert K. Pedersen

**Jorge Antezana**,
Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET
e-mail: atenzana@mate.unlp.edu.ar
**Gustavo Corach**,
Depto. de Matemática, Facultad de Ingeniería UBA, Buenos Aires, Argentina and IAM-CONICET
e-mail: gcorach@fi.uba.ar

**Demetrio Stojanoff**,
Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET
e-mail: demetrio@mate.unlp.edu.ar



**Abstract**

If $\mathcal{H}$ is a Hilbert space, $\mathcal{S} \subseteq \mathcal{H}$ is a closed subspace of $\mathcal{H}$, and $A$ is a positive bounded linear operator on $\mathcal{H}$, the spectral shorted operator $\rho(\mathcal{S}, A)$ is defined as the infimum of the sequence $\Sigma(\mathcal{S}, A^n)^{1/n}$, where $\Sigma(\mathcal{S}, B)$ denotes the shorted operator of $B$ to $\mathcal{S}$. We characterize the left spectral resolution of $\rho(\mathcal{S}, A)$ and show several properties of this operator, particularly in the case that $\dim \mathcal{S} = 1$. We use these results to generalize the concept of Kolmogorov complexity for the infinite dimesional case and for non invertible operators.

**MSC 2000:** Primary 47A30, 47B15.
**Keywords:** shorted operator, spectral order, positive operators, spectral resolutions.

## 1 Introduction

Let $\mathcal{H}$ be a separable Hilbert space and $L(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. Given a positive (i.e. semidefinite non negative) operator $A \in L(\mathcal{H})$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, the shorted operator $\Sigma(\mathcal{S}, A)$ was defined by Krein [8] and Anderson-Trapp [2] by

$$\Sigma(\mathcal{S}, A) = \max\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}\},$$

where the maximum is taken for the natural order relation in $L(\mathcal{H})^+$, the set of positive operators in $L(\mathcal{H})$ (see [2], [14], [15]).

In a previous paper [3], the authors have defined, under the asuumption that $\dim \mathcal{H} < \infty$, the so called spectral shorted matrix:

$$\rho(\mathcal{S}, A) = \lim_{m\to\infty} \Sigma(\mathcal{S}, A^m)^{1/m} = \inf_{m\to\infty} \Sigma(\mathcal{S}, A^m)^{1/m}. \tag{1}$$

This paper is the continuation of [3]. It is devoted to study the natural generalization of $\rho$ to the infinite dimensional setting. If $\dim \mathcal{H} = \infty$ and $A \in L(\mathcal{H})^+$, the operator $\rho(\mathcal{S}, A)$ is also defined by equation (1), under the assumption that the subspace $\mathcal{S}$ is closed. We call this operator the *spectral shorted operator* associated to $\mathcal{S}$ and $A$.

Many properties of the spectral shorted matrices showed in [3] hold also for spectral shorted operators, but some of them must be formulated in terms of the spectral measure of $A$ instead of eigenvalues and eigenspaces, as in [3].

As in the matrix case, the properties of $\rho$ are strongly related with the so called *spectral order* of positive operators. Recall the definition of the spectral order $\preccurlyeq$ in $L(\mathcal{H})^+$: given $A, B \in L(\mathcal{H})^+$, we write $A \preccurlyeq B$ if $A^m \le B^m$ for all $m \ge 1$. The spectral order was extensively studied by M. P. Olson in [11], where the following characterization is proved: given $A, B \in L(\mathcal{H})^+$, then $A \preccurlyeq B$ if and only if $f(A) \le f(B)$ for every non-decreasing map $f : [0, +\infty) \to \mathbb{R}$.

Section 2 contains preliminaries and a brief account of the main properties of the shorting operation, spectral order and spectral resolutions. In section 3 we collect those properties of $\rho$ which can be plainly generalized to the infinite dimensional setting. The most subtle tool is the continuity of the map $t \mapsto t^r$ (for $0 \le r \le 1$) with respect to the strong operator topology on $L(\mathcal{H})^+$. It is used, for instance, for proving that for every $t > 0$,

$$\rho\left(\mathcal{S}, A^t\right) = \rho(\mathcal{S}, A)^t. \tag{2}$$

The spectral order provides the following link with Krein and Anderson-Trapp definition of the shorted operator: $\rho(\mathcal{S}, A)$ is the biggest (in both orders $\le$ and $\preccurlyeq$) element $D$ of $L(\mathcal{H})^+$ such that $D \preccurlyeq A$ and $R(D) \subseteq \mathcal{S}$ (see Theorem 3.5). This shows the monoticity of $\rho(\mathcal{S}, \cdot)$ with respect to the preorder $\preccurlyeq$ and allows us to get some results about limits of spectral shorted operators.

In section 3 we get a complete characterization of $\rho(\mathcal{S}, A)$ in terms of the (left) spectral resolution of $A$: for every $0 < \lambda \in \mathbb{R}$,

$$\aleph_{[\lambda,\infty)}(\rho(\mathcal{S}, A)) = \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}}.$$

This results allows us to get simple proofs in our context of several properties of spectral shorted matrices. For example, given $A \in L(\mathcal{H})^+$ and two closed subspaces $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{H}$,

1. $\rho(\mathcal{S} \cap \mathcal{T}, A) = \rho(\mathcal{T}, \rho(\mathcal{S}, A))$.

2. $\sigma(\rho(\mathcal{S}, A)) \subseteq \sigma(A)$.

3. $f(\rho(\mathcal{S}, A)) = \rho(\mathcal{S}, f(A))$, for every non-decreasing *right continuous* positive function $f$ defined on $[0, +\infty)$.

4. $\lambda_{min}(A)P_{\mathcal{S}} \leq \rho\,(\mathcal{S}, A)$, where $\lambda_{min}(C) = \min \sigma\,(C)$, for $C \in L(\mathcal{H})^+$.
5. If $\rho\,(\mathcal{S}, A)$ is considered as acting in $\mathcal{S}$, then

$$\lambda_{min}(\rho\,(\mathcal{S}, A)) = \min\{\mu \in \sigma\,(A) :\; P_{\mathcal{S}}\, \aleph_{[\mu,\mu+\varepsilon)}(A) \neq 0 \;\; \forall\, \varepsilon > 0\}.$$

The case $\dim \mathcal{S} = 1$ is extensively studied in section 5. If $\mathcal{S}$ is the subspace generalted by the unit vector $\xi$, we denote by $\rho\,(A, \xi)$ the unique positive number such that $\rho\,(\mathcal{S}, A) = \rho\,(A, \xi)\, P_{\mathcal{S}}$. The main results of this secton are:

1. If $A \in L(\mathcal{H})^+$ and $\xi \in \mathcal{H}$ is an unit vector, then

$$\rho\,(A, \xi) = \min \sigma\,(\rho\,(\mathcal{S}, A)) = \min \Big\{\mu \in \sigma\,(A) :\; \aleph_{[\mu,\mu+\varepsilon)}(A)\xi \neq 0 \;\; \forall\, \varepsilon > 0\Big\}.$$

2. $\rho\,(A, \xi) = \max\{\lambda \in \sigma\,(A) : \xi \in R(\aleph_{[\lambda,\infty)}(A))\}$.
3. If $A$ is invertible, then $\rho\,(A, \xi) = \lim\limits_{m\to\infty} \|A^{-m}\xi\|^{-1/m} = \inf\limits_{m\in\mathbb{N}} \|A^{-m}\xi\|^{-1/m}$.
4. If $R(A)$ is closed and $\xi \in R(A)$, then, $\rho\,(A, \xi) = \lim_{m\to\infty} \|(A^\dagger)^m\xi\|^{-1/m}$, where $A^\dagger$ is the Moore-Penrose pseudo-inverse of $A$. If $\xi \notin R(A)$, then $\rho\,(A, \xi) = 0$.
5. If $\sigma_\rho\,(A) = \big\{\rho\,(A, \xi) : \|\xi\| = 1\big\}$, then

$$\sigma_\rho\,(A) = \sigma_+\,(A) \cup \sigma_{pt}(A) = \big\{\lambda \in \sigma\,(A) : \forall\, \varepsilon > 0\;,\; \aleph_{[\lambda,\lambda+\varepsilon)}(A) \neq 0\big\},$$

   where $\sigma_{pt}(A)$ denotes the point spectrum of $A$, i.e the set of eigenvalues of $A$ and $\sigma_+\,(A)$ is the set of points in $\sigma\,(A)$ which are limit point of $\sigma\,(A) \setminus \{\lambda\}$ from the right. This shows that $\sigma_\rho\,(A)$ is allways dense in $\sigma\,(A)$, but $\sigma_\rho\,(A) \neq \sigma\,(A)$ in general.
6. $\rho\,(A, \xi) \neq 0$ if and only if $\xi \in R_0(A) := \bigcup_{\lambda>0} R(\aleph_{[\lambda,\infty)}(A)) \subseteq R(A)$.

In [5], J. I. Fujii and M. Fujii consider the Kolmogorov's complexity

$$K\,(A, \xi) = \lim_{n\to\infty} \frac{\log(\langle A^n\xi, \xi\rangle)}{n} = \log \lim_{n\to\infty} \langle A^n\xi, \xi\rangle^{1/n}\,. \tag{3}$$

for an invertible positive matrix $A$ and a unit vector $\xi$ and show several properties $K$. In [3] we show that, if $\mathcal{S}$ is the subspace generated by $\xi$, then

$$K\,(A, \xi) = \log \rho\,\big(A^{-1}, \xi\big)^{-1}\,.$$

We define a generalized version (for $\dim \mathcal{H} = \infty$ and $A \in L(\mathcal{H})^+$ not necessarily invertible) without logarithms (in order to avoid the value $-\infty$) of the Kolmogorov complexity as follows: given $\xi \in \mathcal{H}$, $\xi \neq 0$ and $A \in L(\mathcal{H})^+$, we denote by

$$k\,(A, \xi) = \lim_{n\to\infty} \langle A^n\xi, \xi\rangle^{1/n}\,,$$

that is, $k\,(A, \xi) = \exp K\,(A, \xi)$ in the cases where $K\,(A, \xi)$ can be defined as in equation (3). Among other properties, we show that: if $\xi \in \mathcal{H}$ and $A \in L(\mathcal{H})^+$, then

1. If $\|\xi\| = 1$, then the sequence $\langle A^n\xi, \xi\rangle^{1/n}$ is increasing. So that, $\lim_{n\to\infty} \langle A^n\xi, \xi\rangle^{1/n}$ exists for every $\xi \in \mathcal{H}$.

2. $k(A, \xi) = k(A, a\xi)$ for every $0 \neq a \in \mathbb{C}$.

3. $k(A, \xi) = k\big(A, \aleph_{[\lambda,\infty)}(A)\xi\big)$ for every $\lambda > 0$ such that $\aleph_{[\lambda,\infty)}(A)\xi \neq 0$.

4. $k(A, \xi) \neq 0$ (i.e. $K(A, \xi) \neq -\infty$) if and only if $P_{\overline{R(A)}}\, \xi \in R_0(A) \setminus \{0\}$.

5. If $\xi \neq 0$, then $k(A, \xi) \in \sigma(A)$. Moreover,

$$\{k(A, \xi) : \xi \neq 0\} = \{\lambda \in \sigma(A) : \aleph_{(\lambda+\varepsilon,\lambda]}(A) \neq 0 \ , \ \forall\, \varepsilon > 0\ \},$$

which is a dense subset of $\sigma(A)$.

6. 
$$\begin{aligned} k(A, \xi) &= \min\{\lambda \in \sigma(A) : \xi \in R(\aleph_{(-\infty,\lambda]}(A))\} \\ &= \max\Big\{\mu \in \sigma(A) :\ \aleph_{(\mu-\varepsilon,\mu]}(A)\xi \neq 0 \ \ \forall\, \varepsilon > 0\Big\} \\ &= \sup\Big\{\mu \in \sigma(A) :\ \aleph_{[\mu,\infty)}(A)\xi \neq 0\Big\} \ . \end{aligned}$$

7. If $R(A)$ is closed, then

   (a) If $\xi \in R(A)$ then $k(A, \xi) = \rho\left(A^\dagger, \xi\right)^{-1}$.

   (b) If $\xi \notin R(A)$, but $P\xi \neq 0$, where $P = P_{R(A)}$, then

$$k(A, \xi) = k(A, P\xi) = \rho\left(A^\dagger, \frac{P\xi}{\|P\xi\|}\right)^{-1} \ .$$

# 2 Preliminaries

For an operator $A \in L(\mathcal{H})$, we denote by $R(A)$ the range of $A$, $N(A)$ the null-space of $A$, $\sigma(A)$ the spectrum of $A$, $A^*$ the adjoint of $A$, $\rho(A)$ the spectral radius of $A$, $\|A\|$ the spectral norm (i.e. the operator norm induced by the norm of the Hilbert space $\mathcal{H}$) of $A$. We denote by $L(\mathcal{H})_{sa}$ the space of selfadjoint operators in $L(\mathcal{H})$ and by $L(\mathcal{H})^+$ the space of positive (i.e. semidefinite non-negative) operators in $L(\mathcal{H})$. If $A \in L(\mathcal{H})_{sa}$, we denote by $\lambda_{min}(A) = \min \sigma(A) = \inf_{\|\xi\|=1}\langle A\xi, \xi\rangle$.

Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, we denote by $P_\mathcal{S}$ the orthogonal (i.e. selfadjoint) projection onto $\mathcal{S}$. If $P$ and $Q$ are orthogonal projections, we denote by $P \wedge Q$ the orthogonal projection onto $R(P) \cap R(Q)$. If $B \in L(\mathcal{H})$ satisfies $P_\mathcal{S} B P_\mathcal{S} = B$, we sometimes consider the compression of $B$ to $\mathcal{S}$, (i.e. the restriction of $B$ to $\mathcal{S}$ as a linear transformation form $\mathcal{S}$ to $\mathcal{S}$), and we say that we consider $B$ as *acting* on $\mathcal{S}$. Several times this is done in order to consider $\sigma(B)$ just in terms of the action of $B$ on $\mathcal{S}$. For example, if $B \geq \lambda P_\mathcal{S}$ for some $\lambda > 0$, then we can deduce that $0 \notin \sigma(B)$, if we consider $B$ as acting on $\mathcal{S}$.

Along this note we use the fact that every closed subspace $\mathcal{S}$ of $\mathcal{H}$ induces a representation of elements of $L(\mathcal{H})$ by $2 \times 2$ block matrices, that is, we shall identify each $A \in L(\mathcal{H})$ with a $2 \times 2$-matrix, let us say $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$. Observe that $\begin{pmatrix} A_{11}^* & A_{21}^* \\ A_{12}^* & A_{22}^* \end{pmatrix}$ is the matrix which represents $A^*$.

We use in this note several standard results of spectral theory, functional calculus and weak convergences of opeartors in $L(\mathcal{H})_{sa}$. About these matters, we refer the reader to the books of Pedersen [13] or Kadison and Ringrose [7]. If $A \in L(\mathcal{H})_{sa}$ we denote by $E_A$ the spectral measure associated to $A$, defined by $E_A(\Delta) = \aleph_\Delta(A)$, for any Borel set $\Delta \subseteq \mathbb{R}$. The sigles SOT are used to mention the *strong operator topology* of $L(\mathcal{H})_{sa}$. In the following subsections, we state explicitely several known results which we shall need in the sequel. Particulrly those we think are not "de libro".

## Shorted operators.

Following Anderson and Trapp [1], [2], we define

**Definition 2.1.** Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then, the *shorted operator* of $A$ to $\mathcal{S}$ is defined by

$$\Sigma(\mathcal{S}, A) = \max\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}\},$$

where the maximum is taken for the natural order relation in $L(\mathcal{H})^+$ (see [2]).

In the next theorem we state some results on shorted operators proved by Anderson and Trapp [2], M.G. Krein [8] and E. L. Pekarev [14] which are relevant in this paper.

**Theorem 2.2.** *Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathcal{H}$ and let $A, B \in L(\mathcal{H})^+$. Then*

1. *If $\mathcal{S} \subseteq \mathcal{T}$, then, $\Sigma(\mathcal{S}, A) \leq \Sigma(\mathcal{T}, A)$.*
2. *$\Sigma(\mathcal{S} \cap \mathcal{T}, A) = \Sigma(\mathcal{S}, \Sigma(\mathcal{T}, A))$.*
3. *If $A \leq B$, then, $\Sigma(\mathcal{S}, A) \leq \Sigma(\mathcal{S}, B)$.*
4. *Let $\mathcal{M} = A^{-1/2}(\mathcal{S})$. Then $\Sigma(\mathcal{S}, A) = A^{1/2} P_{\mathcal{M}} A^{1/2}$.*

There are also some results about the continuity of the shorting operation (see [2], Corollary 2 and 3).

**Proposition 2.3.** *Let $A_n$ $(n \in \mathbb{N})$ be a sequence of positive matrices such that $A_n \underset{n\to\infty}{\overset{SOT}{\searrow}} A$. Then, for every closed subspace $\mathcal{S}$ it holds $\Sigma(\mathcal{S}, A_n) \underset{n\to\infty}{\overset{SOT}{\searrow}} \Sigma(\mathcal{S}, A)$.*

**Proposition 2.4.** *Let $\mathcal{S}_n$ $(n \in \mathbb{N})$ and $\mathcal{S}$ be closed subspaces such that $P_{\mathcal{S}_n} \underset{n\to\infty}{\overset{SOT}{\searrow}} P_{\mathcal{S}}$. Then, for every $A \in L(\mathcal{H})^+$, it holds that $\Sigma(\mathcal{S}_n, A) \underset{n\to\infty}{\overset{SOT}{\searrow}} \Sigma(\mathcal{S}, A)$.*

*Proof.* Since $\{\Sigma(\mathcal{S}_n, A)\}$ is a non-increasing sequence, it has a strong limit, say $L$. As $\Sigma(\mathcal{S}_n, A) \leq A$ for all $n \in \mathbb{N}$, then $L \leq A$. On the other hand, $L \leq \Sigma(\mathcal{S}_n, A)$ implies

$$R(L^{1/2}) \subseteq R\left(\Sigma(\mathcal{S}_n, A)^{1/2}\right) \subseteq \mathcal{S}_n \quad \forall\, n \in \mathbb{N}.$$

Therefore $R(L) \subset \bigcap_{n=1}^{\infty} \mathcal{S}_n = \mathcal{S}$. Finally, if $0 \leq X \leq A$ and $R(X) \subset \mathcal{S}$, then $R(X) \subseteq \mathcal{S}_n$, so that $X \leq \Sigma(\mathcal{S}_n, A)$, for all $n \in \mathbb{N}$. Therefore $X \leq L$. ∎

## Spectral order.

The spectral order was considered by Olson (see [11]) with the purpose of reporting an order relation with respect to which the real vector space of selfadjoint operators form a conditionally complete lattice. Throughout this note we shall only use the spectral order for positive operators, and this is the reason why we take the following statement as definition of the spectral order.

**Definition 2.5.** Let $A, B \in L(\mathcal{H})^+$. We write $A \preccurlyeq B$ if for every $m \in \mathbb{N}$ it holds that $A^m \leq B^m$. The relation $\preccurlyeq$ defined on $L(\mathcal{H})^+$ is a partial order and it is called *spectral order*.

**Examples 2.6.** Given $A, B \in L(\mathcal{H})^+$. Then

1. If $AB = BA$ and $A \leq B$, then, $A \preccurlyeq B$.
2. If $\dim \mathcal{H} = n < \infty$, then $A \preccurlyeq B$ if and only if there is a positive integer $k \leq n$ and an sequence of positive matrices $\{D_i\}_{0 \leq i \leq k}$ such that, $D_0 = A$, $D_k = B$, $D_i \leq D_{i+1}$ and $D_i D_{i+1} = D_{i+1} D_i$ $(i = 0, \cdots, k-1)$ (see [3]). ▲

The next results was proved by Olson in [11].

**Theorem 2.7.** *Let $A, B \in L(\mathcal{H})^+$. The following statements are mutually equivalent.*

**(1)** $A \preccurlyeq B$,

**(2)** $\aleph_{[\lambda,\infty)}(A) \leq \aleph_{[\lambda,\infty)}(B)$ $(0 \leq \lambda < \infty)$,

**(3)** *$f(A) \leq f(B)$ for every non-decreasing continuous function $f$ on $[0, \infty)$.*

The following result about functions which are continuous relative to the S.O.T topology of $L(\mathcal{H})^+$ or $L(\mathcal{H})_{sa}$ is a key tool for the extention of the results about spectral shorted operators from matrices to operators in Hilbest spaces. A proof can be found, for example, in Pedersen's book [12].

**Lemma 2.8.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $|f(t)| \leq \alpha|t| + \beta$ for some positive numbers $\alpha$ and $\beta$. Then, if $\{A_\alpha\}_{\alpha \in \Lambda}$ is a net in $L(\mathcal{H})_{sa}$ such that $A_\alpha \xrightarrow{S.O.T.} A \in L(\mathcal{H})_{sa}$, it holds that $f(A_\alpha) \xrightarrow{S.O.T.} f(A)$, i.e. $f : L(\mathcal{H})_{sa} \to L(\mathcal{H})_{sa}$ is continuous for the S.O.T. topology. In particular $f(t) = t^r$ for $0 \leq r \leq 1$ is S.O.T.-continuous in $L(\mathcal{H})^+$.*

**Proposition 2.9.** *Let $\{A_n\}$ be a sequence in $L(\mathcal{H})^+$ such that $A_{n+1} \preccurlyeq A_n$, $n \in \mathbb{N}$ and $A_n \underset{n\to\infty}{\overset{SOT}{\searrow}} A \in L(\mathcal{H})^+$. Then, for every $k \in \mathbb{N}$, $A_n^k \underset{n\to\infty}{\overset{SOT}{\searrow}} A^k$. In particular, $A \preccurlyeq A_n$, $n \in \mathbb{N}$.*

*Proof.* Fix $k \in \mathbb{N}$. Since the sequence $\{A_n\}$ is non increasing with respect to the spectral order, there exists $B \in L(\mathcal{H})^+$ such that $A_n^k \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} B$. By Lemma 2.8, applied to the map $f(t) = t^{1/k}$, we can deduce that $A_n \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} B^{1/k} = A$. So that, $B = A^k$. ■

### Spectral resolutions

Given $f : \mathbb{R} \to L(\mathcal{H})$, we say that $f$ is a right (resp. left) *spectral resolution* if

1. There exist $m, M \in \mathbb{R}$ such that $f(\lambda) = 0$ for $\lambda < m$ and $f(\lambda) = I$ for $\lambda > M$ (resp. $f(\lambda) = I$ for $\lambda < m$ and $f(\lambda) = 0$ for $\lambda > M$).
2. $f(\lambda)$ is a selfadjoint projection, for every $\lambda \in \mathbb{R}$.
3. If $\lambda \leq \mu$ then $f(\lambda) \leq f(\mu)$ (resp. $f(\lambda) \geq f(\mu)$) as operators.
4. $f$ is continuous on the right (resp. $f$ is continuous on the left).

Under these hypothesis, by the standard spectral theory, there exists an unique $A \in L(\mathcal{H})_{sa}$ such that $f$ is its spectral resolution, i.e.

$$f(\lambda) = E_A(\ (-\infty, \lambda]) = \aleph_{(-\infty,\lambda]}\ (A) \quad \text{(resp. } f(\lambda) = E_A(\ [\lambda, \infty)\ ) = \aleph_{[\lambda,\infty)}(A)\ ). \tag{4}$$

Conversely, if $A \in L(\mathcal{H})_{sa}$, then the map $f$ defined by equation (4) is a right (resp. left) spectral resolution.

The relation between right and left spectral resolutions is given by the following identity: if $A \in L(\mathcal{H})_{sa}$, then $E_A(\ [-\lambda, \infty)\ ) = E_{-A}(\ (-\infty, \lambda])$. On the other hand, if $f$ is a left spectral resolution, then $g(\lambda) = f(-\lambda)$ is a right spectral resolution. Then, if $A$ is the operator associated to $g$, then $-A$ is the operator associated to $f$.

## 3 The spectral shorted operator

In this section we define the spectral shorted operator in the infinite dimensional setting, and we show its basics properties. All results and proofs of this section ar very similar as those which appear in [3] for de finite dimensional case, but using SOT-convergence instead of convergence in norm. The main difference is that, in the proof of Proposition 3.4, we need to apply Lemma 2.8 about SOT-continuity of the map $A \mapsto A^r$ for $0 \leq r \leq 1$. Also Propostion 3.7 is a properly infinite dimensional result.

**Proposition 3.1.** *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace. Then the map $t \mapsto \Sigma(\mathcal{S}, A^t)^{1/t}$, $t \in [1, \infty)$ is non-increasing.*

*Proof.* Fix $t \geq 1$. Then $\Sigma(\mathcal{S}, A^t) \leq A^t$. Since $0 \leq 1/t \leq 1$, by Löwner theorem we can deduce that $\Sigma(\mathcal{S}, A^t)^{1/t} \leq A$. On the other hand $R(\Sigma(\mathcal{S}, A^t)^{1/t}) \subseteq \mathcal{S}$. So, by the definition of shorted operator, $\Sigma(\mathcal{S}, A^t)^{1/t} \leq \Sigma(\mathcal{S}, A)$. Now, given $1 \leq r \leq s$, take $t = s/r \geq 1$. By the previous remarks, applied to $A^r$ and $t$, we have that

$$\Sigma(\mathcal{S}, A^r) \geq \Sigma\left(\mathcal{S}, A^{rt}\right)^{1/t} = \Sigma(\mathcal{S}, A^s)^{r/s}.$$

Since $1/r \leq 1$, by Löwner theorem we have that $\Sigma(\mathcal{S}, A^r)^{1/r} \geq \Sigma(\mathcal{S}, A^s)^{1/s}$. ■

**Definition 3.2.** Let $A \in L(\mathcal{H})^+$, and let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. We denote by

$$\rho(\mathcal{S}, A) = \inf_{t \geq 1} \Sigma(\mathcal{S}, A^t)^{1/t} = \lim_{t \to +\infty} \Sigma(\mathcal{S}, A^t)^{1/t},$$

where the limit is taken in the strong operator topology (SOT).

**Remark 3.3.** Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces.

1. If $A = P_{\mathcal{T}}$, then $\rho(\mathcal{S}, A) = \Sigma(\mathcal{S}, A^t)^{1/t} = P_{\mathcal{S} \cap \mathcal{T}}$, for every $t \in [1, \infty)$.
2. If $AP = PA$, then $\rho(\mathcal{S}, A) = \Sigma(\mathcal{S}, A^t)^{1/t} = PA$, for every $t \in [1, \infty)$.
3. $\rho\left(\mathcal{S}, cA\right) = c\,\rho\left(\mathcal{S}, A\right)$ for every $c \in [0, +\infty)$.
4. If $\mathcal{S} \subseteq \mathcal{T}$, then, $\rho\left(\mathcal{S}, A\right) \leq \rho\left(\mathcal{T}, A\right)$, since $\Sigma\left(\mathcal{S}, A^t\right)^{1/t} \leq \Sigma\left(\mathcal{T}, A^t\right)^{1/t}$ for every $t \geq 1$. ▲

**Proposition 3.4.** *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. Then, for every $t \in (0, \infty)$ it holds that*

$$\rho\left(\mathcal{S}, A\right)^t = \rho\left(\mathcal{S}, A^t\right)$$

*In particular, for every $t \in (0, \infty)$*

$$\rho\left(\mathcal{S}, A\right)^t \leq A^t$$

*Proof.* Firstly, we prove the statement for $t \geq 1$. By Lemma 2.8, the map $x \to x^r$ is continuous in the strong operator topology when $0 \leq r \leq 1$. So, given $t \in (1, \infty)$, since $st \to \infty$ as $s \to \infty$, we have that

$$\rho\left(\mathcal{S}, A^t\right)^{1/t} = \left(\lim_{s \to \infty} \Sigma\left(\mathcal{S}, (A^t)^s\right)^{1/s}\right)^{1/t} = \lim_{s \to \infty} \Sigma\left(\mathcal{S}, A^{st}\right)^{1/st} = \rho\left(\mathcal{S}, A\right),$$

where the limits are taken in the strong operator topology. This proves, for $t \geq 1$, that

$$\rho\left(\mathcal{S}, A^t\right) = \rho\left(\mathcal{S}, A\right)^t. \tag{5}$$

Now, if $t \in (0, 1)$,

$$\rho\left(\mathcal{S}, A^t\right) = \left(\rho\left(\mathcal{S}, A^t\right)^{1/t}\right)^t = \rho\left(\mathcal{S}, (A^t)^{1/t}\right)^t = \rho\left(\mathcal{S}, A\right)^t,$$

where in the second equality, we have used equation (5) for $\dfrac{1}{t} \geq 1$. ■

Recall that given two positive operators $A$ and $B$ we say that

$$A \preccurlyeq B \qquad \text{if} \qquad A^n \leq B^n \quad \forall n \geq 1$$

With respect to this order, the spectral shorted operator has a characterization similar to Anderson-Trapp's definition of shorted operator.

**Theorem 3.5.** *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. If*

$$\mathcal{M}_\rho(\mathcal{S}, A) = \{D \in L(\mathcal{H})^+ : \ D \preccurlyeq A, \ R(D) \subseteq \mathcal{S}\}$$

*then*

$$\rho(\mathcal{S}, A) = \max \mathcal{M}_\rho(\mathcal{S}, A),$$

*where the "maximum" is taken for any of the orders $\leq$ and $\preccurlyeq$.*

*Proof.* Firstly, note that $\rho(\mathcal{S}, A) \in \mathcal{M}_\rho(\mathcal{S}, A)$. In fact, $\rho(\mathcal{S}, A)^m \leq A^m$ for every $m \in \mathbb{N}$ by Proposition 3.4, and $R(\rho(\mathcal{S}, A)) \subseteq \mathcal{S}$ by definition.

Suppose that $D \in \mathcal{M}_\rho(\mathcal{S}, A)$. Fix $m \in \mathbb{N}$. As $D^m \leq A^m$, it holds that $\Sigma(\mathcal{S}, D^m)^{1/m} \leq \Sigma(\mathcal{S}, A^m)^{1/m}$. Since $\Sigma(\mathcal{S}, D^m)^{1/m} = D$, taking $m \to \infty$ we have $D \leq \rho(\mathcal{S}, A)$. This shows that $\rho(\mathcal{S}, A) = \max \mathcal{M}_\rho(\mathcal{S}, A)$ for the usual order.

Note also that, if $D \in \mathcal{M}_\rho(\mathcal{S}, A)$, then for every $k \in \mathbb{N}$, $D^k \preccurlyeq A^k$ and $D^k \in \mathcal{M}_\rho(\mathcal{S}, A^k)$. By the previous case, applied to $A^k$, one gets

$$D^k \leq \rho\left(\mathcal{S}, A^k\right) = \rho(\mathcal{S}, A)^k \ , \quad k \in \mathbb{N}.$$

Hence $D \preccurlyeq \rho(\mathcal{S}, A)$. ■

**Corollary 3.6.** *Let $A$ and $B$ be positive operators such that $A \preccurlyeq B$ and $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces such that $\mathcal{S} \subseteq \mathcal{T}$. Then $\rho(\mathcal{S}, A) \preccurlyeq \rho(\mathcal{T}, B)$.*

*Proof.* It is enough to note that $\mathcal{M}_\sigma(\mathcal{S}, A) \subseteq \mathcal{M}_\sigma(\mathcal{T}, B)$. ■

Another application of Theorem 3.5 is the following result about the convergence of sequences of spectral shorted operators.

**Proposition 3.7.** *Let $\{A_n\}$ be a sequence in $L(\mathcal{H})^+$ such that $A_{n+1} \preccurlyeq A_n$, $n \in \mathbb{N}$ and $A_n \xrightarrow[n\to\infty]{S.O.T.} A$, and let $\{\mathcal{S}_n\}$ be a sequence of subspaces such that $\mathcal{S}_{n+1} \subseteq \mathcal{S}_n$. Then*

$$\rho(\mathcal{S}_n, A_n) \underset{n\to\infty}{\overset{SOT}{\searrow}} \rho(\mathcal{S}, A),$$

*where $\mathcal{S} = \bigcap_{n=1}^{\infty} \mathcal{S}_n$.*

*Proof.* By Corollary 3.6, for every $n \in \mathbb{N}$, $\rho(\mathcal{S}_{n+1}, A_{n+1}) \leq \rho(\mathcal{S}_n, A_n)$. Then there is a positive operator $L$ such that $\rho(\mathcal{S}_n, A_n) \xrightarrow[n\to\infty]{S.O.T.} L$. On one hand, by Proposition 2.9, $A \preccurlyeq A_n$, $n \in \mathbb{N}$. As, in addition, $\mathcal{S} \subseteq \mathcal{S}_n$, we have that $\rho(\mathcal{S}, A) \leq \rho(\mathcal{S}_n, A_n)$, $n \in \mathbb{N}$. This shows that $\rho(\mathcal{S}, A) \leq L$. On the other hand, for every $n > m$ and $k \geq 1$, by Corollary 3.6 and the definition of spectral shorted operators,

$$L \leq \rho(\mathcal{S}_n, A_n) \leq \rho(\mathcal{S}_m, A_n) \leq \Sigma\left(\mathcal{S}_m, A_n^k\right)^{1/k}. \tag{6}$$

Now fix $k \geq 1$. By Proposition 2.9, $A_n^k \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} A^k$. Therefore, by Lemma 2.8,

$$\Sigma\left(\mathcal{S}_m, A_n^k\right)^{1/k} \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} \Sigma\left(\mathcal{S}_m, A^k\right)^{1/k}. \tag{7}$$

In a similar way, using Proposition 2.4, we have that

$$\Sigma\left(\mathcal{S}_n, A^k\right)^{1/k} \underset{n\to\infty}{\overset{\text{SOT}}{\searrow}} \Sigma\left(\mathcal{S}, A^k\right)^{1/k}. \tag{8}$$

Hence, joining equations (6) (7) and (8), we obtain $L \leq \Sigma\left(\mathcal{S}, A^k\right)^{1/k}$. Finally, since the last inequality is true for every $k$, by taking limit we have that $L \leq \rho\left(\mathcal{S}, A\right)$. ■

As the following example shows, the last Proposition fails, in general, if the sequence of subspaces is not non-increasing.

**Example 3.8.** Let $\mathcal{H}$ be a separable Hilbert space and $A$ a positive and injective operator such that $R(A^{1/2}) \neq \mathcal{H}$. Let $\mathcal{L}$ be a proper dense subspace of $\mathcal{H}$ such that $R(A^{1/2}) \cap \mathcal{L} = \{0\}$. Take an orthonormal basis $\{e_n\}$ of $\mathcal{H}$ contained in $\mathcal{L}$, and define $\mathcal{S}_n = \langle e_1, \ldots, e_n\rangle$. Then, $P_{\mathcal{S}_n} \underset{n\to\infty}{\overset{\text{SOT}}{\nearrow}} I$, but, $\rho\left(\mathcal{S}_n, A\right) = \Sigma\left(\mathcal{S}_n, A\right) = 0$ for all $n \in \mathbb{N}$, because, as it was proved in [2], $R(\Sigma\left(\mathcal{S}_n, A\right)^{1/2}) = R(A^{1/2}) \cap \mathcal{S}_n = \{0\}$. ▲

# 4 Main properties of $\rho\left(\mathcal{S}, A\right)$.

Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. It is shown in [3] that, if $\dim \mathcal{H} < \infty$ and $0 < \lambda \in \mathbb{R}$, then

$$\bigoplus_{\mu\geq\lambda} \ker(\rho(\mathcal{S}, A) - \mu I) = \mathcal{S} \cap \bigoplus_{\mu\geq\lambda} \ker(A - \mu I).$$

This can be reformulated, in terms of spactral measures, as

$$\aleph_{[\lambda,\infty)}(\rho\left(\mathcal{S}, A\right)) = \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}}.$$

This formula, which allows to compute the spectrum and the eigenvectors of $\rho\left(\mathcal{S}, A\right)$, gives the complete characterization of $\rho\left(\mathcal{S}, A\right)$ in the matrix case.

In the infinite dimensional case, a similar formula can be proved following the same methods (with considerable more effort). Instead, it seems more convenient to construct an operator by means of the left spectral resolution given by

$$f(\lambda) = \begin{cases} \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}} & \lambda > 0 \\ I & \lambda \leq 0 \end{cases} \tag{9}$$

and then to show that its associated operator agrees with $\rho\left(\mathcal{S}, A\right)$. This can be done by using the characterization of $\rho\left(\mathcal{S}, A\right)$ given in Theorem 3.5. Note that the verification of the fact that $f$ is, indeed, a left spectral resolution is apparent from the fact that $\lambda \mapsto \aleph_{[\lambda,\infty)}(A)$ is the left spectral resolution of $A$.

**Theorem 4.1.** Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then $\rho(\mathcal{S}, A)$ is the operator defined by the left spectral resolution $f$ defined in equation (9). In other words, for $0 < \lambda \in \mathbb{R}$,

$$\aleph_{[\lambda,\infty)}(\rho(\mathcal{S}, A)) = \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}}.$$

*Proof.* Let $B$ be the operator defined by the spectral resolution $f$. By Theorem 2.7, it is clear that $B \preccurlyeq A$ and every $D \in \mathcal{M}_\rho(\mathcal{S}, A)$ satisfies $D \preccurlyeq B$. Indeed, suppose that $0 \leq D \preccurlyeq A$ and $R(D) \subseteq \mathcal{S}$. Then, for $\lambda > 0$, $\aleph_{[\lambda,\infty)}(D) \leq \aleph_{[\lambda,\infty)}(A)$ and

$$\aleph_{[\lambda,\infty)}(D) \leq \aleph_{(0,\infty)}(D) \leq P_{\overline{R(D)}} \leq P_{\mathcal{S}}.$$

Therefore $\aleph_{[\lambda,\infty)}(D) \leq \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}} = \aleph_{[\lambda,\infty)}(B)$. Since $\aleph_{[\lambda,\infty)}(D) = I = \aleph_{[\lambda,\infty)}(B)$ for $\lambda \leq 0$, we get that $D \preccurlyeq B$ by Theorem 2.7. Finally, since

$$\aleph_{[\lambda,\infty)}(\|A\| P_{\mathcal{S}}) = \begin{cases} 0 & \|A\| < \lambda \\ P_{\mathcal{S}} & 0 < \lambda \leq \|A\| \\ I & \lambda \leq 0 \end{cases},$$

we deduce that $B \preccurlyeq \|A\| \; P_{\mathcal{S}}$ and, in particular, $R(B) \subseteq \mathcal{S}$. Then, by Theorem 3.5,

$$B = \max \mathcal{M}_\rho(\mathcal{S}, A) = \rho(\mathcal{S}, A).$$

■

**Corollary 4.2.** *Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces of $\mathcal{H}$. Then*

$$\rho(\mathcal{S} \cap \mathcal{T}, A) = \rho(\mathcal{T}, \rho(\mathcal{S}, A)).$$

*Proof.* Note that both operators have, as left spectral resolution, the map

$$f(\lambda) = \begin{cases} \aleph_{[\lambda,\infty)}(A) \wedge P_{\mathcal{S}} \wedge P_{\mathcal{T}} & \lambda > 0 \\ I & \lambda \leq 0 \end{cases}.$$

■

**Remark 4.3.** Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces of $\mathcal{H}$. Then

$$\rho(\mathcal{S} \cap \mathcal{T}, A) \leq \rho(\mathcal{T}, \Sigma(\mathcal{S}, A)).$$

Indeed, it can be deduced from inequalities

$$\Sigma\left(\mathcal{S} \cap \mathcal{T}, A^{2^m}\right) \leq \Sigma\left(\mathcal{T}, \Sigma\left(\mathcal{S}, A^{2^m}\right)\right) \leq \Sigma\left(\mathcal{T}, \Sigma(\mathcal{S}, A)^{2^m}\right) \qquad \forall m \in \mathbb{N}.$$

Note that the mentioned statement can not be deduced from Corollary 4.2. ▲

**Proposition 4.4.** *If $A \in L(\mathcal{H})^+$ and $\mu = \min \sigma(A)$, then*

$$\mu P \leq \rho(\mathcal{S}, A).$$

*In particular, if $A$ is invertible then $\rho(\mathcal{S}, A)$ is invertible if it is considered as acting on $\mathcal{S}$.*

*Proof.* Note that $\mu^m = \min\sigma(A^m)$ for all $m \in \mathbb{N}$. Then $\mu^m P_{\mathcal{S}} \le \mu^m I \le A^m$ for all $m \in \mathbb{N}$. So that, $\mu P_{\mathcal{S}} \preccurlyeq A$ and the result follows by Theorem 3.5. ■

**Remark 4.5.** Given an operator $A \in L(\mathcal{H})^+$, then $r \notin \sigma(A)$ if and only if there exist an $\varepsilon > 0$ such that $\aleph_{[r-\varepsilon,+\infty)}(A) = \aleph_{[r+\varepsilon,+\infty)}(A)$. ▲

**Proposition 4.6.** *Let $A \in L(\mathcal{H})^+$. Then, if $\rho(\mathcal{S}, A)$ is considered as acting on $\mathcal{S}$, it holds*

$$\sigma(\rho(\mathcal{S}, A)) \subseteq \sigma(A).$$

*Proof.* By Proposition 4.4, if $0 \notin \sigma(A)$ then $0 \notin \sigma(\rho(\mathcal{S}, A))$. On the other hand, if $r > 0$ and $r \notin \sigma(A)$, then, by Remark 4.5, there exists $\varepsilon > 0$ such that $\aleph_{[r-\varepsilon,+\infty)}(A) = \aleph_{[r+\varepsilon,+\infty)}(A)$. Hence,

$$\aleph_{[r-\varepsilon,+\infty)}(\rho(\mathcal{S}, A)) = P_{\mathcal{S}} \wedge \aleph_{[r-\varepsilon,+\infty)}(A) = P_{\mathcal{S}} \wedge \aleph_{[r+\varepsilon,+\infty)}(A) = \aleph_{[r+\varepsilon,+\infty)}(\rho(\mathcal{S}, A)).$$

Thus, $r \notin \sigma(\rho(\mathcal{S}, A))$. ■

**Proposition 4.7.** *Let $A \in L(\mathcal{H})^+$, $\mathcal{S}$ a closed subspace and $f : [0,+\infty) \to [0,+\infty)$ a non-decreasing right continuous function. Then,*

$$f(\rho(\mathcal{S}, A)) = \rho(\mathcal{S}, f(A)) \tag{10}$$

*Proof.* Given $\lambda \ge 0$, since $f$ is non-decreasing and right continuous there exist $\eta \ge 0$ such that

$$\{\mu : f(\mu) \ge \lambda\} = [\eta, +\infty) \quad \text{and, for every} \quad C \in L(\mathcal{H})^+, \quad \aleph_{[\lambda,\infty)}(f(C)) = \aleph_{[\eta,\infty)}(C).$$

If $\eta = 0$, then $\aleph_{[\lambda,\infty)}(f(\rho(\mathcal{S}, A))) = \aleph_{[\lambda,\infty)}(\rho(\mathcal{S}, f(A))) = I$. On the other hand, if $\eta > 0$,

$$\begin{aligned}\aleph_{[\lambda,\infty)}(f(\rho(\mathcal{S}, A))) &= \aleph_{[\eta,\infty)}(\rho(\mathcal{S}, A)) = \aleph_{[\eta,\infty)}(A) \wedge P_{\mathcal{S}} \\ &= \aleph_{[\lambda,\infty)}(f(A)) \wedge P_{\mathcal{S}} = \aleph_{[\lambda,\infty)}(\rho(\mathcal{S}, f(A))),\end{aligned}$$

which shows that $f(\rho(\mathcal{S}, A))$ and $\rho(\mathcal{S}, f(A))$ have the same (left) spectral resolution. Hence $f(\rho(\mathcal{S}, A)) = \rho(\mathcal{S}, f(A))$ ■

## Computation of $\min\sigma(\rho(\mathcal{S}, A))$.

**Proposition 4.8.** *Let $A \in L(\mathcal{H})^+$. Then, if $\rho(\mathcal{S}, A)$ is considered as acting on $\mathcal{S}$, it holds*

$$\min\sigma(\rho(\mathcal{S}, A)) = \max\{\lambda \ge 0 : A^m \ge \lambda^m P_{\mathcal{S}},\ \forall\ m \in \mathbb{N}\}. \tag{11}$$

*Proof.* Note that $A^m \ge \lambda^m P_{\mathcal{S}}$, $m \in \mathbb{N}$, if and only if $\lambda P_{\mathcal{S}} \preccurlyeq A$. On the other hand, since $P_{\mathcal{S}}$ and $\rho(\mathcal{S}, A)$ commute, $\lambda P_{\mathcal{S}} \le \rho(\mathcal{S}, A)$ if and only if $\lambda P_{\mathcal{S}} \preccurlyeq \rho(\mathcal{S}, A)$ if and only if $\lambda P_{\mathcal{S}} \in \mathcal{M}_\rho(\mathcal{S}, A)$ if and only if $\lambda P_{\mathcal{S}} \preccurlyeq A$.

■

**Theorem 4.9.** *Let $A \in L(\mathcal{H})^+$. Then, if $\rho(\mathcal{S}, A)$ is considered as acting on $\mathcal{S}$,*

$$\begin{aligned} \min \sigma(\rho(\mathcal{S}, A)) &= \max\{\lambda \geq 0 : P_{\mathcal{S}} \leq \aleph_{[\lambda,\infty)}(A)\} \\ &= \min\{\mu \in \sigma(A) : \ R(\aleph_{[\mu,\mu+\varepsilon)}(A)) \not\subseteq \mathcal{S}^\perp \ \ \forall\, \varepsilon > 0\} \qquad (12) \\ &= \min\{\mu \in \sigma(A) : \ P_{\mathcal{S}}\ \aleph_{[\mu,\mu+\varepsilon)}(A) \neq 0 \ \ \forall\, \varepsilon > 0\}. \end{aligned}$$

*Proof.* For any $B \in L(\mathcal{S})^+$, $\min \sigma(B) = \max\{\lambda \geq 0 : \aleph_{[\lambda,\infty)}(B) = I_{\mathcal{S}}\}$. Applying this identity to our problem, we get $\lambda_0 = \min \sigma(\rho(\mathcal{S}, A)) = \max\{\lambda \geq 0 : P_{\mathcal{S}} \leq \aleph_{[\lambda,\infty)}(A)\}$. Then $P_{\mathcal{S}} \leq \aleph_{[\lambda_0,\infty)}(A)$ but $P_{\mathcal{S}} \not\leq \aleph_{[\lambda_0+\varepsilon,\infty)}(A)$ for every $\varepsilon > 0$. So that, $\lambda_0 \in \{\mu \in \sigma(A) : P_{\mathcal{S}}\ \aleph_{[\mu,\mu+\varepsilon)}(A) \neq 0 \ \ \forall\, \varepsilon > 0\}$, since if $P_{\mathcal{S}}\ \aleph_{[\lambda_0,\lambda_0+\varepsilon)}(A) = 0$, then

$$P_{\mathcal{S}}\ \aleph_{[\lambda_0+\varepsilon,\infty)}(A) = P_{\mathcal{S}} \left(\aleph_{[\lambda_0,\infty)}(A) - \aleph_{[\lambda_0,\lambda_0+\varepsilon)}(A)\right) = P_{\mathcal{S}}\ \aleph_{[\lambda_0,\infty)}(A) = P_{\mathcal{S}},$$

i.e. $P_{\mathcal{S}} \leq \aleph_{[\lambda_0+\varepsilon,\infty)}(A)$. If $\lambda_0 = 0$, then equation (12) is clear, since $[\lambda_0, \lambda_0 + \varepsilon)$ is an open subset of $\sigma(\rho(\mathcal{S}, A))$. If $\lambda_0 > 0$, let $0 \leq \lambda < \lambda_0$ and $0 < \varepsilon < \lambda_0 - \lambda$. Then $\lambda + \varepsilon \leq \lambda_0$. Since $\lambda_0 = \max\{\lambda \geq 0 : P_{\mathcal{S}} \leq \aleph_{[\lambda,\infty)}(A)\}$, it holds that $P_{\mathcal{S}}\aleph_{[\lambda,\infty)}(A) = P_{\mathcal{S}}\aleph_{[\lambda+\varepsilon,\infty)}(A) = P_{\mathcal{S}}$. Hence

$$P_{\mathcal{S}} = P_{\mathcal{S}}\aleph_{[\lambda,\infty)}(A) = P_{\mathcal{S}}\aleph_{[\lambda,\lambda+\varepsilon)}(A) + P_{\mathcal{S}}\aleph_{[\lambda+\varepsilon,\infty)}(A) = P_{\mathcal{S}}\aleph_{[\lambda,\lambda+\varepsilon)}(A) + P_{\mathcal{S}}.$$

Therefore $P_{\mathcal{S}}\aleph_{[\lambda,\lambda+\varepsilon)}(A) = 0$, showing equation (12). ■

# 5 The case $\dim \mathcal{S} = 1$.

**Definition 5.1.** Suppose that $\dim \mathcal{S} = 1$ and let $\xi \in \mathcal{S}$ an unit vector. For every $A \geq 0$ there exist $\lambda, \mu \geq 0$ such that $\rho(\mathcal{S}, A) = \lambda P_{\mathcal{S}}$ and $\Sigma(\mathcal{S}, A) = \mu P_{\mathcal{S}}$. Denote $\rho(A, \xi) = \lambda$ and $\Sigma(A, \xi) = \mu$.

**Remark 5.2.** Let $\mathcal{S}$ be the subspace generated by the unit vector $\xi \in \mathcal{H}$. There are several ways to compute $\rho(A, \xi)$ in terms of $\rho(\mathcal{S}, A)$, and similarly $\Sigma(A, \xi)$ in terms of $\Sigma(\mathcal{S}, A)$. For example:

1. By Theorem 4.9,

$$\begin{aligned} \rho(A, \xi) &= \min \sigma(\rho(\mathcal{S}, A)) = \min\left\{\mu \in \sigma(A) : \ P_{\mathcal{S}}\aleph_{[\mu,\mu+\varepsilon)}(A) \neq 0 \ \ \forall\, \varepsilon > 0\right\} \\ &= \min\left\{\mu \in \sigma(A) : \ \aleph_{[\mu,\mu+\varepsilon)}(A)\xi \neq 0 \ \ \forall\, \varepsilon > 0\right\} \end{aligned} \quad . \qquad (13)$$

2. By Proposition 4.8

$$\rho(A, \xi) = \max\{\lambda \geq 0 : \langle A^n \eta, \eta\rangle \geq \lambda^n |\langle \xi, \eta\rangle|^2 \ , \ \ \forall\, n \in \mathbb{N}, \ \eta \in \mathcal{H}\}.$$

3. Also $\rho(A, \xi) = \|\rho(\mathcal{S}, A)\, \xi\| = \langle \rho(\mathcal{S}, A)\, \xi, \xi\rangle$. Similar formulae hold for $\Sigma(A, \xi)$.

4. By Proposition 4.6, $\rho(A,\xi) \in \sigma(A)$. Moreover, by Theorem 4.1 (or Theorem 4.9),

$$\rho(A,\xi) = \max\{\lambda \in \sigma(A) : \xi \in R(\aleph_{[\lambda,\infty)}(A))\}. \tag{14}$$

The following result relates the spectral short of operator to one dimensional subspaces and the spectral order.

**Proposition 5.3.** *Let $A$, $B \in L(\mathcal{H})^+$. Then $A \preccurlyeq B$ if and only if $\rho(A,\xi) \leq \rho(B,\xi)$ for every unit vector $\xi \in \mathcal{H}$.*

*Proof.* One implication follows from Corollary 3.6. On the other hand, suppose that $\rho(A,\xi) \leq \rho(B,\xi)$ for every unit vector $\xi \in \mathcal{H}$. Given $\lambda \geq 0$ such that $\aleph_{[\lambda,\infty)}(A) \neq 0$, let $\zeta \in R(\aleph_{[\lambda,\infty)}(A))$. By equation (14), $\lambda \leq \rho(A,\zeta)$. Since $\rho(A,\zeta) \leq \rho(B,\zeta)$, by equation (14) we have that $\zeta \in R(\aleph_{[\lambda,\infty)}(B))$. Hence $R(\aleph_{[\lambda,\infty)}(A)) \subseteq R(\aleph_{[\lambda,\infty)}(B))$ for every $\lambda \geq 0$. By Theorem 2.7, we deduce that $A \preccurlyeq B$. ■

**Proposition 5.4.** *Let $A \in L(\mathcal{H})^+$ and let $\mathcal{S}$ be the subspace of $\mathcal{H}$ generated by the unit vector $\xi$. If $A$ is invertible, then for $m \in \mathbb{N}$,*

$$\Sigma\left(A^{2m},\xi\right)^{1/2m} = \|A^{-m}\xi\|^{-1/m} = \langle A^{-2m}\xi,\xi\rangle^{-1/2m}, \tag{15}$$

*and*

$$\rho(A,\xi) = \lim_{m\to\infty} \|A^{-m}\xi\|^{-1/m} = \inf_{m\in\mathbb{N}} \|A^{-m}\xi\|^{-1/m} \tag{16}$$

*If $R(A)$ is closed, then*

1. *If $\xi \notin R(A)$, then $\rho(A,\xi) = 0$.*
2. *If $\xi \in R(A)$ and $B = A^\dagger$, then $\rho(A,\xi) = \lim_{m\to\infty} \|B^m\xi\|^{-1/m} = \inf_{m\in\mathbb{N}} \|B^m\xi\|^{-1/m}$.*

*Proof.* Using Theorem 4.9, the closed range case easily reduces to the invertible case, by considering $A$ as acting on $R(A)$, because $A^\dagger$ acts on $R(A)$ as the inverse of $A$. Note that, if $R(A)$ is closed, then there exists $\varepsilon > 0$ such that $\aleph_{[0,\varepsilon)}(A) = P_{N(A)}$. Therefore $\xi \notin R(A)$ implies that $P_{\mathcal{S}}\aleph_{[0,\varepsilon)}(A) \neq 0$, and, by Remark 5.2, that $\rho(A,\xi) = 0$.

Suppose that $A$ is invertible. For $m \in \mathbb{N}$, denote by $\eta_m = A^{-m/2}\xi$. Fix $m \in \mathbb{N}$. By Theorem 2.2, if $\mathcal{M}_m = A^{-m/2}(\mathcal{S})$, then $\Sigma(\mathcal{S},A^m) = A^{m/2}P_{\mathcal{M}_m}A^{m/2}$, and

$$\Sigma(A^m,\xi) = \|\Sigma(\mathcal{S},A^m)\,\xi\| = \|A^{m/2}P_{\mathcal{M}_m}A^{m/2}\xi\|.$$

Note that $\mathcal{M}_m$ is the subspace generated by $\eta_m$, so that $P_{\mathcal{M}_m}\rho = \|\eta_m\|^{-2}\langle\rho,\eta_m\rangle\eta_m$, $\rho \in \mathcal{H}$. Then

$$\begin{aligned}\Sigma(A^m,\xi) = \|A^{m/2}P_{\mathcal{M}_m}A^{m/2}\ \xi\| &= \left\|A^{m/2}\left(\|\eta_m\|^{-2}\langle A^{m/2}\ \xi,\eta_m\rangle\eta_m\right)\right\| \\ &= \|\eta_m\|^{-2}\|\langle\xi,\xi\rangle\ \xi\| = \|\eta_m\|^{-2}.\end{aligned}$$

Therefore $\Sigma(A^{2m},\xi) = \|A^{-m}\xi\|^{-2}$, so that

$$\Sigma\left(A^{2m},\xi\right)^{1/2m} = \|A^{-m}\xi\|^{-1/m}\ , \quad m \in \mathbb{N}.$$

Equation (16) follows using Remark 5.2 and the definition of $\rho(\mathcal{S},A)$. ■

**Remark 5.5.** Equation (15) and, consequently, Proposition 5.4, can also be deduced from the following formula: for every invertible $B \in L(\mathcal{H})^+$ and $\xi \in \mathcal{H}$ with $\|\xi\| = 1$,

$$\Sigma\,(B, \xi) = \langle B^{-1}\xi, \xi\rangle^{-1}.$$

This formula is the one dimensional case of the characterization of Schur complements in terms of the block representation of the inverse of an operator (see [9] Lemma 4.7 or, for a matrix version, Horn-Johnson book [6]). ▲

Let $A \in L(\mathcal{H})^+$. We shall denote by

$$\sigma_\rho\,(A) = \big\{\rho\,(A, \xi) : \|\xi\| = 1\big\}.$$

By Proposition 4.6, we have that $\sigma_\rho\,(A) \subseteq \sigma\,(A)$. If $\dim \mathcal{H} < \infty$, it was shown in [3] (see also [5]) that $\sigma_\rho\,(A) = \sigma\,(A)$. We shall see that this property fails in general. First we fix some notations:

1. For $B \in L(\mathcal{H})^+$ we denote by

$$\begin{aligned} \sigma_+\,(A) \;&= \big\{\lambda \in \sigma\,(A) : \exists\ (\mu_n)_{n\in\mathbb{N}} \textit{ in } \sigma\,(A) \textit{ such that } \mu_n > \lambda \textit{ and } \mu_n \searrow_{n\to\infty} \lambda\big\} \\ &= \big\{\lambda \in \sigma\,(A) : \forall\ \varepsilon > 0\ ,\ \aleph_{(\lambda,\lambda+\varepsilon)}(A) \neq 0\big\}, \end{aligned}$$

   i.e. those points $\lambda \in \sigma\,(A)$ which are limit point of $\sigma\,(A) \setminus \{\lambda\}$ from the right.

2. $\sigma_{pt}(A) = \big\{\lambda \in \sigma\,(A) : N(A - \lambda I) \neq \{0\}\big\}$, the point spectrum of $A$.

**Proposition 5.6.** *Let $A \in L(\mathcal{H})^+$. Then*

$$\sigma_\rho\,(A) = \sigma_+\,(A) \cup \sigma_{pt}(A) = \big\{\lambda \in \sigma\,(A) : \forall\ \varepsilon > 0\ ,\ \aleph_{[\lambda,\lambda+\varepsilon)}(A) \neq 0\big\}.$$

*In particular, this shows that $\sigma_\rho\,(A)$ is dense in $\sigma\,(A)$.*

*Proof.* Let $\lambda \in \sigma\,(A)$ and let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\sigma\,(A)$ such that $\mu_n \searrow_{n\to\infty} \lambda$. Denote by $\lambda_0 = \mu_1 + 1$ and $\lambda_n = \frac{1}{2}(\mu_{n+1} + \mu_n)$, $n \in \mathbb{N}$. Note that, since $\mu_n \in (\lambda_n, \lambda_{n-1})$, then $\aleph_{(\lambda_n,\lambda_{n-1})}(A) \neq 0$. We take, for every $n \in \mathbb{N}$, an unit vector $\xi_n \in R(\aleph_{(\lambda_n,\lambda_{n-1})}(A))$. Consider the unit vector

$$\xi = \sum_{n\in\mathbb{N}} \frac{\xi_n}{2^n}\ .$$

Recall formula (14), wich says that $\rho\,(A, \xi) = \max\{\mu \in \sigma\,(A) : \xi \in R(\aleph_{[\mu,\infty)}(A))\}$. It is clear by construction of $\xi$ that in our case we get $\rho\,(A, \xi) = \lambda$, because $\lambda = \inf_n \mu_n = \inf_n \lambda_n$. If $\lambda \in \sigma_{pt}(A)$, just take $\xi \in N(A - \lambda I)$ and clearly $\rho\,(A, \xi) = \Sigma\,(A, \xi) = \lambda$.

Now suppose that $\lambda \in \sigma\,(A)$ but $\lambda \notin \sigma_+\,(A) \cup \sigma_{pt}(A)$. This means that there exists $\varepsilon > 0$ such that $\aleph_{[\lambda,\lambda+\varepsilon)}(A) = 0$. Therefore, for any unit vector $\xi$, it is impossible that

$$\lambda = \max\{\mu \in \sigma\,(A) : \xi \in R(\aleph_{[\mu,\infty)}(A))\},$$

because if $\xi \in R(\aleph_{[\lambda,\infty)}(A))$, then $\xi \in R(\aleph_{[\lambda+\varepsilon,\infty)}(A))$. ■

**Remark 5.7.** If $A \in L(\mathcal{H})^+$ is not invertible, then $0 \in \sigma(A)$. If 0 were an isolated point of $\sigma(A)$ then $A$ must have closed range. So that, $N(A) \neq \{0\}$. Otherwise $\aleph_{(0,\varepsilon)}(A) \neq 0$ for every $\varepsilon > 0$. This shows that $0 \in \sigma_\rho(A)$. More generally, for $A \in L(\mathcal{H})^+$, it holds that $\lambda_{min}(A) = \min \sigma(A) \in \sigma_\rho(A)$. On the other hand, by Proposition 5.6, $\|A\| \in \sigma_\rho(A)$ if and only if $\|A\|$ is an eigenvalue of $A$.

**Remark 5.8.** For $A \in L(\mathcal{H})^+$, we shall denote by $R_0(A)$ the subspace

$$R_0(A) = \bigcup_{\lambda>0} R(\aleph_{[\lambda,\infty)}(A)).$$

If $R(A)$ is closed, then $R_0(A) = R(A)$, since 0 is an isolated point of $\sigma(A)$. But in other case, $R_0(A)$ is properly included in $R(A)$, but it is still a dense subspace of $\overline{R(A)}$. We are interested in this subspace because, by formula (14), if $\xi \in \mathcal{H}$ an unit vector, then $\rho(A,\xi) \neq 0$ if and only if $\xi \in R_0(A)$. ▲

## 5.1 Kolmogorov's complexity

Given an invertible matrix $A \in L(\mathbb{C}^m)^+$ and $\xi \in \mathbb{C}^m$ a unit vector, J. I. Fujii and M. Fujii [5] define the Kolmogorov's complexity:

$$K(A,\xi) = \lim_{n\to\infty} \frac{\log(\langle A^n\xi,\xi\rangle)}{n} = \log \lim_{n\to\infty} \langle A^n\xi,\xi\rangle^{1/n}. \tag{17}$$

Using formula (15), we can see that the limit is, in fact, a supremum; and we have the identity

$$K(A,\xi) = \log \rho\left(A^{-1/2},\xi\right)^{-2} = \log \rho\left(A^{-1},\xi\right)^{-1}. \tag{18}$$

This shows, using formulae (13) and (14), the following formula:

$$\begin{aligned} \exp K(A,\xi) &= \min\left\{\lambda \in \sigma(A) : \xi \in R(\aleph_{(-\infty,\lambda]}(A))\right\} \\ &= \max\left\{\mu \in \sigma(A) :\ \aleph_{(\mu-\varepsilon,\mu]}(A)\xi \neq 0 \ \ \forall\, \varepsilon > 0\right\}. \end{aligned} \tag{19}$$

With these identities in mind we generalize the notion of Kolmogorov's complexity in two directions: firstly we define it for infinite dimensional Hilbert spaces; secondly we remove the hypothesis of invertibility of $A$. Note that the own notion of spectral shorted operator is, in some sense, a generalization of the Kolmogorov's complexity relative to arbitrary (not necesarily onedimensional) closed subspaces of a Hilbert space $\mathcal{H}$.

If $\mathcal{H}$ is a Hilbert space and $A \in L(\mathcal{H})^+$ is invertible, then we just have to define $K(A,\xi)$ as in equation (18) or, equivalently, equation (19). It is easy to see that this is equivalent to define it as in the finite dimensional setting, as in equation (17). We should mention that some of the properties of $K(A,\xi)$ proved by J. I. Fujii and M. Fujii fail if $\mathcal{H}$ is infinite dimensional. As an example, the identity

$$\sigma(A) = \left\{\exp(K(A,\xi)) : \|\xi\| = 1\right\}.$$

**Definition 5.9.** Given $\xi \in \mathcal{H}$ and $A \in L(\mathcal{H})^+$, we denote by

$$k(A,\xi) = \lim_{n\to\infty} \langle A^n\xi,\xi\rangle^{1/n},$$

that is, $k(A,\xi) = \exp K(A,\xi)$ in the cases where $K(A,\xi)$ is defined.

**Remark 5.10.** If $\xi \in \mathcal{H}$ and $A \in L(\mathcal{H})^+$, then

1. If $\|\xi\| = 1$, then the sequence $\langle A^n\xi,\xi\rangle^{1/n}$ is increasing. So that, $\lim_{n\to\infty}\langle A^n\xi,\xi\rangle^{1/n}$ exists for every $\xi \in \mathcal{H}$.
2. $k(A,\xi) = k(A,a\xi)$ for every $0 \neq a \in \mathbb{C}$.
3. $k(A,\xi) = k\big(A,\aleph_{[\lambda,\infty)}(A)\xi\big)$ for every $\lambda > 0$ such that $\aleph_{[\lambda,\infty)}(A)\xi \neq 0$.

Indeed, by Hölder inequality for states (also by Jensen inequality, see [4]), if $\|\xi\| = 1$, $p \geq 1$ and $1/p + 1/q = 1$, then

$$\langle A^p\xi,\xi\rangle^{1/p}\langle I^q\xi,\xi\rangle^{1/q} = \langle A^p\xi,\xi\rangle^{1/p} \geq \langle A\xi,\xi\rangle.$$

Applying this inequality to $A^n$ with $p = (n+1)/n$ one gets that $\langle A^n\xi,\xi\rangle^{1/n} \leq \left\langle A^{n+1}\xi,\xi\right\rangle^{1/n+1}$. Item 2 follows from the fact that $|a|^{2/n} \xrightarrow[n\to\infty]{} 1$. To show 3, suppose that $\|\xi\| = 1$ and denote by $\xi_1 = \aleph_{[\lambda,\infty)}(A)\xi$ and $\xi_2 = \xi - \xi_1$. Then, since $\aleph_{[\lambda,\infty)}(A)$ commutes with $A$, for every $n \in \mathbb{N}$,

$$\begin{aligned}\langle A^n\xi_1,\xi_1\rangle &\leq \langle A^n\xi_1,\xi_1\rangle + \langle A^n\xi_2,\xi_2\rangle = \langle A^n\xi,\xi\rangle\\ &\leq \langle A^n\xi_1,\xi_1\rangle + \lambda^n \leq (1+\|\xi_1\|^{-2})\langle A^n\xi_1,\xi_1\rangle.\end{aligned}$$

This shows that $k(A,\xi) = k(A,\xi_1)$, since $(1+\|\xi_1\|^{-2})^{1/n} \xrightarrow[n\to\infty]{} 1$. ▲

Recall that, for $A \in L(\mathcal{H})^+$, we denote by $R_0(A) = \bigcup_{\lambda>0} R(\aleph_{[\lambda,\infty)}(A))$.

**Proposition 5.11.** *Let $A \in L(\mathcal{H})^+$ and $0 \neq \xi \in \mathcal{H}$. Then $k(A,\xi) \neq 0$ if and only if $P_{\overline{R(A)}}\,\xi \in R_0(A)$. Moreover, equation (19) holds in general:*

$$\begin{aligned}k(A,\xi) &= \min\left\{\lambda \in \sigma(A) : \xi \in R(\aleph_{(-\infty,\lambda]}(A))\right\}\\ &= \max\left\{\mu \in \sigma(A) : \ \aleph_{(\mu-\varepsilon,\mu]}(A)\xi \neq 0 \ \ \forall\,\varepsilon > 0\right\} \qquad (20)\\ &= \sup\left\{\mu \in \sigma(A) : \ \aleph_{[\mu,\infty)}(A)\xi \neq 0\right\}.\end{aligned}$$

*Proof.* Let $\lambda = \sup\left\{\mu \in \sigma(A) : \ \aleph_{[\mu,\infty)}(A)\xi \neq 0\right\}$.
If $\mu > \lambda$, then $\xi \in R(\aleph_{(-\infty,\mu]}(A))$, so that $\langle A^n\xi,\xi\rangle \leq \mu^n\|\xi\|^2$ for $n \in \mathbb{N}$, and $k(A,\xi) \leq \mu$. On the other hand, if $\mu < \lambda$ then $\aleph_{[\mu,\infty)}(A)\xi = \xi_1 \neq 0$, and, by Remark 5.10, $k(A,\xi) = k(A,\xi_1) \geq \mu$, since $\langle A^n\xi_1,\xi_1\rangle \geq \mu^n\|\xi_1\|^2$ for every $n \in \mathbb{N}$. This shows that $k(A,\xi) = \lambda$. The other equalities are straightforward, by spectral theory. ■

By Proposition 4.6, we have that $\sigma_\rho(A) \subseteq \sigma(A)$ and, therefore, if $A$ is invertible,

$$\left\{k(A,\xi) : \|\xi\| \neq 0\right\} = \left\{\rho\left(A^{-1},\xi\right)^{-1} : \|\xi\| = 1\right\} \subseteq \sigma\left(A^{-1}\right)^{-1} = \sigma(A).$$

As we shall see below, the reverse inclusion fails in general:

**Proposition 5.12.** *Let $A > 0$. Then*

$$\begin{aligned}\left\{k(A,\xi) : \|\xi\| \neq 0\right\} &= \sigma_-(A) \cup \sigma_{pt}(A)\\ &= \left\{\lambda \in \sigma(A) : \aleph_{(\lambda+\varepsilon,\lambda]}(A) \neq 0 \ , \ \forall\, \varepsilon > 0 \ \right\},\end{aligned}$$

*where $\sigma_-(A)$ is the set of points in $\sigma(A)$ which are limit point of $\sigma(A) \setminus \{\lambda\}$ from the left. The set $\left\{k(A,\xi) : \|\xi\| = 1\right\}$ is also dense in $\sigma(A)$.*

*Proof.* It is a consequence of Proposition 5.6 (applied to $A^{-1}$) and the identity

$$\left\{k(A,\xi) : \|\xi\| \neq 0\right\} = \left\{k(A,\xi) : \|\xi\| = 1\right\} = \left\{\rho\left(A^{-1},\xi\right)^{-1} : \|\xi\| = 1\right\}.$$

■

**Remarks 5.13.**

1. Proposition 5.12 is also true for a general $A \in L(\mathcal{H})^+$. The proof is similar to the proof of Proposition 5.6, by using equation (20) instead of (14).

2. Let $\mathcal{H} = \ell^2(\mathbb{N})$, denote by $e_n$, $n \in \mathbb{N}$, the cannonical orthonormal basis of $\mathcal{H}$, and consider the diagonal invertible operators $A, B \in L(\mathcal{H})^+$ defined by

$$A(e_n) = \left(2 + \frac{1}{n}\right)e_n \quad , \quad B(e_n) = \left(2 - \frac{1}{n}\right)e_n \ , \quad n \in \mathbb{N}.$$

   It is easy to see, using Propositions 5.6 and 5.12, that $2 \notin \left\{k(A,\xi) : \|\xi\| = 1\right\}$ and $2 \notin \sigma_\rho(B)$.

3. If $C \in L(\mathcal{H})^+$, then $\|C\| \in \left\{k(C,\xi) : \|\xi\| = 1\right\}$ and $\lambda_{min}(C) \in \sigma_\rho(C)$. On the other hand, if $A$ and $B$ are the operators of the previous example, $\|B\| = 2 \notin \sigma_\rho(B)$ and $\lambda_{min}(A) = 2 \notin \left\{k(A,\xi) : \|\xi\| = 1\right\}$.

**Remark 5.14 (Operators with closed range).** Suppose that $A \in L(\mathcal{H})^+$ and $R(A)$ is closed.

1. If $\xi \in R(A)$ is an unit vector, then, by Proposition 5.4, $k(A,\xi) = \rho\left(A^\dagger,\xi\right)^{-1}$.

2. If $\xi \notin N(A)$ and $\xi \notin R(A)$, then the behaviors of the Kolmogorov complexity and the spectral shorted operator, relative to $\xi$ are different. By Proposition 5.4, $\rho(A,\xi) = \rho\left(A^\dagger,\xi\right) = 0$. On the other hand, if $P = P_{R(A)}$, then $P\xi \neq 0$ and

$$k(A,\xi) = \lim_{n\to\infty} \langle A^n P\xi, P\xi\rangle^{1/n} = k\left(A, \frac{P\xi}{\|P\xi\|}\right) = \rho\left(A^\dagger, \frac{P\xi}{\|P\xi\|}\right)^{-1} \neq 0.$$

# References


[1] W. N. Anderson, Shorted operators, SIAM J. Appl. Math. 20 (1971), 520-525.

[2] W. N. Anderson and G. E. Trapp, Shorted operators II, SIAM J. Appl. Math. 28 (1975), 60-71.

[3] J. Antezana, G. Corach and D. Stojanoff, Spectral shorted matrices, Linear Algebra Appl. 381 (2004), 197-217.

[4] J. Antezana, P. Massey and D. Stojanoff, Jensen inequalities and majorization, preprint.

[5] Jun Ichi Fujii and Masatoshi Fujii, Kolmogorov's complexity for positive definite matrices, Special issue dedicated to Professor T. Ando. Linear Algebra Appl. 341 (2002), 171-180.

[6] R.A. Horn and C.R. Johnson, Matrix analysis, Cambridge University Press, 1985.

[7] R. V. Kadison and J. R. Ringrose, Fundamentals of the theory of operator algebras. Vol. I. Pure and Applied Mathematics, 100. Academic Press, Inc., New York, 1983.

[8] M. G. Krein, The theory of self-adjoint extensions of semibounded Hermitian operators and its applications, Mat. Sb. (N. S.) 20 (62) (1947), 431-495

[9] P. Massey and D. Stojanoff, Generalized Schur Complements and P-complementable Operators, Linear Alg. Appl. (to appear).

[10] Chi-Kwong Li and R. Mathias, Extremal characterizations of the Schur complement and resulting inequalities, SIAM Rev. 42 (2000), no. 2, 233–246.

[11] M. P. Olson, The selfadjoint operators of a von Neumann algebra form a conditionally complete lattice, Proc. Amer. Math. Soc., 28 (1971) 537-544.

[12] G. K. Pedersen, $C^*$-algebras and their automorphism groups. London Mathematical Society Monographs, 14. Academic Press, London-New York, 1979.

[13] G. K. Pedersen, Analysis now. Graduate Texts in Mathematics, 118. Springer-Verlag, New York, 1989.

[14] E. L. Pekarev, Shorts of operators and some extremal problems, Acta Sci. Math. (Szeged) 56 (1992), 147-163.

[15] E. L. Pekarev and J. L. Smul'jan, Parallel addition and parallel substraction of operators, (Russian, English) Math. USSR, Izv. 10(1976), 351-370

[16] J. L. Smul'jan, A Hellinger operator integral. (Russian) Mat. Sb. (N.S.) 49 (91) 1959 381-430. English transl. AMS Transl. 22 (1962), 289-337.