III Jornadas de Fundamentos de Cuántica

November 22nd, 2013
Córdoba, Argentina
FaMAF

BEYOND HEISENBERG:

Entropic and Fidelity-Based Uncertainty Relations

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THE UNCERTAINTY PRINCIPLE

Heisenberg, 1927: Given two (or more) quantum observables one cannot predict with certainty and simultaneously the outcomes of both

UNCERTAINTY RELATION: GENERAL FORM

$$\mathcal{U}(\mathcal{A}, \mathcal{B}, \rho) \geq \mathfrak{B}(\mathcal{A}, \mathcal{B})$$

- state-independent
- = 0 only if A and B share at least one eigenstate (commute: particular case)
- >0 in other case

Uncertainty measure associated to the probability distributions of the outcomes

UNCERTAINTY RELATIONS

PRELIMINARIES AND NOTATION

Observables

$$\mathcal{A} = \sum_{i=1}^{N} a_i |a_i\rangle\langle a_i|$$

$$\mathcal{B} = \sum_{j=1}^{N} b_j |b_j\rangle\langle b_j|$$

(discrete, non-degenerate spectra)

T: unitary transformation matrix

$$T_{ji} = \langle b_j | a_i \rangle$$

Overlap:

$$c \equiv \max_{ji} |T_{ji}| \in \left[\frac{1}{\sqrt{N}}, 1\right]$$

complementary observables (e.g. σ_X and σ_Z)

share at least one eigenstate (e.g. $[\mathcal{A}, \mathcal{B}] = 0$)

Quantum states

$$\rho \geq 0 \;\text{ and }\; \mathrm{Tr}\rho = 1$$

$$\rho = |\Psi\rangle\langle\Psi| \quad \textit{Pure states}$$

Probability vectors

$$p(\mathcal{A}, \rho) = [\, p_1(\mathcal{A}, \rho) \cdots p_N(\mathcal{A}, \rho) \,]^t$$

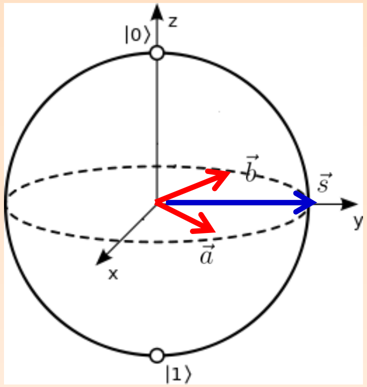
$$p_i(\mathcal{A}; \rho) = \langle a_i | \rho | a_i \rangle \underbrace{\rightarrow}_{\text{pure state}} |\langle a_i | \Psi \rangle|^2$$

HEISENBERG-ROBERTSON RELATION (1929)

$$V(\mathcal{A}; \rho) V(\mathcal{B}; \rho) \geq \frac{1}{4} |\langle [\mathcal{A}, \mathcal{B}] \rangle_\rho|$$

$$V(\mathcal{A}; \rho) = \sum_i (a_i - \langle \mathcal{A} \rangle_\rho)^2 \, p_i(\mathcal{A}; \rho)$$

$$\langle \mathcal{A} \rangle_\rho = \sum_i a_i \, p_i(\mathcal{A}; \rho)$$



qubit system $(N = 2)$

$$\rho = \frac{1}{2}(I + \vec{s} \cdot \vec{\sigma}), \; \vec{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z), \; \|\vec{s}\| \leq 1$$

$$\mathcal{A} = \vec{a} \cdot \vec{s}, \; \|\vec{a}\| = 1 \qquad \mathcal{B} = \vec{b} \cdot \vec{s}, \; \|\vec{b}\| = 1$$

HR UR:

$$\sqrt{1 - (\vec{a} \cdot \vec{s})^2} \sqrt{1 - (\vec{b} \cdot \vec{s})^2} \geq \left| (\vec{a} \times \vec{b}) \cdot \vec{s} \right|$$

PART I: ENTROPIC UNCERTAINTY RELATIONS

S. Zozor, G.M. Bosyk, and M. Portesi, *General entropic uncertainty relations for N-level systems,* to be submitted (2013)
S. Zozor, G.M. Bosyk, and M. Portesi, *On a generalized entropic uncertainty relation for the qubit,* J. Phys. A: Math. Theo. **46** 465301 (2013)

ENTROPY AND UNCERTIANTY RELATIONS

RÉNYI ENTR0PY

$$H_\lambda[p] = \frac{1}{1-\lambda} \ln \sum_{i=1}^{N} p_k^\lambda \,, \quad \lambda \geq 0$$

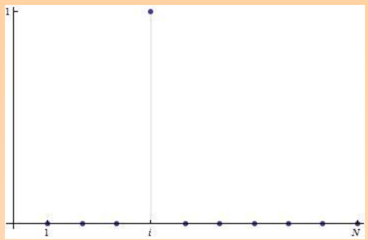
p: probability vector $\{p \in \mathcal{P} = [p_1 \cdots p_N]^t \in [0,1]^N,\ \sum_k p_k = 1\}$

◇Shannon entropy: $H_1[p] = -\sum_i p_i \ln p_i$

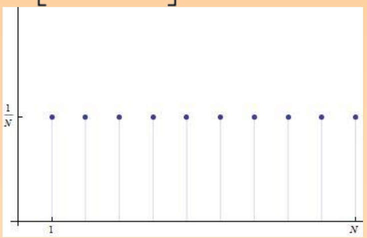
◇Colision entropy: $H_2[p] = -\ln \sum_i p_i^2$

◇Min entropy: $H_\infty[p] = -\ln \max_i p_i$

$p_k = \delta_{ki}$: $H_\lambda[p] = 0$



$p = \left[\frac{1}{N} \cdots \frac{1}{N}\right]^t$: $H_\lambda[p] = \ln N$



◇ $H_\lambda[p]$ is Schur-concave

◇ $H_\lambda[p]$ is concave in p for $\lambda \in [0,1]$

◇ $H_\lambda[p]$ for fixed p decreases with λ

GOAL AND PREVIUOS RESULTS

$$\boxed{H_\alpha\left[p(\mathcal{A},\rho)\right] + H_\beta\left[p(\mathcal{B},\rho)\right] \geq \overline{\mathfrak{B}}_{\alpha,\beta} > 0, \quad (\alpha,\beta) \in \mathbb{R}_+^2}$$

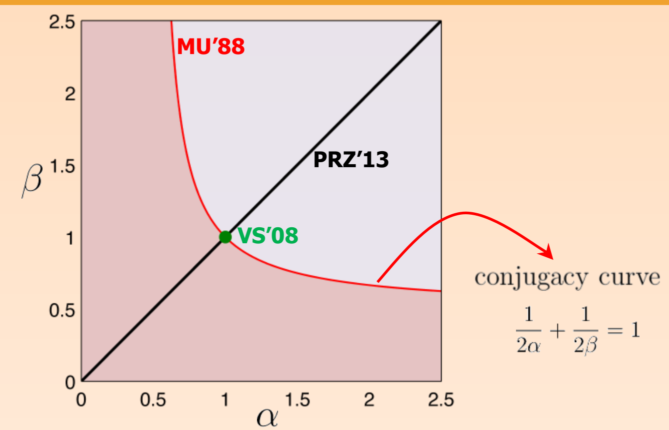
$T-$optimal bound

$$\mathfrak{B}_{\alpha,\beta}(T) = \min_\rho \left(H_\alpha\left[p(\mathcal{A},\rho)\right] + H_\beta\left[p(\mathcal{B},\rho)\right]\right) \geq \overline{\mathfrak{B}}_{\alpha,\beta}$$

Solved only for qubit systems ($N=2$),
SZ, GMB, MP, JPA **46**, 465301 (2013)

$c-$optimal bound

$$\widetilde{\mathfrak{B}}_{\alpha,\beta}(c,N) = \min_{T \in \mathrm{U}(N), \max|T_{ji}|=c} \mathfrak{B}_{\alpha,\beta}(T)$$



H. Maassen and J.B.M. Uffink, Phys. Rev. Lett. **60**, 1103 (1988)

J.L. de Vicente and J. Sanchez-Ruiz, Phys. Rev. A **77**, 042110 (2008)

Z. Puchała, Ł. Rudnicki, and K. Życzkowski, J. Phys. A: Math. Theor. **46**, 272002 (2013)

METHOD: CONSTRAINED MINIMIZATION

$$H_\alpha\left[p(\mathcal{A},\rho)\right] + H_\beta\left[p(\mathcal{B},\rho)\right] \ge \overline{\mathfrak{B}}_{\alpha,\beta}(c) > 0, \quad (\alpha,\beta)\in R_+^2$$

Landau-Pollak inequality (LPI): $\arccos\sqrt{P_{\mathcal{A};\rho}} + \arccos\sqrt{P_{\mathcal{B};\rho}} \ge \arccos c$

maxima probabilities $P_{\mathcal{A};\rho} = \max_i \langle a_i|\rho|a_i\rangle$ and $P_{\mathcal{B};\rho} = \max_j \langle b_j|\rho|a_j\rangle$

Step 1

$$H_{\lambda,\min}(P) = \min_{p\in\mathcal{P}} H_\lambda(p) \ \text{ s.t. } \max_k p_k = P \in \left[\frac{1}{N}, 1\right]$$

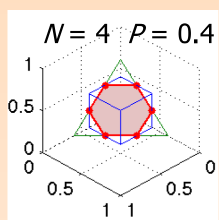
Equivalent problem:

$p = [P\, q_1 \cdots q_{N-1}]^t$ and find $q \in \mathbb{R}^{N-1}$ that maximizes

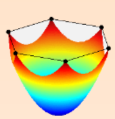
$$f_\lambda(q) = \frac{\sum_k q_k^\lambda - (1-P)}{\lambda - 1} \quad \text{s.t.} \quad \begin{array}{l} q \in \mathcal{HC}_P \equiv [0\,;\, P]^{N-1} \\ q \in \mathcal{HP}_P \equiv \left\{ q : \sum_k q_k = 1-P \right\} \end{array}$$

- $\mathcal{HC}_P \cap \mathcal{HP}_P$: convex polytope of pure points

$$\Big[\underbrace{P\cdots P}_{M \text{ times}} \quad 1-MP \quad \underbrace{0\cdots 0}_{N-M-1 \text{ times}}\Big]^t, \quad M = \left\lfloor \frac{1}{P} \right\rfloor$$



- f_λ convex: maximum at pure points

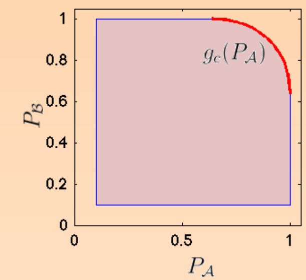


$$H_{\lambda,\min}(P) = \frac{\ln\left[MP^\lambda + (1-MP)^\lambda\right]}{1-\lambda}$$

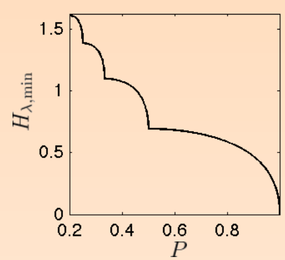
$$H_\alpha\left[p(\mathcal{A},\rho)\right] + H_\beta\left[p(\mathcal{B},\rho)\right] \ge H_{\alpha,\min}(P_{\mathcal{A};\rho}) + H_{\beta,\min}(P_{\mathcal{B};\rho})$$

Step 2

$$\overline{\mathfrak{B}}_{\alpha,\beta}(c) = \left\{ \begin{array}{l} \min\limits_{(P_\mathcal{A},P_\mathcal{B})\in\left[\frac{1}{N}\,;\,1\right]^2} H_{\alpha,\min}(P_\mathcal{A}) + H_{\beta,\min}(P_\mathcal{B}) \\ \text{s.t. LPI} \end{array} \right.$$



- $H_{\lambda,\min}(P)$ decreases vs P



- fix $P_\mathcal{B}$ (resp. $P_\mathcal{A}$) $\Rightarrow$ minimum at $P_\mathcal{B} = g_c(P_\mathcal{A})$

RESULTS AND EXAMPLES

Proposition: $\forall(\alpha,\beta)\in\mathbb{R}_+^2$

$$H_\alpha\left[p(\mathcal{A},\rho)\right]+H_\beta\left[(p(\mathcal{B},\rho)\right]\geq\overline{\mathfrak{B}}_{\alpha,\beta}(c)=\min_{\theta\in[0\,;\,\gamma]}\left(\frac{\ln\mathcal{D}_\alpha(\theta)}{1-\alpha}+\frac{\ln\mathcal{D}_\beta(\gamma-\theta)}{1-\beta}\right)$$

where $\gamma\equiv\arccos c$ and $\mathcal{D}_\lambda(\theta)\equiv\left\lfloor\frac{1}{\cos^2\theta}\right\rfloor(\cos^2\theta)^\lambda+\left(1-\left\lfloor\frac{1}{\cos^2\theta}\right\rfloor\cos^2\theta\right)^\lambda$

Corollary 1:

$$c>\frac{1}{\sqrt{2}}\Rightarrow\widetilde{\mathfrak{B}}_{\alpha,\beta}(c)=\overline{\mathfrak{B}}_{\alpha,\beta}(c)$$

$c-$optimum bound and equivalent to the qubit case

Corollary 2:

$$\overline{\mathfrak{B}}_{\alpha,\beta}(c)\geq\overline{\mathfrak{B}}^{(D)}\equiv2\ln\left(\frac{2}{1+c}\right)$$

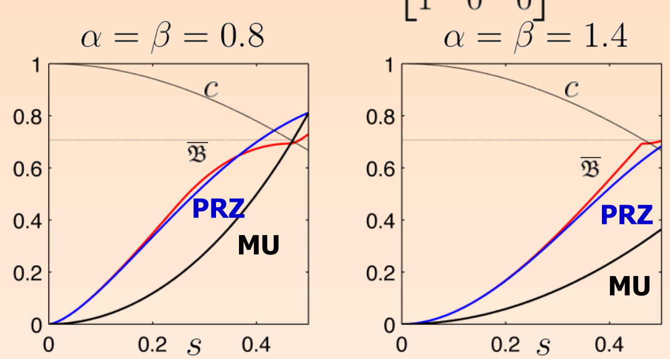
improvement of Deutsch bound

Corollary 3:

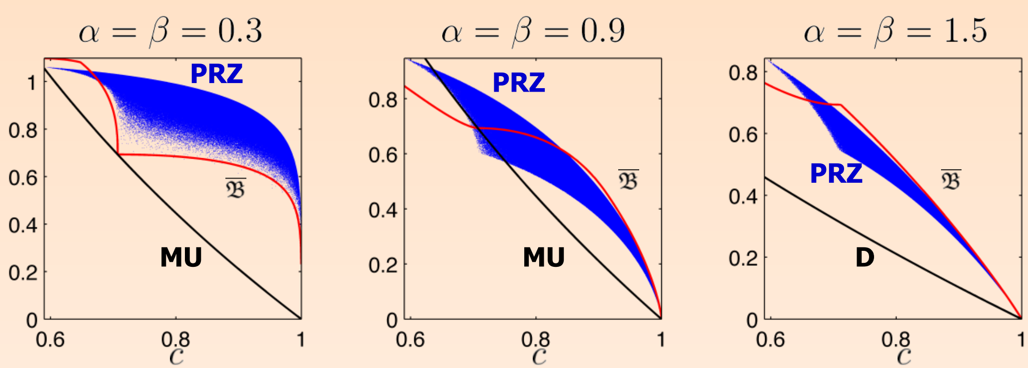
$$c\leq\frac{1}{2}\Rightarrow\overline{\mathfrak{B}}_{\alpha,\beta}(c)\leq\overline{\mathfrak{B}}^{(MU)}\equiv-2\ln c$$

no improvement of Maassen-Uffink bound

Example1: $T=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}^s$



Example 2: $T\sim U(3)$



PART II: FIDELITY-BASED UNCERTAINTY RELATIONS

G.M. Bosyk, T.M. Osán, P.W Lamberti and M. Portesi, *Geometric formulation of the uncertainty principle*, arXiv:1308.4029v2 [quant-ph] (2013)

LANDAU-POLLAK INEQUALITY FOR PURE STATES

$$\arccos\sqrt{P_{\mathcal{A};\Psi}} + \arccos\sqrt{P_{\mathcal{B};\Psi}} \geq \arccos c$$

H.J. Landau and H.O. Pollak, Bell Syst. Tech. J. **40**, 65 (1961)
J. Uffink, Ph. D. thesis, University of Utrecht (1990)

maxima probabilities

$$P_{\mathcal{A};\Psi} = \max_i |\langle a_i|\Psi\rangle|^2$$

$$P_{\mathcal{B};\Psi} = \max_j |\langle b_j|\Psi\rangle|^2$$

◇ if $c < 1$ then $P_{\mathcal{A};\Psi}$ and $P_{\mathcal{B};\Psi}$ cannot both be equal to one

◇ if $c = \frac{1}{\sqrt{N}}$ and $P_{\mathcal{A};\Psi} = 1$ then $P_{\mathcal{B};\Psi} = \frac{1}{N}$

Sketch of the proof:

Step 1: consider Wootters metric between state $|\Psi\rangle$ and $|a_i\rangle,\ |b_j\rangle$ — eigenstates of $\mathcal{A}$ and $\mathcal{B}$

Wotters metric: $d_{\mathrm{W}}(\varphi,\psi) = \arccos|\langle\varphi|\psi\rangle|$

Step 2: apply the triangle inequality

probabilities of the i-th and j-th outcomes of $\mathcal{A}$ and $\mathcal{B}$, resp.; T_{ij}

$$\arccos|\langle a_i|\Psi\rangle| + \arccos|\langle b_j|\Psi\rangle| \geq \arccos|\langle a_i|b_j\rangle|$$

Step 3: choose the eigenstates that maximize the probabilities $\arccos\sqrt{P_{\mathcal{A};\Psi}} + \arccos\sqrt{P_{\mathcal{B};\Psi}} \geq \arccos|\langle a_{i'}|b_{j'}\rangle|$

Step 4: use decreasesing property of arccos $\arccos|\langle a_{i'}|b_{j'}\rangle| \geq \arccos\max_{ij}|\langle a_i|b_j\rangle|$

FIDELITY & PURIFICATION

FIDELITY

measure of similarity of quantum states

$$F(\rho,\sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$$

- normalization $0 \leq F(\rho,\sigma) \leq 1$
- identity of indiscernibles $F(\rho,\sigma) = 1 \text{ iff } \rho = \sigma$
- symmetry $F(\rho,\sigma) = F(\sigma,\rho)$
- both pure sates $F(|\psi\rangle\langle\psi|, |\varphi\rangle\langle\varphi|) = |\langle\psi|\varphi\rangle|^2$
- one pure state $F(|\psi\rangle\langle\psi|, \sigma) = \langle\psi|\sigma|\psi\rangle$

PURIFICATION

mixed quantum states $\rightarrow$ pure quantum states

$$\rho \in B(\mathcal{H}) \rightarrow |\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}_{\mathrm{aux}} : \mathrm{Tr}_{\mathrm{aux}}(|\Psi\rangle\langle\Psi|) = \rho$$

Link between fidelity and purifications

$$F(\rho,\sigma) = \max_{|\varphi\rangle} |\langle\psi|\varphi\rangle|^2$$

all purifcations of σ

fixed purification ρ

LANDAU-POLLAK INEQUALITY FOR MIXED STATES

We prove:

$$\arccos\sqrt{P_{\mathcal{A};\rho}} + \arccos\sqrt{P_{\mathcal{B};\rho}} \geq \arccos c$$

maxima probabilities

$$P_{\mathcal{A};\rho} = \max_i \langle a_i|\rho|a_i\rangle$$
$$P_{\mathcal{B};\rho} = \max_j \langle b_j|\rho|a_j\rangle$$

Sketch of the proof:

Step 1: consider Wootters metric between $|s\rangle$, $|r\rangle$ and $|t\rangle$ purifications of σ, ρ and τ, resp.

Step 2: use triangle inequality $\arccos|\langle s|r\rangle| + \arccos|\langle t|r\rangle| \geq \arccos|\langle s|t\rangle|$

Step 3: choose the purifications that maximize the overlaps with ρ: $F(\sigma,\rho) = \max_{|s\rangle}|\langle s|r\rangle|^2 = |\langle\tilde{s}|r\rangle|^2$ and $F(\tau,\rho) = \max_{|t\rangle}|\langle t|r\rangle|^2 = |\langle\tilde{t}|r\rangle|^2$

$$\arccos\sqrt{F(\sigma,\rho)} + \arccos\sqrt{F(\tau,\rho)} \geq \arccos|\langle\tilde{s}|\tilde{t}\rangle|$$

Step 4: use decreasing property of arccos $\arccos|\langle\tilde{s}|\tilde{t}\rangle| \geq \arccos\sqrt{F(\sigma,\tau)}$

we proof triangle inequality for the angle!!!

Step 5: use that $\sigma = |a_i\rangle\langle a_i|$, $\tau = |b_j\rangle\langle b_j|$ ⇒ $F(\sigma,\rho) = \langle a_i|\rho|a_i\rangle = p_i(\mathcal{A};\rho)$, $F(\tau,\rho) = \langle b_j|\rho|b_j\rangle = p_j(\mathcal{B};\rho)$ ⇒ $\arccos\sqrt{p_i(\mathcal{A};\rho)} + \arccos\sqrt{p_j(\mathcal{B};\rho)} \geq \arccos|\langle a_i|b_j\rangle|$

Step 6 and 7: similar to LPI for pure states, i.e, choose max probabilities and use that arccos decreases

FIDELITY-BASED UNCERTAINTY RELATIONS

Metric between quantum states

- non-negativy $d(\rho,\sigma) \geq 0$ and $d(\rho,\sigma) = 0$ iff $\rho = \sigma$
- symmetry $d(\rho,\sigma) = d(\sigma,\rho)$
- triangle inequality $d(\sigma,\rho) + d(\tau,\rho) \geq d(\sigma,\tau)$

Fidelity-Based metrics

$d(\rho,\sigma) = f\left(F(\rho,\sigma)\right)$

with $f \downarrow$ and $f(x) = 0$ iff $x = 1$

Examples:

- Angle $d_{\mathrm{A}}(\rho,\sigma) = \arccos\sqrt{F(\rho,\sigma)}$
- Bures $d_{\mathrm{B}}(\rho,\sigma) = \sqrt{2 - 2\sqrt{F(\rho,\sigma)}}$
- Root-infidelity $d_{\mathrm{RI}}(\rho,\sigma) = \sqrt{1 - F(\rho,\sigma)}$

Uncertainty measure

$$\mathcal{U}(\mathcal{A};\rho) = f\left(P_{\mathcal{A};\rho}\right)$$

- ◇ $\mathcal{U}(\mathcal{A};\rho) \geq 0$
- ◇ $\mathcal{U}(\mathcal{A};\rho)$ is decreasing in terms of $P_{\mathcal{A};\rho}$
- ◇ $\mathcal{U}(\mathcal{A};\rho) = 0$ if and only if $P_{\mathcal{A};\rho} = 1$
- ◇ $\arg\max_{P_{\mathcal{A};\rho}} \mathcal{U}(\mathcal{A};\rho) = \frac{1}{N}$

MAIN RESULT

$\mathcal{A}$ and B observables, and ρ a denisty operator:

$$\boxed{\mathcal{U}(\mathcal{A};\rho) + \mathcal{U}(\mathcal{B};\rho) \geq f(c^2)} \text{ where } c = \max_{ij} |\langle b_j|a_i\rangle| \in \left[\frac{1}{\sqrt{N}}, 1\right]$$

- ◇ family of uncertainty relations
- ◇ universal (state-independent) lower bound
- ◇ for $c < 1$: uncertainty-sum is strictly greater than zero (non-trivial)
- ◇ for $c = \frac{1}{\sqrt{N}}$ (complementary observables): certainty on one implies maximum ignorance on the other

COMPARISON FOR KNOWN METRICS

Angle (A): $\arccos\sqrt{P_{\mathcal{A};\rho}} + \arccos\sqrt{P_{\mathcal{B};\rho}} \geq \arccos c$

Bures (B): $\sqrt{1 - 1\sqrt{P_{\mathcal{A};\rho}}} + \sqrt{1 - 1\sqrt{P_{\mathcal{B};\rho}}} \geq \sqrt{1-c}$

Root-infidelity (RI): $\sqrt{1 - P_{\mathcal{A};\rho}} + \sqrt{1 - P_{\mathcal{B};\rho}} \geq \sqrt{1-c^2}$

Order relation: $\mathcal{D}_{\mathrm{A},c} \subseteq \mathcal{D}_{\mathrm{B},c} \subseteq \mathcal{D}_{\mathrm{RI},c}$

LPI is the tightest UR!!!

◇ if $P_{\mathcal{A};\rho} = 1$ (certainty), then $P_{\mathcal{B};\rho} \leq g_{\lambda,c}(1) = c^2$ for any metric

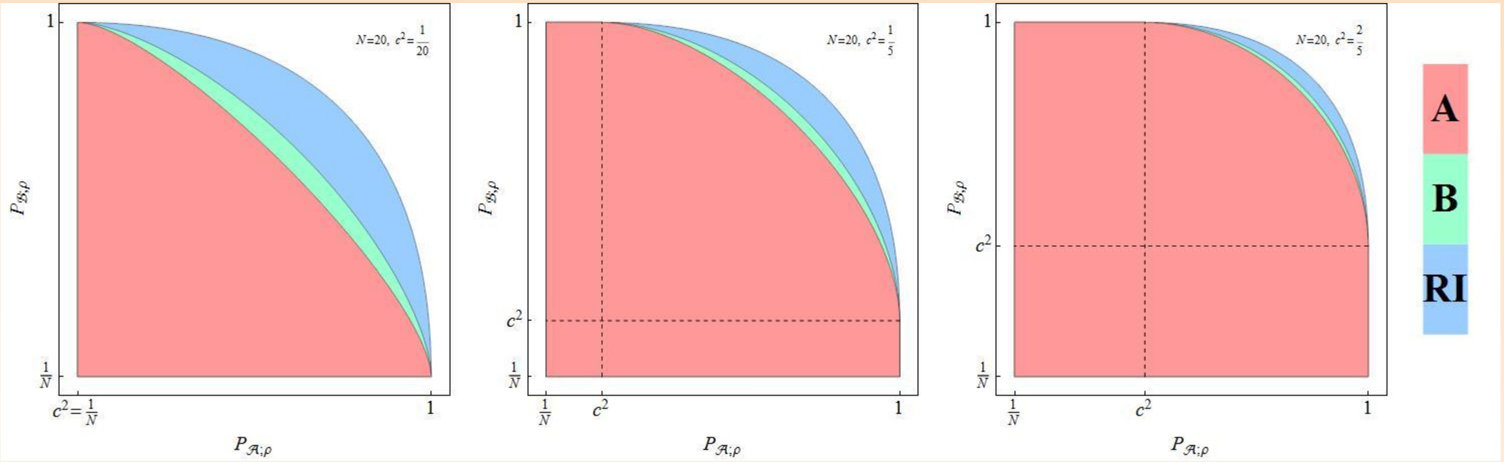
◇ when $c = 1$ trvial case, $g_1(P) = 1$ for each metric (there is no restriction)

Set of values of $P_{\mathcal{A};\rho}$ and $P_{\mathcal{B};\rho}$ allowed by each UR:

$$\mathcal{D}_c = \left\{ (P_{\mathcal{A};\rho}, P_{\mathcal{B};\rho}) \in \left[\frac{1}{N}, 1\right]^2 : \ P_{\mathcal{B};\rho} \leq g_c(P_{\mathcal{A};\rho}) \right\}$$

$$g_c(P) = \begin{cases} 1 & \text{if } \frac{1}{N} \leq P \leq c^2 \\ h_c(P) & \text{if } c^2 \leq P \leq 1 \end{cases} \quad \text{for each metric}$$

Metric	$h_c(P)$
Angle (A)	$\left(\sqrt{1-P}\sqrt{1-c^2} + c\sqrt{P}\right)^2$
Bures (B)	$\left(\sqrt{P} + 2\sqrt{1-\sqrt{P}}\sqrt{1-c} + c - 1\right)^2$
Root-infidelity (RI)	$P + 2\sqrt{1-P}\sqrt{1-c^2} + c^2 - 1$



SUMMARY

- We revisit the formulation of Uncertainty Principle:
  - going beyond Heisenberg-like uncertainty relations

- General Entropic Uncertainty Relations for N-level systems
  - ✓ lower bound for the sum of Rényi entropies of arbitrary entropic indices
  - ✓ c-optimal bound for overlap > $1/\sqrt{2}$ (improves all c-dependent bounds, e.g. Deutsch, MU, etc)
  - ✓ improves PRZ bound in several situations
  - ✓ easy to calculate (solve a one-dimensional minimization problem)
  - x open: extension to POVM measurements (LPI to POVM?)

- Geometric formulation of Uncertainty Principle
  - ✓ extension of LPI to mixed states
  - ✓ family of fidelity-based uncertainty relations
  - x conjecture: angle metric gives the tightest uncertainty relation (LPI) within this framework
  - x open: extension to POVM measurement