Heat trace asymptotics and the Gauss–Bonnet theorem for general connections

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Abstract

We examine the local super trace asymptotics for the de Rham complex defined by an arbitrary super connection on the exterior algebra. We show, in contrast to the situation in which the connection in question is the Levi-Civita connection, that these invariants are generically non-zero in positive degree and that the critical term is not the Pfaffian.

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1. Introduction

1.1. The Chern–Gauss–Bonnet theorem

Throughout this paper, we will let $\mathcal{M} := (M, g)$ be a compact smooth Riemannian manifold, without boundary, of dimension m. Let $\chi(M)$ be the Euler characteristic of M. If m is odd, then $\chi(M)$ is zero and if m is even, then $\chi(M)$ is given by integrating a suitable expression in the curvature tensor which can be described as follows. Let $\{e_i\}$ be a local orthonormal frame for the tangent bundle and let R_{ijkl} denote the components of the curvature tensor. We adopt the *Einstein* convention and sum over repeated indices. With our sign convention, the components of the *Ricci tensor* are given by setting $\rho_{ij} = R_{ikkj}$, and the *scalar curvature* is given by contracting the Ricci tensor and setting $\tau = \rho_{ii}$; τ is positive on the unit sphere in Euclidean space. If $m = 2\bar{m}$, then the *Pfaffian* (or *Euler form*) is given by setting

$$E_m := \frac{1}{(-8\pi)^{\bar{m}}\bar{m}!} g(e^{i_1} \wedge \cdots \wedge e^{i_m}, e^{j_1} \wedge \cdots \wedge e^{j_m}) R_{i_1 i_2 j_1 j_2} \ldots R_{i_{m-1} i_m j_{m-1} j_m}. \quad (1)$$

We have, for example,

$$E_2 = \frac{\tau}{4\pi} \quad \text{and} \quad E_4 = \frac{\tau^2 - 4|\rho|^2 + |R|^2}{32\pi^2}.$$

One sets $E_m = 0$ if m is odd. Let dx denote the Riemannian unit of volume on M:

$$dx = g dx^1, \ldots, dx^m, \quad \text{where} \quad g := \sqrt{\det(g_{ij})}.$$

Chern [6] established the following result which generalizes the classical two-dimensional Gauss–Bonnet formula.

Theorem 1. *If (M, g) is a compact Riemannian manifold, then*

$$\chi(M) = \int_M E_m \, dx.$$

1.2. Heat trace asymptotics

Let D be an operator of Laplace type on the space of smooth sections $C^\infty(V)$ to some vector bundle V over M, i.e. locally D has the form

$$D = -\{g^{ij} \, \mathrm{Id} \partial_{x_i} \partial_{x_j} + A^i \partial_{x_i} + B\}. \tag{2}$$

Here, A^i and B are suitable locally defined endomorphisms of the vector bundle V. For example, the scalar Laplacian $\Delta_{\mathcal{M}}^0$ is of this form since we may express

$$\Delta_{\mathcal{M}}^0 = -g^{-1} \partial_{x_i} g g^{ij} \partial_{x_j}.$$

Let e^{-tD} be the fundamental solution of the heat equation. If f is a smooth 'smearing' function, which is used to localize matter, then there is a complete asymptotic expansion as $t \downarrow 0$ of the form

$$\mathrm{Tr}_{L^2}\{f \, e^{-tD}\} \sim \sum_{n=0}^{\infty} t^{(n-m)/2} \int_M a_n(x, D) f(x) \, dx.$$

We set $f = 1$ to define the global heat trace asymptotics

$$a_n(D) := \int_M a_n(x, D) \, dx$$

and expand

$$\mathrm{Tr}_{L^2}\{e^{-tD}\} \sim \sum_{n=0}^{\infty} t^{(n-m)/2} a_n(D). \tag{3}$$

The local invariants $a_n(x, D)$ vanish for n odd and for n even are given by a local formula which is homogeneous of order n in the derivatives of the total symbol of D. The asymptotic series given in equation (3) was first studied by Minakshisundaram and Pleijel [24] for the scalar Laplacian, and the existence of the full asymptotic series in a very general setting was due to Seeley [27, 28]. This asymptotic formula can be generalized to the case of manifolds with a boundary if suitable boundary conditions are imposed; Kennedy, Critchley and Dowker [22] is a seminal work in this regard—the invariants a_n for n odd appear for manifolds with a boundary, so this is a convenient formalism.

1.3. The heat equation and the Chern–Gauss–Bonnet formula

We restrict ourselves for the moment to the case in which $D = \Delta_{\mathcal{M}}^p$ is the Laplacian on the space $C^\infty(\Lambda^p(M))$ of smooth p-forms. McKean and Singer [23] observed that

$$\sum_{p=0}^{m} (-1)^p \mathrm{Tr}_{L^2}\left(e^{-t\Delta_{\mathcal{M}}^p}\right) = \chi(M). \tag{4}$$

They wrote: '*Because of the complete cancellation of the time-dependent part of the alternating sum Z, it is natural to hope that some fantastic cancellation will also take place in the small, i.e. in the alternating pole sum*'. In other words, they conjectured that the following identity involving the super trace holds:

$$a_n(x, d + \delta) := \sum_{p=0}^{m} (-1)^p a_n\left(x, \Delta_{\mathcal{M}}^p\right) = \begin{cases} E_m & \text{if} \quad n = m \\ 0 & \text{if} \quad n < m \end{cases}. \tag{5}$$

Here, d denotes exterior differentiation, δ denotes the dual, interior multiplication and

$$\Delta_{\mathcal{M}}^p := d_{p-1}\delta_{p-1} + \delta_p d_p.$$

McKean and Singer established equation (5) in the special case that $m = 2$; subsequently, Patodi [26] established this identity for arbitrary m and thereby gave a heat equation proof of the Chern–Gauss–Bonnet theorem. By imposing absolute boundary conditions, manifolds with boundary can be considered [14]. We refer to the previous related work [18–20].

1.4. The de Rham complex

Introduce a $\mathbb{Z}_2$ grading on the exterior algebra by setting

$$\Lambda^e(M) = \oplus_p \Lambda^{2p}(M) \quad \text{and} \quad \Lambda^o(M) = \oplus_p \Lambda^{2p+1}(M).$$

We then have an elliptic complex of Dirac type

$$(d + \delta)_{\mathcal{M}}^{e/o} : C^\infty(\Lambda^{e/o}) \to C^\infty(\Lambda^{o/e}),$$

where the associated operators of Laplace type are given by

$$\Delta_{\mathcal{M}}^e = (d + \delta)_{\mathcal{M}}^o (d + \delta)_{\mathcal{M}}^e = \oplus_p \Delta_{\mathcal{M}}^{2p},$$
$$\Delta_{\mathcal{M}}^o = (d + \delta)_{\mathcal{M}}^e (d + \delta)_{\mathcal{M}}^o = \oplus_p \Delta_{\mathcal{M}}^{2p+1}.$$

We now review a bit of classical tensor calculus. Let $\xi \in T^*M$ be a cotangent vector. Let $\text{ext}(\xi)$ denote left exterior multiplication by ξ and let $\text{int}(\xi)$ be the dual, interior multiplication. We then have

$$\text{ext}(\xi)\omega := \xi \wedge \omega \quad \text{and} \quad g(\text{ext}(\xi)\omega, \tilde{\omega}) = g(\omega, \text{int}(\xi)\tilde{\omega}).$$

If ∇^g denotes the Levi-Civita connection, then the exterior differentiation d and the adjoint interior differentiation δ are given by the formulas

$$d\omega = \text{ext}(dx^i)\nabla_{\partial_{x_i}}^g \omega \quad \text{and} \quad \delta\omega = -\text{int}(dx^i)\nabla_{\partial_{x_i}}^g \omega.$$

We define $\gamma(\xi) := \text{ext}(\xi) - \text{int}(\xi)$ to give the exterior algebra the structure of a *Clifford module*

$$\gamma(\xi)\gamma(\eta) + \gamma(\eta)\gamma(\xi) = -2g(\xi, \eta) \,\text{Id}.$$

Let ∇ be a connection on the exterior algebra bundle $\Lambda(M)$. We suppose that ∇ is a *super connection*, i.e. that $\nabla = \nabla^e \oplus \nabla^o$ restricts to define connections on Λ^e and Λ^o separately, but aside from that we impose no restrictions on ∇. In physics, these connections play a role in the context of dark matter and supergravity; see, e.g., [2, 12, 25]. For such a connection, one defines

$$(d + \delta)_\nabla^{e/o} := \gamma(dx^i)\nabla_{\partial_{x_i}}^{e/o} : C^\infty(\Lambda^{e/o}) \to C^\infty(\Lambda^{o/e}),$$

$$\Delta_\nabla^{e/o} := (d + \delta)_\nabla^{o/e}(d + \delta)_\nabla^{e/o} \text{ and}$$

$$a_n(x, (d + \delta)_\nabla) := a_n\left(x, \Delta_\nabla^e\right) - a_n\left(x, \Delta_\nabla^o\right).$$

Equation (5) generalizes to this setting to become:

Theorem 2. *Adopt the notation established above. Then,*

$$\mathrm{Tr}_{L^2}\big(\mathrm{e}^{-t\Delta_\nabla^e}\big) - \mathrm{Tr}_{L^2}\big(\mathrm{e}^{-t\Delta_\nabla^o}\big) = \chi(M).$$

While this is perhaps not surprising, we shall give the proof in section 3, as the usual proofs in the literature (see, for example, the discussion in [16]) assume that the operators in question are self-adjoint and it is necessary to remove this restriction.

If D is an operator of Laplace type on a smooth vector bundle V, then

$$a_0(x, D) = (4\pi)^{-m/2} \dim(V).$$

Thus, in particular

$$a_0(x, (d+\delta)_\nabla) = (4\pi)^{-m/2} \{\dim(\Lambda^e) - \dim(\Lambda^o)\} = 0.$$

The local 'fantastic cancellation' does not hold in this setting for the higher order heat trace asymptotics. We shall establish the following result in section 3.

Theorem 3. *Let $\mathcal{M} = (\mathbb{T}^m, g_0)$ be the flat torus. Fix a point $x \in \mathbb{T}^m$. There exists a super connection ∇ on $\Lambda^*(M)$ so that $(d+\delta)_\nabla$ is self-adjoint and so that $a_{2n}(x, (d+\delta)_\nabla) \neq 0$ for $n \geqslant 1$ for some point x of $\mathbb{T}^m$.*

Remark 1. Let ∇ be the connection of theorem 3. We note that E_m vanishes for the flat metric; thus, in particular, if m is even, then $a_m(x, (d+\delta)_\nabla) \neq E_m$.

If ∇ is a connection on TM, we can extend ∇ to act on the cotangent bundle by requiring that the duality equation is satisfied:

$$X\langle Y, \omega\rangle = \langle \nabla_X Y, \omega\rangle + \langle X, \nabla_X \omega\rangle \; \forall X, Y \in C^\infty(TM), \; \omega \in C^\infty(T^*M),$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between TM and T^*M. Then there is a unique extension of ∇ to a connection on the full exterior algebra which satisfies the Leibnitz rule:

$$\nabla_X(\omega_1 \wedge \omega_2) = (\nabla_X \omega_1) \wedge \omega_2 + \omega_1 \wedge (\nabla_X \omega_2) \; \forall \, X \in C^\infty(TM), \; \omega_i \in C^\infty(\Lambda M).$$

Since such an induced connection restricts to a connection on each $\Lambda^p M$ separately, necessarily any induced connection is a super connection. However, since an arbitrary super connection need not obey the Leibnitz rule, not every super connection is induced from an underlying connection on the tangent bundle; in particular, the connections used to establish theorem 3 will not satisfy the Leibnitz property.

In this setting, one has a weaker version of the McKean–Singer local vanishing theorem.

Theorem 4. *Let ∇ be a connection on ΛM which is induced from an underlying connection on TM. Then, $a_n(x, (d+\delta)_\nabla) = 0$ for $3n < 2m$.*

1.5. General elliptic complexes

Instead of considering the *de Rham complex* one can consider more generally a first-order partial differential operator

$$A : C^\infty(V_1) \to C^\infty(V_2), \tag{6}$$

where V_1 and V_2 are smooth vector bundles over M. We assume that V_1 and V_2 are equipped with fiber metrics and form the formal adjoint $A^* : C^\infty(V_2) \to C^\infty(V_1)$. We say that equation (6) defines an *elliptic complex of Dirac type* if the associated second-order operators $D_1 := A^*A$ and $D_2 := AA^*$ are of Laplace type. We define

$$a_n(x, A) := a_n(x, D_1) - a_n(x, D_2).$$

4

Note that in the case of the de Rham complex, the associated Laplacians are given by $D_1 = \Delta^e_{\mathcal{M}}$ and $D_2 = \Delta^o_{\mathcal{M}}$. Equation (5) generalizes to this setting to yield

$$\int_M a_n(x, A)\, dx = \begin{cases} \text{index}(A) & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}. \tag{7}$$

Atiyah, Patodi and Singer [3] showed that if A was the operator of the signature complex with coefficients in an auxiliary bundle W, then $a_n(x, A) = 0$ for $n < m$ while $a_m(x, A)$ gave the classical formula in terms of the Hirzebruch L polynomial and the Chern forms of the coefficient bundle W; a similar result held for the twisted spin complex. This result then led to a heat equation proof of the Atiyah–Singer theorem in full generality; the Lefschetz formulas and other results have also been established using these methods. We refer to [16] for more details; the field is a vast one and it is not possible to give a full account in a short note such as this.

If (M, g, J) is a Kähler manifold, the same approach led to a proof of the Riemann–Roch formula. However, if the metric in question was not Kähler, the local invariants of the heat equation did not give rise to the classical formula; there were divergence terms [13, 21]. Rather than vanishing for $n < m$, in fact one could construct examples where $a_n(x, \partial + \delta'')$ was non-zero for certain values of $n < m$. But it remained an open problem to construct index problems where the local index formula did not vanish for a_2; such examples are provided by theorem 3.

The following result, which generalizes equation (7) to the situation not involving adjoints, will be used in the proof of theorem 2:

Theorem 5. *Let $A_\pm : C^\infty(V_\pm) \to C^\infty(V_\mp)$ be first-order partial differential operators over a compact Riemannian manifold without boundary such that $D_\pm := A_\mp A_\pm$ are operators of Laplace type on $C^\infty(V_\pm)$. Set $a_n(x, A_+, A_-) := a_n(x, D_+) - a_n(x, D_-)$. Then,*

$$\int_M a_n(x, A_+, A_-)dx = \begin{cases} index(A_+) & if \quad n = m \\ 0 & if \quad n \neq m \end{cases}.$$

1.6. Outline of the paper

In section 2, we shall summarize briefly the theory of compact operators on Hilbert space that we shall need; we omit the proofs as they are standard and refer instead to [7] and [32] for further details. In section 3, we will use these results to establish theorems 2 and 5. In section 4, we recall some formulas for the heat trace asymptotics which we use in section 5 to complete the proof of theorem 3. In section 6, we prove theorem 4.

2. Spectral theory of compact operators in a separable Hilbert space

Lemma 1. *Let $\sigma(K)$ denote the spectrum of a compact operator K on an infinite-dimensional separable Hilbert space.*

(i) *Let $0 \neq \mu \in \sigma(K)$. Then*

    (a) *μ is an eigenvalue of K;*
    (b) *there exists an integer $\nu_K(\mu)$ so that $\ker(K - \mu)^{\nu_K(\mu)} = \ker(K - \mu)^{\nu_K(\mu)+1}$.*

(ii) *The eigenvalues can only accumulate at 0.*

(iii) *$0 \in \sigma(K)$ and $\sigma(K)$ is countable.*

Let $0 \neq \mu \in \sigma(K)$, where K is a compact operator on an infinite-dimensional separable Hilbert space. By lemma 1, the eigenvalues only accumulate at 0. The *Riesz projection* is defined by setting

$$\pi_K(\mu) := \frac{1}{2\pi\sqrt{-1}} \int_\gamma (\Theta - K)^{-1} \, d\Theta,$$

where γ is any simple closed curve about μ in the complex plane which encloses no other eigenvalues of K. Let $E_K(\mu) := \text{Range}\{\pi_K(\mu)\}$. We set $\pi_K(\mu) = 0$ and $E_K(\mu) = \{0\}$, if $\mu \notin \sigma(K)$.

Lemma 2. *Adopt the notation established above. Then*

(i) *if $0 \neq \mu \in \sigma(K)$, then*

    (a) $\pi_K(\mu)^2 = \pi_K(\mu)$;

    (b) $E_K(\mu)$ *is a finite-dimensional space*;

    (c) $E_K(\mu) = \ker(\mu - K)^{\nu_K(\mu)}$.

(ii) *If $0 \neq \mu_i \in \sigma(K)$ with $\mu_1 \neq \mu_2$, then $\pi_K(\mu_1)\pi_K(\mu_2) = 0$.*

(iii) *Let $E_K(0) := \cap_{0 \neq \mu \in \sigma(K)} \ker(\pi_K(\mu))$. Then*

    (a) $KE_K(0) \subset E_K(0)$;

    (b) $\sigma(K|_{E_K(0)}) = \{0\}$.

3. Heat trace formulas for the index

3.1. Spectral theory of operators of Laplace type

Let D be an operator of Laplace type on the space of smooth sections to a vector bundle V over a compact Riemannian manifold. Standard elliptic theory, see for example the discussion in [16], shows that the spectrum of D (viewed as an unbounded operator in Hilbert space) is contained in a cone about the positive real axis with arbitrarily small slope. Thus, in particular, there exists $\kappa \gg 0$ so that the operator $(D + \kappa)$ is invertible; $K := (D + \kappa)^{-1}$ is then a compact operator on $L^2(V)$ (viewed as a Banach space). We have

$$D\phi = \lambda\phi \Leftrightarrow (D + \kappa)\phi = (\lambda + \kappa)\phi \Leftrightarrow (D + \kappa)^{-1}\phi = (\lambda + \kappa)^{-1}\phi.$$

Thus, the spectrum of D is a countable set which only accumulates at ∞. Set $\sigma(D) = \{\lambda_n\}$, set $\nu_D(\lambda_n) := \nu_K((\lambda_n + \kappa)^{-1})$ and set

$$E_D(\lambda_n) := \{\phi : (D - \lambda_n)^{\nu_D(\lambda_n)}\phi = 0\} = E_K((\lambda_n + \kappa)^{-1}).$$

Elliptic regularity then shows $E_D(\lambda_n) \subset C^\infty(V)$. Note that $0 \in \sigma(K)$ plays no role and the analysis of section 2 shows that there is a direct sum decomposition (which is not an orthogonal direct sum decomposition in general) of the form

$$L^2(V) = \oplus_n E_D(\lambda_n).$$

It is then immediate that

$$\text{Tr}_{L^2}(e^{-tD}) = \sum_n e^{-t\lambda_n} \dim(E_D(\lambda_n)).$$

3.2. Proof of theorem 5

Let $A_\pm : C^\infty(V_\pm) \to C^\infty(V_\mp)$ be first-order partial differential operators over a compact Riemannian manifold without boundary such that $D_\pm := A_\mp A_\pm$ are the operators of Laplace type on $C^\infty(V_\pm)$. Since $A_\pm D_\pm = D_\mp A_\pm$,

$$A_\pm : E_{D_\pm}(\lambda_n) \to E_{D_\mp}(\lambda_n).$$

If $\lambda_n \neq 0$, then $D_\pm = A_\mp A_\pm$ is invertible on $E_{D_\pm}(\lambda_n)$; consequently, $A_\pm$ is an isomorphism. The decomposition

$$L^2(V_\pm) = E_{D_\pm}(0) \oplus_{\lambda_n \neq 0} E_{D_\pm}(\lambda_n)$$

shows that

$$\ker(A_\pm) = E_{D_\pm}(0) \quad \text{and} \quad \text{range}(A_\mp) = \oplus_{\lambda_n \neq 0} E_{D_\pm}(\lambda_n).$$

Consequently,

$$\text{index}(A_+) = \dim\{E_{D_+}(0)\} - \dim\{E_{D_-}(0)\}$$

and the usual cancellation argument shows that

$$\begin{aligned}
\text{Tr}_{L^2}(e^{-tD_+}) - \text{Tr}_{L^2}(e^{-tD_-}) &= \sum_n e^{-t\lambda_n}[\dim\{E_{D_+}(\lambda_n)\} - \dim\{E_{D_-}(\lambda_n)\}] \\
&= \dim E_{D_+}(0) - \dim E_{D_-}(0) = \text{index}(A_+).
\end{aligned}$$

3.3. The proof of theorem 2

The condition that $A_\mp A_\pm$ are of Laplace type is a condition only on the leading order symbols of $A_\pm$; it is unchanged by zeroth-order perturbations of these operators. Furthermore, since $a_m(x, A_+, A_-)$ is given by a local formula, $\text{index}(A_+)$ varies continuously if we perturb the zeroth-order symbol; as the index is integer valued, it is therefore unchanged by zeroth-order perturbations. Consequently, if we are dealing with the generalized de Rham complex, the index must in fact be $\chi(M)$ and theorem 2 as follows.

4. Local formulas for the heat trace asymptotics

Let D be an operator of Laplace type on $C^\infty(V)$. We adopt the notation of equation (2). There is a unique connection ∇ on V and a unique endomorphism E of V so that we may express D using the Bochner formalism

$$Du = -\{g^{ij}u_{;ij} + Eu\},$$

where $u_{;ij}$ denotes the second covariant differentiation of u with respect to the connection ∇. Let $\Gamma_{uv}{}^w$ be the Christoffel symbols of the Levi-Civita connection and let ω be the connection 1-form of ∇. Using the notation established above, one then has [16] that

$$\begin{aligned}
\omega_i &= \tfrac{1}{2} g_{ij}\{A^j + g^{lk}\Gamma_{lk}{}^j \text{Id}\}, \\
E &= B - g^{ij}\{\partial_{x_i}\omega_j + \omega_i\omega_j - \omega_k\Gamma_{ij}{}^k\}.
\end{aligned} \tag{8}$$

Let Ω be the curvature of the connection ∇. The following is well known—see, for example, the discussion in [17] and the references therein:

$$a_0(x, D) = (4\pi)^{-m/2} \text{Tr}\{\text{Id}\},$$

$$a_2(x, D) = (4\pi)^{-m/2} \tfrac{1}{6} \text{Tr}\{6E + \tau \text{Id}\},$$

$$\begin{aligned}
a_4(x, D) = (4\pi)^{-m/2} \tfrac{1}{360} \text{Tr}\{&60E_{;kk} + 60\tau E + 180E^2 + 12\tau_{;kk} \text{Id} \\
&+ 5\tau^2 \text{Id} - 2|\rho|^2 \text{Id} + 2|R|^2 \text{Id} + 30\Omega_{ij}\Omega_{ij}\}.
\end{aligned} \tag{9}$$

Partial information is available for all the terms [5]:

$$a_{2n}(x, D) = \frac{(-1)^n n!}{(2n+1)!}\{-n\Delta^{n-1}\tau \operatorname{Tr}(\mathrm{Id}) - (4n+2)\Delta^{n-1}\operatorname{Tr}(E)\} + \cdots. \tag{10}$$

We refer to [15, 29] for a discussion of a_6 and note that formulae for a_8 and a_{10} are available in this setting [1, 4, 30].

5. Super trace asymptotics for the de Rham complex on a flat torus

Let $\mathcal{M} = (\mathbb{T}^m, g)$ be the flat m-dimensional torus with the usual periodic parameters $(x^1, \ldots, x^m)$ so that $g(\partial_{x_i}, \partial_{x_j}) = \delta_{ij}$; the analysis is simplified by taking a flat structure but a similar analysis holds in the general setting. Let $V = \Lambda(M)$ be the exterior algebra and let $\Xi := +\mathrm{Id}$ on Λ^e and $\Xi := -\mathrm{Id}$ on Λ^o be the chirality operator which defines the super trace. Let $\gamma^i := \mathrm{ext}(\mathrm{d}x^i) - \mathrm{int}(\mathrm{d}x^i)$ and θ be the connection 1-form of a connection ∇ on $\Lambda(\mathbb{T}^m)$; ∇ is a super connection if and only if $\theta\Xi = \Xi\theta$. Set

$$A_\nabla := \gamma^i(\partial_{x_i} + \theta_i).$$

Set $\theta_{j/i} := \partial_{x_i}\theta_j$.

Lemma 3. *If ∇ is a super connection on $\Lambda(\mathbb{T}^m)$, then*

(i) $A_\nabla : C^\infty(\Lambda^{e/o}) \to C^\infty(\Lambda^{o/e})$ is an elliptic complex of Dirac type;
(ii) the operator A is self-adjoint if and only if $\theta_i^*\gamma^i = -\gamma^i\theta_i$;
(iii) $\operatorname{Tr}\{E\Xi\} = -\operatorname{Tr}\{\gamma^i\gamma^j\theta_{j/i}\Xi\}$.

Proof. The first two assertions are immediate from the definitions that we have given. Since $g_{ij} = \delta_{ij}$, we can raise and lower indices freely and use equation (8) to compute

$$D = \gamma^i(\partial_{x_i} + \theta_i)\gamma^j(\partial_{x_j} + \theta_j)$$
$$= -\left\{\partial_{x_i}^2 - (\gamma^i\theta_i\gamma^j + \gamma^j\gamma^i\theta_i)\partial_{x_j} - (\gamma^i\theta_i\gamma^j\theta_j + \gamma^i\gamma^j\theta_{j/i})\right\}$$
$$\omega_j = -\tfrac{1}{2}\left(\gamma^i\theta_i\gamma^j + \gamma^j\gamma^i\theta_i\right)$$
$$E = -(\gamma^i\theta_i\gamma^j\theta_j + \gamma^i\gamma^j\theta_{j/i}) + \tfrac{1}{2}(\gamma^i\theta_{i/j}\gamma^j + \gamma^j\gamma^i\theta_{i/j})$$
$$- \tfrac{1}{4}(\gamma^i\theta_i\gamma^j + \gamma^j\gamma^i\theta_i)(\gamma^k\theta_k\gamma^j + \gamma^j\gamma^k\theta_k).$$

We now take the super trace and show that most of the terms vanish. We use the identities $\Xi\theta_j = \theta_j\Xi$, $\gamma^i\Xi = -\Xi\gamma^i$ and $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ to compute

$$-\operatorname{Tr}\{\gamma^i\theta_i\gamma^j\theta_j\Xi\} = -\operatorname{Tr}\{-\gamma^i\theta_i\Xi\gamma^j\theta_j\} = -\operatorname{Tr}\{-\gamma^j\theta_j\gamma^i\theta_i\Xi\} = 0,$$

$$\tfrac{1}{2}\operatorname{Tr}\{\gamma^i\theta_{i/j}\gamma^j\Xi + \gamma^j\gamma^i\theta_{i/j}\Xi\} = \tfrac{1}{2}\operatorname{Tr}\{\gamma^i\theta_{i/j}\gamma^j\Xi + \gamma^i\theta_{i/j}\Xi\gamma^j\}$$
$$= \tfrac{1}{2}\operatorname{Tr}\{\gamma^i\theta_{i/j}\gamma^j\Xi - \gamma^i\theta_{i/j}\gamma^j\Xi\} = 0.$$

$$-\tfrac{1}{4}\operatorname{Tr}\{\gamma^i\theta_i\gamma^j\gamma^k\theta_k\gamma^j\Xi + \gamma^i\theta_i\gamma^j\gamma^j\gamma^k\theta_k\Xi + \gamma^j\gamma^i\theta_i\gamma^k\theta_k\gamma^j\Xi + \gamma^j\gamma^i\theta_i\gamma^j\gamma^k\theta_k\Xi\}$$
$$= -\tfrac{1}{4}\operatorname{Tr}\{\gamma^i\theta_i\gamma^j\gamma^k\theta_k\gamma^j\Xi + \gamma^i\theta_i\gamma^j\gamma^j\gamma^k\theta_k\Xi - \gamma^j\gamma^j\gamma^i\theta_i\gamma^k\theta_k\Xi - \gamma^i\theta_i\gamma^j\gamma^k\theta_k\gamma^j\Xi\}$$
$$= -\tfrac{1}{4}\operatorname{Tr}\{-\gamma^i\theta_i\gamma^k\theta_k\Xi + \gamma^i\theta_i\gamma^k\theta_k\Xi\} = 0.$$

We take $\theta_1 := f(x_2)\gamma^1\gamma^2\Xi$. Then $\theta_1\Xi = \Xi\theta_1$ so θ defines a super connection. Furthermore,

$$\theta_1^* = f(x_2)\Xi\gamma^2\gamma^1 = -f(x_2)\gamma^1\gamma^2\Xi,$$

$$-\theta_1^*\gamma^1 = f(x_2)\gamma^1\gamma^2\Xi\gamma^1 = \gamma^1 f(x_2)\gamma^1\gamma^2\Xi = \gamma^1\theta_1,$$

so A_∇ is self-adjoint. We have

$$\mathrm{Tr}\{\gamma^i\gamma^j\theta_{j/i}\Xi\} = \mathrm{Tr}\{\gamma^2\gamma^1\gamma^1\gamma^2\Xi\Xi\partial_{x_2}f\} = \dim\{\Lambda(\mathbb{T}^m)\}\partial_{x_2}f.$$

We choose f so all the derivatives of f are non-zero at some point x of $\mathbb{T}^m$. We then have $\{\Delta^{n-1}\mathrm{Tr}(E)\}(x) \neq 0$ for $n \geqslant 1$. Theorem 3 now follows from equations (9) and (10). □

6. The proof of theorem 4

Let g be a Riemannian metric on M and ∇ be a connection on TM. We expand

$$\left(\nabla_{\partial_{x_i}} - \nabla^g_{\partial_{x_i}}\right)\partial_{x_j} = \theta_{ij}{}^k\partial_{x_k}.$$

We extend ∇ to a connection on all of ΛM using the Leibnitz rule as discussed previously. Fix a point x of M. We normalize the choice of the coordinate system so that $g(x)_{ij} = \delta_{ij}$ and so that the first derivatives of the metric $g_{ij/k} := \partial_{x_k}g_{ij}$ vanish at x. We introduce formal variables $g_{ij/\alpha}$ and $\theta_{ij}{}^k{}_{/\alpha}$ for the derivatives ∂_x^α of the metric and of the normalized connection 1-form. We let $\mathcal{I}_m$ be the space of all polynomials which are invariant, i.e. which are independent of the particular coordinate system chosen for evaluation.

There is a natural grading which is defined on $\mathcal{I}_m$. We set

$$\deg(g_{ij/\alpha}) = |\alpha| \quad \text{and} \quad \deg(\theta_{ij}{}^k{}_{/\alpha}) := 1 + |\alpha|.$$

Let $\mathcal{I}_{m,n} \subset \mathcal{I}_m$ consist of those invariants which are homogeneous of order n. This grading can also be expressed in a coordinate-free fashion by noting that an invariant is homogeneous of order n if and only if $P(c^2g) = c^{-n}P(g)$. In particular, there are no non-trivial invariants of odd order. Thus, we may decompose

$$\mathcal{I}_m = \oplus_n \mathcal{I}_{m,n}.$$

The proof of theorem 4 will rest upon the following result.

Lemma 4. *Adopt the notation established above:*

(i) $a_n(x, (d+\delta)_\nabla)$ defines an element $\varepsilon_{m,n} \in \mathcal{I}_{m,n}$;
(ii) there is a natural restriction map $r : \mathcal{I}_{m,n} \to \mathcal{I}_{m-1,n} \to 0$;
(iii) $r(\varepsilon_{m,n}) = 0$;
(iv) r is injective from $\mathcal{I}_{m,n}$ to $\mathcal{I}_{m-1,n}$ if $3n < 2m$.

Proof. Assertion (i) follows from the Seeley calculus; a_n is given by a local formula which is homogeneous of degree n in the derivatives of the total symbol of the operator. Weyl's theory of invariants of the orthogonal group [31] permits us to express the spaces $\mathcal{I}_{m,n}$ tensorially in terms of contractions of indices in polynomials of the covariant derivatives of the curvature tensor and of θ where the sum ranges over indices $1 \leqslant i_\mu \leqslant m$. If $P \in \mathcal{I}_{m,n}$, then $r(P)$ is defined by letting the corresponding sum range over indices $1 \leqslant i_\mu \leqslant m-1$; it now follows that r is surjective.

It is worth illustrating this argument with an example. The scalar curvature

$$\tau_m := \sum_{i,j=1}^m R_{ijji}$$

defines an element of $\mathcal{I}_{m,2}$ for any m. The sequence $\{\tau_m\}$ satisfies $r(\tau_m) = \tau_{m-1}$ and for this reason one usually does not subscript to explicitly demonstrate the dimension m.

There is an alternative description of the restriction map which is useful. Let (N, g_N) be a Riemannian manifold of dimension $m - 1$ which is equipped with a connection ∇_N on the tangent bundle giving rise to a structure $\theta_N = \{\theta_i\}_{i=1}^{m-1}$. Let s be the usual periodic parameter on the circle. We let $M := N \times S^1$, $g_M = g_N + ds^2$ and $\theta_M = \{\theta_i\}_{i=1}^{m-1}$ define ∇_M, with the connection ∇_M on $T(N \times S^1)$ being flat in the S^1 direction. Since the structures are flat in S^1, we have

$$r(P)(N, g_N, \theta_N)(x) = P(M, g_M, \theta_M)(x, s), \quad \text{for any} \quad s \in S^1.$$

Clifford multiplication $\gamma(ds) := \text{ext}(ds) - \text{int}(ds)$ defines an isomorphism from $\Lambda^e(M)$ to $\Lambda^o(M)$ which intertwines the associated Laplacians since the structures are induced from an underlying structure on N. Thus, the super trace of the heat asymptotics cancels and assertion (iii) follows.

Let $P \in \mathcal{I}_{m,n}$. If A is a monomial of order n, let $c(P, A)$ be the coefficient of A in P; we say A *is a monomial of* P if $c(P, A) \neq 0$. Express A in the form

$$A = g_{i_1 j_1 / \alpha_1} \cdots g_{i_\ell j_\ell / \alpha_\ell} \theta_{k_1 l_1}{}^{n_1}{}_{/\beta_1} \cdots \theta_{k_t l_t}{}^{n_t}{}_{/\beta_t}.$$

By hypothesis,

$$|\alpha_i| \geqslant 2 \quad \text{for} \quad 1 \leqslant i \leqslant \ell, \tag{11}$$

since we have normalized the coordinate system so the first derivatives of the metric vanish. Furthermore, the order is given by

$$n = |\alpha_1| + \cdots + |\alpha_\ell| + (|\beta_1| + 1) + \cdots + (|\beta_t| + 1). \tag{12}$$

Let $\deg_a(A)$ be the number of times that an index a appears in A;

$$\deg_a(A) = \delta_{i_1, a} + \delta_{j_1, a} + \alpha_1(a) + \cdots + \delta_{i_\ell, a} + \delta_{j_\ell, a} + \alpha_\ell(a)$$
$$+ \delta_{k_1, a} + \delta_{l_1, a} + \delta_{n_1, a} + \beta_1(a) + \cdots + \delta_{k_t, a} + \delta_{l_t, a} + \delta_{n_t, a} + \beta_t(a).$$

If A is a monomial of P, we replace x_a by $-x_a$ to see that

$$\deg_a(A) \equiv 0 \mod 2. \tag{13}$$

If $r(P) = 0$, then $\deg_m(A) > 0$ since the restriction map is defined by setting to zero those monomials which involve the final index. Permuting the coordinate indices and applying equation (13) then yields

$$\deg_a(A) \geqslant 2 \quad \text{for} \quad 1 \leqslant a \leqslant m. \tag{14}$$

Suppose $r(P) = 0$. Let A be a monomial of P. We use equation (11), equation (12) and equation (14) to estimate

$$2m \leqslant \sum_{a=1}^{m} \deg_a(A) = 2\ell + 3t + |\alpha_1| + \cdots + |\alpha_\ell| + |\beta_1| + \cdots + |\beta_t|$$
$$= 2\ell + 2t + n \leqslant |\alpha_1| + \cdots + |\alpha_\ell| + 2t + n \leqslant 2n + t \leqslant 3n. \tag{15}$$

This shows that r is injective if $3n < 2m$ which establishes assertion (iv). □

7. Dedication

Stuart Dowker has made deep contributions to many areas of mathematics and physics and remains active presently (see, for example, [8, 9]). The second and third authors have been honored to have collaborated with Stuart on several papers (see, for example, [10, 11]) and the remaining authors have used his results extensively. We hope this paper serves as a fitting tribute to our colleague and friend and all the authors join in dedicating this paper to him.

Acknowledgments

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