

Canonical sphere bundles of the Grassmann manifold.

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Abstract

For a given Hilbert space $\mathcal{H}$, consider the space of self-adjoint projections $\mathcal{P}(\mathcal{H})$. In this paper we study the differentiable structure of a canonical sphere bundle over $\mathcal{P}(\mathcal{H})$ given by

$$\mathcal{R} = \{ (P, f) \in \mathcal{P}(\mathcal{H}) \times \mathcal{H} : Pf = f, \|f\| = 1 \}.$$

We establish the smooth action on $\mathcal{R}$ of the group of unitary operators of $\mathcal{H}$, therefore $\mathcal{R}$ is an homogeneous space. Then we study the metric structure of $\mathcal{R}$ by endowing it first with the uniform quotient metric, which is a Finsler metric, and we establish minimality results for the geodesics. These are given by certain one-parameter groups of unitary operators, pushed into $\mathcal{R}$ by the natural action of the unitary group. Then we study the restricted bundle $\mathcal{R}_2^+$ given by considering only the projections in the restricted Grassmannian, locally modelled by Hilbert-Schmidt operators. Therefore we endow $\mathcal{R}_2^+$ with a natural Riemannian metric that can be obtained by declaring that the action of the group is a Riemannian submersion. We study the Levi-Civita connection of this metric and establish a Hopf-Rinow theorem for $\mathcal{R}_2^+$, again obtaining a characterization of the geodesics as the image of certain one-parameter groups with special speeds. ¹

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators and $\mathcal{P}(\mathcal{H})$ the set of all orthogonal projections on $\mathcal{H}$. Throughout, we identify $\mathcal{P}(\mathcal{H})$ with the Grassmann manifold of $\mathcal{H}$ (i.e. the set consisting of all closed subspaces of $\mathcal{H}$). In this paper, we study the differential and metric structure of the following set

$$\mathcal{R} = \{ (P, f) \in \mathcal{P}(\mathcal{H}) \times \mathcal{H} : Pf = f, \|f\| = 1 \},$$

which is the total space of the canonical sphere bundle given by $\pi_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{P}(\mathcal{H})$, $\pi_{\mathcal{R}}(P, f) = P$. Clearly, the unitary group $\mathcal{U}(\mathcal{H})$ acts on $\mathcal{R}$:

$$U \cdot (P, f) = (UPU^*, Uf), \quad U \in \mathcal{U}(\mathcal{H}), \quad (P, f) \in \mathcal{R}.$$

As we shall see below, the connected components of $\mathcal{R}$ turn out to be a smooth homogeneous spaces of $\mathcal{U}(\mathcal{H})$. Thus, the present work can be seen as a contribution to the geometry of infinite dimensional homogeneous spaces arising in operator theory and operator algebras, which has been a subject of study in different settings, for instance Grassmann manifolds [1, 6, 9, 16, 21],

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Stiefel manifolds [4, 8] and orbits of selfadjoint operators [2, 7]. Also abstract homogeneous spaces are considered in [3, 11, 18]; for more examples and a detailed account we refer to the book [5]. On the other hand, the definition of the bundle $(\pi_{\mathcal{R}}, \mathcal{R}, \mathcal{P}(\mathcal{H}))$ is motivated from classical concepts in topology. Canonical bundles over finite dimensional Grassmann manifolds played a fundamental role in classification problems of vector bundles [19], meanwhile results on classification of sphere bundles goes back to the works [24, 25].

The contents of this paper are as follows. In Section 2 we show that the connected components of $\mathcal{R}$ coincide with orbits given by the action defined above. This is indeed a direct consequence of the following fact: given $(P_0, f_0) \in \mathcal{R}$, the map

$$\rho_{(P_0, f_0)} : \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{R}, \quad \rho_{(P_0, f_0)}(U) = (UP_0U^*, Uf_0)$$

has continuous local cross sections. In Section 3 we also use this result to prove that each orbit is a smooth homogeneous space of $\mathcal{U}(\mathcal{H})$ and a submanifold of $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$. The latter means that the quotient topology in $\mathcal{R}$ coincides with the topology inherited from $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$, and that tangent spaces, which can be described as

$$(T\mathcal{R})_{(P_0, f_0)} = \{ (XP_0 - P_0X, Xf_0) : X = -X^* \in \mathcal{B}(\mathcal{H}), \},$$

are closed and complemented in $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$. Section 4 contains the results on the metric geometry of $\mathcal{R}$. As a homogeneous space, the natural way to endow each component of $\mathcal{R}$ is to push down a metric from $\mathcal{U}(\mathcal{H})$. If we consider in $\mathcal{U}(\mathcal{H})$ the Finsler metric defined by using the operator norm $\| \cdot \|$, then the Finsler metric on $\mathcal{R}$ is given as follows: for $(XP_0 - P_0X, Xf_0) \in (T\mathcal{R})_{(P_0, f_0)}$,

$$\|(XP_0 - P_0X, Xf_0)\|_{(P_0, f_0)} := \inf \{ \|X + Y\| : Y = -Y^*, YP_0 = P_0Y, Yf_0 = 0 \}.$$

This is the quotient norm in $(T\mathcal{R})_{(P_0, f_0)} \simeq \mathcal{B}_{ah}(\mathcal{H})/\mathfrak{g}$, where $\mathcal{B}_{ah}(\mathcal{H})$ is the space of anti-Hermitian operators and $\mathfrak{g}$ is the Lie algebra of the isotropy group. Indeed, note that $Y = -Y^*$, $YP_0 = P_0Y$ and $Yf_0 = 0$ if and only if $Y \in \mathfrak{g}$. We show that the quotient norm is always attained, or in the terminology of [11], there exists a *minimal lifting* for each tangent vector. To prove this, we observe that the quotient norm can be interpreted as a two-step best approximation problem using 3×3 block operator matrices. In the first step, we use Krein's extension theorem (see [17, 23, 10]).

The Finsler metric defined in $\mathcal{R}$ allows us to measure the length of curves. If $\gamma : [0, 1] \rightarrow \mathcal{R}$ is a piecewise smooth curve, then its length is given by

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt.$$

We prove that the initial value problem has a solution: given $(P_0, f_0) \in \mathcal{R}$ and a tangent vector $(XP_0 - P_0X, Xf_0) \in (T\mathcal{R})_{(P_0, f_0)}$, there exists $X_0^* = -X_0$ such that the curve

$$\delta(t) = e^{tX_0} \cdot (P_0, f_0) = (e^{tX_0}P_0e^{-tX_0}, e^{tX_0}f_0)$$

satisfies $\delta(0) = (P_0, f_0)$, $\dot{\delta}(0) = (XP_0 - P_0X, Xf_0)$ and $\delta(t)$ has minimal length in some interval. This problem was already solved in [21] for the Grassmann manifold $\mathcal{P}(\mathcal{H})$, where the quotient Finsler metric coincides with the metric given by the usual operator norm. Remarkably, it was also solved for smooth homogeneous spaces which are the quotient of the unitary group of a von

Neumann algebra by the unitary group of a sub-algebra [11]. Although our homogeneous space does not fit in this category, our proof is adapted from that work.

Section 5 is devoted to the Riemannian geometry of the canonical sphere bundle of the restricted (or Sato) Grassmannian [1, 6, 20]. The total space of this bundle is

$$\mathcal{R}_2^+ = \{(P, f) \in \mathcal{R} : P \in \mathcal{P}_+^2(\mathcal{H})\},$$

where $\mathcal{P}_+^2(\mathcal{H})$ is the connected component containing a fixed projection P_+ in the restricted Grassmannian. The group $\mathcal{U}_2(\mathcal{H})$ of unitaries which are Hilbert-Schmidt perturbations of the identity acts transitively on $\mathcal{P}_+^2(\mathcal{H})$. We observe that $\mathcal{R}_2^+$ is also acted upon transitively by $\mathcal{U}_2(\mathcal{H})$, and has structure of smooth homogeneous space of $\mathcal{U}_2(\mathcal{H})$. It is a submanifold of $\mathcal{B}_2(\mathcal{H}) \times \mathcal{H}$, where $\mathcal{B}_2(\mathcal{H})$ denotes the Hilbert-Schmidt operators on $\mathcal{H}$. We endow $\mathcal{R}_2^+$ with two different Riemannian metrics: the ambient metric and the reductive (or quotient) metric. These metrics are shown to be equivalent and complete. The geodesics of the reductive metric are computed. Any geodesic is minimal up to a critical time value (which is characterized). The main result in this section establishes an estimate of the geodesic radius, which is at least $\pi/4$ (points of $\mathcal{R}_2^+$ at distance less than $\pi/4$ are joined by a unique minimal geodesic).

2 Transitivity and local cross sections for the action

Given $P_0 \in \mathcal{P}(\mathcal{H})$, we define

$$\mathcal{R}_{P_0} = \{(P, f) \in \mathcal{R} : P \text{ is unitary equivalent to } P_0\}.$$

It is well known that P is unitary equivalent to P_0 if and only if $\dim R(P_0) = \dim R(P)$ and $\dim N(P_0) = \dim N(P)$.

Proposition 2.1. *Let $P_0 \in \mathcal{P}(\mathcal{H})$. The action of $\mathcal{U}(\mathcal{H})$ on $\mathcal{R}_{P_0}$ is transitive. If $f_0 \in R(P_0)$, then the map*

$$\rho_{(P_0, f_0)} : \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{R}_{P_0}, \quad \rho_{(P_0, f_0)}(U) = (UP_0U^*, Uf_0)$$

has continuous local cross sections.

Proof. Let $f_0 \in R(P_0)$ with $\|f_0\| = 1$, and $(P, f) \in \mathcal{R}_{P_0}$. There exists $U \in \mathcal{U}(\mathcal{H})$ such that $UP_0U^* = P$. Then U^*f and f_0 belong to the unit sphere of $R(P_0)$. Then there exists a unitary operator V_0 in $\mathcal{U}(R(P_0))$ such that $V_0f_0 = U^*f$. V_0 can be extended to a unitary operator V in $\mathcal{H}$, defining it as the identity on $R(P_0)^\perp$. Note that V commutes with P_0 . Then $W = UV$ satisfies

$$Wf_0 = UVf_0 = UV_0f_0 = UU^*f = f \text{ and } WP_0W^* = UVP_0V^*U^* = UP_0U^* = P.$$

The same procedure provides continuous local cross sections for $\rho_{(P_0, f_0)}$. Consider the open subset of $\mathcal{P}(\mathcal{H})$:

$$\{P \in \mathcal{P}(\mathcal{H}) : \|P - P_0\| < 1\}.$$

It is well known that if $\|P - P_0\| < 1$ then there exists $\mu_{P_0}(P)$ a unitary operator which is continuous (and smooth) in the parameter P , such that

$$\mu_{P_0}(P_0) = 1 \text{ and } \mu_{P_0}(P)P_0\mu_{P_0}^*(P) = P.$$

Then one has that the following set

$$\mathcal{B}_{(P_0, f_0)} = \{(P, f) \in \mathcal{R} : \|P - P_0\| < 1 \text{ and } \|\mu_{P_0}(P)^* f - f_0\| < 2\}$$

is a open subset of $\mathcal{R}$ (and of $\mathcal{R}_{P_0}$). Moreover, for any pair of unit vectors ξ, η in an arbitrary Hilbert space $\mathcal{H}$, satisfying that $\|\xi - \eta\| < 2$, there exists $\nu_{\xi, \eta} \in \mathcal{U}(\mathcal{H})$, which is a continuous and smooth map in terms of ξ and η , such that $\nu_{\xi, \eta} \xi = \eta$. Explicit formulas for ν can be obtained in several ways. For instance, if $x, y, z \in \mathcal{H}$, and $(x \otimes y)(z) = \langle z, y \rangle x$ denotes the elementary rank one operator, we use $\nu = \nu_{\xi, \eta}$ the unitary operator given by rotation in the plane generated by ξ, η

$$\nu = 1 + (c - 1)\xi \otimes \xi + (\bar{c} - 1)\xi' \otimes \xi' + s(\xi' \otimes \xi - \xi \otimes \xi') = \begin{pmatrix} c & s & 0 \\ -s & \bar{c} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $c = \langle \eta, \xi \rangle$, $s = \sqrt{1 - |c|^2} > 0$ (assuming $\|\xi - \eta\| < 2$), and $\xi' = \frac{1}{s}(\eta - c\xi)$.

If $(P, f) \in \mathcal{B}_{(P_0, f_0)}$, then f_0 and $\mu_{P_0}(P)^* f$ are unit vectors in $R(P_0)$. Let $V_{(P_0, f_0)}(P, f)$ be the unitary operator which acts as $\nu_{f_0, \mu_{P_0}(P)^* f}$ on $R(P_0)$ and as the identity in $R(P_0)^\perp$. Then

$$\sigma_{(P_0, f_0)} : \mathcal{B}_{(P_0, f_0)} \rightarrow \mathcal{U}(\mathcal{H}), \quad \sigma_{(P_0, f_0)}(P, f) = \mu_{P_0}(P) V_{(P_0, f_0)}(P, f)$$

is a continuous local cross section for $\rho_{(P_0, f_0)}$. □

Corollary 2.2. *The connected components of $\mathcal{R}$ are $\mathcal{R}_{P_0}$, $P_0 \in \mathcal{P}(\mathcal{H})$, where two projections P_0 and P_1 define the same component if and only if they are unitarily equivalent.*

3 Regular structure

Let us prove that $\mathcal{R}$ is a C^∞ differentiable manifold, and that the map $\pi_{\mathcal{R}}$ is a C^∞ fibre bundle. In order to establish this assertion, the following lemma will be useful (see [22]).

Lemma 3.1. *Let G be a Banach-Lie group acting smoothly on a Banach space X . For a fixed $x_0 \in X$, denote by $\rho_{x_0} : G \rightarrow X$ the smooth map $\rho_{x_0}(g) = g \cdot x_0$. Suppose that*

1. ρ_{x_0} is an open mapping, regarded as a map from G onto the orbit $\{g \cdot x_0 : g \in G\}$ of x_0 (with the relative topology of X).
2. The differential $d(\rho_{x_0})_1 : (TG)_1 \rightarrow X$ splits: its nullspace and range are closed complemented subspaces.

Then the orbit $\{g \cdot x_0 : g \in G\}$ is a smooth (analytic) submanifold of X , and the map

$$\rho_{x_0} : G \rightarrow \{g \cdot x_0 : g \in G\}$$

is a smooth submersion.

Denote by $\mathcal{B}_{ah}(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}) : X^* = -X\}$, the space of anti-Hermitian operators, which is the Banach-Lie algebra of $\mathcal{U}(\mathcal{H})$, and let $\mathcal{B}_h(\mathcal{H}) = \{X \in \mathcal{B}(\mathcal{H}) : X^* = X\}$ be the space of selfadjoint (or Hermitian) operators.

Proposition 3.2. $\mathcal{R}$ is a C^∞ submanifold of $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$. For any $(P_0, f_0) \in \mathcal{R}$, the map

$$\rho_{(P_0, f_0)} : \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{R}_{P_0}$$

is a C^∞ -submersion, and in fact all manifolds and maps are real analytic.

Proof. By Proposition 2.1, $\rho = \rho_{(P_0, f_0)}$ has a local continuous cross section σ such that $\sigma((P_0, f_0)) = 1$, and $\rho \circ \sigma = id_V$ where $V \subset \mathcal{R}_{P_0}$ is an open neighbourhood of (P_0, f_0) . Let $Z \subset \mathcal{U}(\mathcal{H})$ be an open neighbourhood of 1, note that $\rho^{-1}(\rho(Z)) = ZK$ where K is the isotropy group of ρ . Therefore

$$\rho(Z) = \sigma^{-1} \rho^{-1} \rho(Z) = \sigma^{-1}(ZK)$$

is open in $\mathcal{R}_{P_0}$ since the section σ is continuous and $ZK = \cup_{k \in K} Zk$ is open in $\mathcal{U}(\mathcal{H})$. This proves that ρ is locally open around $1 \in \mathcal{U}(\mathcal{H})$; by the transitivity of the action of ρ in $\mathcal{R}_{P_0}$, it follows that $\rho : \mathcal{U}(\mathcal{H}) \rightarrow \mathcal{R}_{P_0}$ is an open mapping.

Now by elementary computations, $\delta_{(P_0, f_0)} = d(\rho_{(P_0, f_0)})_1 : \mathcal{B}_{ah}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \times \mathcal{H}$ is given by

$$\delta_{(P_0, f_0)}(X) = (XP_0 - P_0X, Xf_0).$$

The nullspace of $\delta_{(P_0, f_0)}$ consists of anti-Hermitian operators X which commute with P_0 and satisfy $Xf_0 = 0$. Since $f_0 \in R(P_0)$, this set consists of 2×2 matrices in terms of the decomposition $\mathcal{H} = R(P_0) \oplus N(P_0)$ which are of the form

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix},$$

with X_{ii} anti-Hermitian and $X_{11}f_0 = 0$. Furthermore, X_{11} can be written as a matrix in terms of the decomposition $R(P_0) = \langle f_0 \rangle \oplus \langle f_0 \rangle^\perp$,

$$X_{11} = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & x'_{11} & x'_{12} & \dots \\ 0 & x'_{21} & x'_{22} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix},$$

where X' is an anti-Hermitian operator acting in $R(P_0) \ominus \langle f_0 \rangle$. A natural supplement for the nullspace $\delta_{(P_0, f_0)}$, as matrices in the decomposition

$$\mathcal{H} = \langle f_0 \rangle \oplus (R(P_0) \ominus \langle f_0 \rangle) \oplus N(P_0)$$

is the set of matrices of the form

$$\begin{pmatrix} it & \vec{f} & Y \\ -\vec{f}^t & 0 & \\ -Y^* & 0 & \end{pmatrix}, \quad (1)$$

with $t \in \mathbb{R}$.

The range of $\delta_{(P_0, f_0)}$ is given by

$$(T\mathcal{R})_{(P_0, f_0)} = \{ (XP_0 - P_0X, Xf_0) : X = -X^* \} \subseteq \mathcal{B}_h(\mathcal{H}) \times \mathcal{H}.$$

To prove that this subspace is complemented in $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$, it is sufficient to prove that it is complemented in $\mathcal{B}_h(\mathcal{H}) \times \mathcal{H}$. This can be done using the projection $\mathcal{E} : \mathcal{B}_h(\mathcal{H}) \times \mathcal{H} \rightarrow \mathcal{B}_h(\mathcal{H}) \times \mathcal{H}$ defined by

$$\mathcal{E}((X, h)) = (P_0^\perp X P_0 + P_0 X P_0^\perp, i\text{Im} \langle h, f_0 \rangle f_0 + (P_0 - (f_0 \otimes f_0))h + P_0^\perp X f_0). \quad (2)$$

Clearly, $\mathcal{E}$ is continuous, linear, and it is not difficult to check that $\mathcal{E}^2 = \mathcal{E}$ and

$$\mathcal{E}((X P_0 - P_0 X, X f_0)) = (X P_0 - P_0 X, X f_0)$$

for any anti-Hermitian operator X . Let us prove that the range of $\mathcal{E}$ is contained in $(T\mathcal{R})_{(P_0, f_0)}$. Given $X = X^*$, $h \in \mathcal{H}$, we have to construct an operator $Y = -Y^*$ such that $\mathcal{E}((X, h)) = ((Y P_0 - P_0 Y, Y f_0))$. First we use the following fact: given vectors f, g in a Hilbert space $\mathcal{L}$, with $\|f\| = 1$ there is an operator $z \in \mathcal{B}_{ah}(\mathcal{L})$ such that $zf = g$ if and only if $\langle g, f \rangle = -\langle f, g \rangle$. In this case, the operator z can be taken as

$$z = g \otimes f + \langle f, g \rangle f \otimes f - f \otimes g.$$

Using this fact with $f = f_0$ and $g = i\text{Im} \langle h, f_0 \rangle f_0 + (P_0 - (f_0 \otimes f_0))h$, we have an operator $z \in \mathcal{B}_{ah}(R(P_0))$ satisfying $zf_0 = i\text{Im} \langle h, f_0 \rangle f_0 + (P_0 - (f_0 \otimes f_0))h$. Now suppose that the matrix of X with respect to P_0 is

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{12}^* & x_{22} \end{pmatrix},$$

then take

$$Y = \begin{pmatrix} z & -x_{12} \\ x_{12}^* & 0 \end{pmatrix}.$$

It follows that $Y P_0 - P_0 Y = P_0^\perp X P_0 + P_0 X P_0^\perp$, and

$$\begin{aligned} Y f_0 &= P_0 Y P_0 f_0 + P_0^\perp Y P_0 f_0 = z f_0 + P_0^\perp X f_0 \\ &= i\text{Im} \langle h, f_0 \rangle f_0 + (P_0 - (f_0 \otimes f_0))h + P_0^\perp X f_0. \end{aligned}$$

This proves that $\mathcal{E}$ is a projection with range $(T\mathcal{R})_{(P_0, f_0)}$. $\square$

Remark 3.3. The range of $\delta_{(P_0, f_0)}$ consists of pairs $(X P_0 - P_0 X, X f_0)$, the left hand part of this pair is a selfadjoint operator which is co-diagonal with respect to P_0 , $P_0 Z P_0 = P_0^\perp Z P_0^\perp = 0$. It is easy to see that any selfadjoint operator with this property is of the form $[X, P_0]$ for some $X^* = -X$. Indeed, the conditions are equivalent to $Z = Z P_0 + P_0 Z$ as seen from the computation

$$Z = Z P_0 + (P_0^\perp + P_0) Z P_0^\perp = Z P_0 + P_0^\perp Z P_0^\perp + P_0 Z P_0^\perp = Z P_0 + 0 + P_0 Z - P_0 Z P_0 = Z P_0 + P_0 Z.$$

Therefore if we take $X = Z P_0 - P_0 Z = [Z, P_0]$, then clearly $X^* = -X$ and $[X, P_0] = Z P_0 + P_0 Z = Z$ as claimed. Note that X is not unique, it can be modified by adding to it any skew-adjoint operator commuting with P_0 .

Proposition 3.4. *The map*

$$\pi_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{P}(\mathcal{H}), \quad \pi_{\mathcal{R}}(P, f) = P$$

is a locally trivial fibre bundle.

Proof. Fix $(P_0, f_0) \in \mathcal{R}$, and consider the open subsets

$$\mathcal{V}_{(P_0, f_0)} = \{(P, f) \in \mathcal{R} : \|P - P_0\| < 1\} \text{ and } \mathcal{V}_{P_0} = \{P \in \mathcal{P}(\mathcal{H}) : \|P - P_0\| < 1\}$$

and the map

$$\mathcal{V}_{(P_0, f_0)} \rightarrow \mathcal{V}_{P_0} \times \mathbb{S}(R(P_0)), (P, f) \mapsto (P, \mu_{P_0}^*(P)f),$$

where μ_{P_0} is the $\mathcal{U}(\mathcal{H})$ -valued map defined in the proof of Proposition 2.1. Clearly $\mu_{P_0}^*(P)f \in R(P_0)$, and therefore this map is well defined, with inverse $(P, h) \mapsto (P, \mu_{P_0}(P)h)$, which is a local trivialization of the map $\pi_{\mathcal{R}}$. $\square$

4 The quotient metric in $\mathcal{R}$

In [11] Durán, Mata-Lorenzo and Recht presented a program to study homogeneous spaces which are quotients of the unitary group of a C^* -algebra by the unitary group of a sub- C^* -algebra. This program does not apply exactly to the homogeneous space $\mathcal{R}$, since the isotropy subalgebra of $\mathcal{U}(\mathcal{H})$ is not the unitary group of a C^* -algebra. Nevertheless, the main ideas of their approach apply in our context with minor modifications.

Durán, Mata-Lorenzo and Recht start with a natural idea: since $\mathcal{R}$ is a quotient, the tangent space $(T\mathcal{R})_{(P_0, f_0)}$ identifies with the quotient

$$(T\mathcal{R})_{(P_0, f_0)} \simeq \mathcal{B}_{ah}(\mathcal{H}) / \ker \delta_{(P_0, f_0)} = \mathcal{B}_{ah}(\mathcal{H}) / \{X \in \mathcal{B}_{ah}(\mathcal{H}) : [X, P_0] = 0, Xf_0 = 0\}.$$

This is a quotient of Banach spaces, so one endows it with the quotient norm: if $V \in (T\mathcal{R})_{(P_0, f_0)}$

$$|V|_{(P_0, f_0)} = \inf\{\|X\| : X \in \mathcal{B}_{ah}(\mathcal{H}), \delta_{(P_0, f_0)}(X) = V\}. \quad (3)$$

The elements X in $\mathcal{B}_{ah}(\mathcal{H})$ such that $\delta_{(P_0, f_0)}(X) = V$, will be called *liftings* of $V \in (T\mathcal{R})_{(P_0, f_0)}$. Let us define minimal liftings:

Definition 4.1. *Let $V \in (T\mathcal{R})_{(P_0, f_0)}$. A lifting X_0 of V is called a minimal lifting if it achieves the quotient norm, i.e.*

$$\|X_0\| = \inf\{\|X\| : X \text{ is a lifting of } V\} = |V|_{(P_0, f_0)}.$$

It can be proved that minimal liftings exist for any tangent vector. We shall postpone the proof of this fact, before let us further characterize minimal liftings.

Given a vector $V \in (T\mathcal{R})_{(P_0, f_0)}$, by the form of the kernel of $\delta_{(P_0, f_0)}$ described in Section 3, it is clear that different liftings of V have the following common form:

$$X_* = \begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{x}^t & * & \\ -A^* & & ** \end{pmatrix}, \quad (4)$$

where $x_0 \in \mathbb{R}$, and $*$ and $**$ are to be filled with anti-Hermitian operators in $R(P_0)\ominus < f_0 >$ and $N(P_0)$, respectively. Therefore the problem of finding a minimal lifting consists in filling the spots $*$ and $**$ in order that the completed matrix has the least possible norm. This problem has certain resemblance to M.G. Krein's extension problem (see [17], [23, p. 336] or [10]). In fact, as it will be shown next, it can be tackled using this method. In our context, Krein's extension

problem consists in completing the $*$ spot in the matrix operator (in terms of the decomposition $\mathcal{H} = R(P_0) \oplus N(P_0)$)

$$\begin{pmatrix} B & A \\ -A^* & * \end{pmatrix}$$

in order that the matrix has the least possible norm. We shall divide our quest into two steps.

1. First fill the space $*$ in the first row of X_* :

$$\begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{\bar{x}}^t & * & \end{pmatrix} \quad (5)$$

with $* = Y_0$, in order that the row has the least possible norm.

2. Next, fill the $**$ spot in the matrix

$$\begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{\bar{x}}^t & Y_0 & \\ -A^* & & ** \end{pmatrix} \quad (6)$$

Proposition 4.2. *Suppose that Y_0 and Z_0 are operators acting respectively in $R(P_0)\ominus < f_0 >$ and $N(P_0)$, where Y_0 optimizes the norm of the row (5), and Z_0 optimizes the norm of the matrix (6). Then*

$$X_0 = \begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{\bar{x}}^t & Y_0 & \\ -A^* & & Z_0 \end{pmatrix} \quad (7)$$

is a minimal lifting. Any minimal lifting can be obtained in this fashion.

Proof. By hypothesis, for any anti-Hermitian operator Y in $R(P_0)\ominus < f_0 >$, one has that

$$\left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{\bar{x}}^t & Y_0 & \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{\bar{x}}^t & Y & \end{pmatrix} \right\|.$$

Since Z_0 is a solution of Krein's extension problem, for our optimal choice of Z_0 we have

$$\left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{\bar{x}}^t & Y_0 & \\ -A^* & & Z_0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{\bar{x}}^t & Y_0 & \end{pmatrix} \right\|.$$

On the other hand, for any block matrix operator, the norm of the first row is less or equal than the norm of the full matrix:

$$\left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{\bar{x}}^t & Y & \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{\bar{x}}^t & Y & \\ -A^* & & Z \end{pmatrix} \right\|,$$

for any anti-Hermitian operator Z in $N(P_0)$, which proves our first assertion.

Let X_1 be a minimal lifting,

$$\begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{\bar{x}}^t & Y_1 & \\ -A^* & & Z_1 \end{pmatrix}.$$

Then, if Y_0 and Z_0 are obtained as above

$$\left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{x}^t & Y_1 & \\ & -A^* & Z_1 \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{x}^t & Y_1 & \\ & -A^* & Z_1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{x}^t & Y_0 & \\ & -A^* & Z_0 \end{pmatrix} \right\|.$$

This last norm equals

$$\left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{x}^t & Y_0 & \end{pmatrix} \right\|.$$

From these inequalities it follows that also Y_1 minimizes the norm of the first row, and therefore also Z_1 is a solution of Krein's extension problem for the matrix

$$\begin{pmatrix} ix_0 & \vec{x} & A \\ -\vec{x}^t & Y_1 & \\ -A^* & & ** \end{pmatrix}.$$

□

Therefore to obtain minimal liftings, it suffices to solve the first row problem (5). Let us show that solutions to this problem always exist.

Proposition 4.3. *There exists an anti-Hermitian operator Y_0 in $R(P_0)\ominus < f_0 >$ such that*

$$\left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{x}^t & Y_0 & \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{x}^t & Y & \end{pmatrix} \right\|$$

for any other anti-Hermitian operator Y in $R(P_0)\ominus < f_0 >$.

Proof. Denote by ι the infimum of all possible completions of the first row matrix (5). Let Y_n be a minimizing sequence of anti-Hermitian operators in $R(P_0)\ominus < f_0 >$:

$$\left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{x}^t & Y_n & \end{pmatrix} \right\| \rightarrow \iota.$$

Note that

$$\|Y_n\| \leq \left\| \begin{pmatrix} ix_0 & \vec{x} & A \\ \vec{x}^t & Y_n & \end{pmatrix} \right\|.$$

Therefore the sequence Y_n is norm bounded, and therefore it has a weak operator convergent subsequence Y_{n_k} , $< Y_{n_k}\xi, \eta > \rightarrow < Y_0\xi, \eta >$. Clearly Y_0 is anti-Hermitian. We claim that Y_0 is a solution to the problem. Indeed, for any $\epsilon > 0$, there exists a unit vector ξ_ϵ such that $|< Y_0\xi_\epsilon, \xi_\epsilon >| \geq \|Y_0\| - \epsilon$. Then

$$\|Y_{n_k}\| \geq |< Y_{n_k}\xi_\epsilon, \xi_\epsilon >| \rightarrow |< Y_0\xi_\epsilon, \xi_\epsilon >| \geq \|Y_0\| - \epsilon.$$

Then $\iota \geq \|Y_0\| - \epsilon$ for any ϵ , and thus $\iota = \|Y_0\|$. □

The following result is again adapted from [11], with minor modifications which allow to treat the case when the isotropy group of the unitary action is an arbitrary Banach-Lie group (and not the unitary group of a sub-C*-algebra).

Proposition 4.4. *Let X_0 be a lifting of $V \in (T\mathcal{R})_{(P_0, f_0)}$. Then X_0 is a minimal lifting if and only if there exists a representation π of $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{L}$, and a unit vector $\xi \in \mathcal{L}$ such that*

1. $\pi(X_0^2)\xi = -\|X_0\|^2\xi$
2. $\operatorname{Re} \langle \pi(X_0)\xi, \pi(B)\xi \rangle = 0$, for all $B \in \ker \delta_{(P_0, f_0)}$.

Proof. Suppose first that conditions 1. and 2. hold. Then for any $B \in \ker \delta_{(P_0, f_0)}$,

$$\|X_0 + B\|^2 \geq \|\pi(X_0 + B)\xi\|^2 = \|\pi(X_0)\xi\|^2 + \|\pi(B)\xi\|^2 \geq \|\pi(X_0)\xi\|^2 = -\langle \pi(X_0^2)\xi, \xi \rangle = \|X_0\|^2.$$

Conversely, suppose that X_0 is a minimal lifting. Let $\mathcal{S}$ be the real subspace of $\mathcal{B}_h(\mathcal{H})$ spanned by $X_0^2 + \|X_0\|^2 1$ and the operators of the form $BX_0 + X_0B$, for $B \in \ker \delta_{(P_0, f_0)}$ (note that since B and X_0 are anti-Hermitian, these operators are indeed selfadjoint). We claim that the minimality condition of X_0 , implies that

$$\mathcal{S} \cap Gl(\mathcal{H})^+ = \emptyset.$$

Suppose that this intersection is non empty, i.e. there exists $s \in \mathbb{R}$ and $b \in \ker \delta_{(P_0, f_0)}$ such that

$$s(X_0^2 + \|X_0\|^2 1) + BX_0 + X_0B \geq r1$$

for some $r > 0$. We may suppose $s > 0$, otherwise add $(1 - s)(X_0^2 + \|X_0\|^2 1)$ to the above inequality. Dividing by s we get

$$X_0^2 + \|X_0\|^2 1 + BX_0 + X_0B \geq r1 \tag{8}$$

for some (other) $B \in \ker \delta_{(P_0, f_0)}$ and $r > 0$. In particular this implies that

$$X_0^2 + BX_0 + X_0B \geq (r - \|X_0\|^2)1,$$

and therefore the spectrum of $X_0^2 + BX_0 + X_0B$ satisfies

$$\sigma(X_0^2 + BX_0 + X_0B) \subset (-\|X_0\|^2, +\infty).$$

Also, for $n \geq 1$,

$$n(X_0^2 + \|X_0\|^2 1) + BX_0 + X_0B \geq X_0^2 + \|X_0\|^2 1 + BX_0 + X_0B \geq r1$$

and thus

$$X_0^2 + \|X_0\|^2 1 + \frac{B}{n}X_0 + X_0\frac{B}{n} \geq \frac{r}{n}1.$$

Since $X_0^2 + \frac{B}{n}X_0 + X_0\frac{B}{n} \rightarrow X_0^2$, by the semi-continuity of the spectrum, given the open neighbourhood $(-\infty, \|X_0\|^2)$ of X_0^2 (recall that $X_0 \leq 0$), there exists n_0 such that for $n \geq n_0$

$$\sigma\left(X_0^2 + \frac{B}{n}X_0 + X_0\frac{B}{n}\right) \subset (-\infty, \|X_0\|^2).$$

In particular, there exists $B \in \ker \delta_{(P_0, f_0)}$ of arbitrary small norm such that

$$\sigma(X_0^2 + BX_0 + X_0B) \subset (-\|X_0\|^2, \|X_0\|^2),$$

i.e. $\|X_0^2 + BX_0 + X_0B\| < \|X_0\|^2$. We claim that this inequality contradicts the minimality of X_0 . Indeed, this follows from the following result:

Lemma 4.5. *If $\|X_0 + B\| \geq \|X_0\|$ for all $B \in \ker \delta_{(B_0, f_0)}$, then $\|X_0^2 + BX_0 + X_0B\| \geq \|X_0\|^2$ for all $B \in \ker \delta_{(B_0, f_0)}$.*

Proof. Consider for $t \in (0, 1)$ the function

$$g(t) = -X_0^2 + \frac{1}{t}\{(X_0 + tB)^*(X_0 + tB) + X_0^2\}.$$

We claim that $\|g(t)\| \geq \|X_0\|^2$. Otherwise, the convex combination

$$tg(t) + (1-t)(-X_0^2)$$

would have norm strictly less than $\|X_0\|^2$. Note that this convex combination equals $(X_0 + tB)^*(X_0 + tB)$. Thus we would have $\|X_0\|^2 > \|(X_0 + tB)^*(X_0 + tB)\| = \|X_0 + tB\|^2$, which contradicts the minimality of X_0 . Then

$$\| -X_0^2 - X_0B - BX_0 - tB \| \geq \|X_0\|^2$$

for all $t \in (0, 1)$. If $t \rightarrow 0$ this yields

$$\|X_0^2 + X_0B + BX_0\| \geq \|X_0\|^2,$$

which proves the lemma. $\square$

Our claim follows: $\mathcal{S} \cap GL(\mathcal{H})^+ = \emptyset$. $Gl(\mathcal{H})^+$ is a convex open subset of $\mathcal{B}_h(\mathcal{H})$, $\mathcal{S}$ is a subspace of $\mathcal{B}_h(\mathcal{H})$. Therefore, by the Hahn-Banach theorem there exists a (real) linear functional φ_0 in $\mathcal{B}_h(\mathcal{H})$ such that

$$\varphi_0(\mathcal{S}) = 0 \text{ and } \varphi_0(Gl(\mathcal{H})^+) > 0.$$

This functional extends to a complex linear functional φ defined in $\mathcal{B}(\mathcal{H})$. The fact that $\varphi(Gl^+(\mathcal{H})) > 0$ implies that φ is positive and bounded. We may normalize it so that it has norm 1. Consider π the Gelfand-Naimark-Segal representation induced by φ , with cyclic vector ξ . Let us prove that π and ξ satisfy the conditions in the statement of the proposition. Since φ is zero at the generators of $\mathcal{S}$,

$$\varphi(X_0^2 + \|X_0\|^2) = 0, \text{ i.e. } \varphi(X_0)^2 = -\|X_0\|^2$$

and

$$\varphi(X_0B + BX_0) = 0.$$

The first condition means that $\langle \pi(X_0^2)\xi, \xi \rangle = -\|X_0\|^2$. Thus one has equality in the Cauchy-Schwarz inequality

$$\|X_0\|^2 = \langle \pi(X_0^2)\xi, \xi \rangle = |\langle \pi(X_0^2)\xi, \xi \rangle| \leq \|\pi(X_0^2)\| \|\xi\|^2 \leq \|X_0\|^2.$$

Then $\pi(X_0^2)\xi$ is a multiple of ξ , and thus $\pi(X_0^2)\xi = -\|X_0\|\xi$. The second condition

$$0 = \varphi(X_0B + BX_0) = \varphi(X_0B) + \varphi((X_0B)^*) = \varphi(X_0B) + \overline{\varphi(X_0B)},$$

i.e.

$$0 = \operatorname{Re} \varphi(X_0B) = \operatorname{Re} \langle \pi(X_0B)\xi, \xi \rangle = -\operatorname{Re} \langle \pi(B)\xi, \pi(X_0)\xi \rangle.$$

Considering $iB \in \ker \delta_{(P_0, f_0)}$, one also gets that $\operatorname{Im} \langle \pi(B)\xi, \pi(X_0)\xi \rangle = 0$, which finishes the proof. $\square$

Theorem 4.6. *Let $V \in (T\mathcal{R})_{(P_0, f_0)}$ with $|V|_{(P_0, f_0)} = 1$. Let X_0 be a minimal lifting of V . Then the curve*

$$\delta(t) = e^{tX_0} \cdot (P_0, f_0) = (e^{tX_0} P_0 e^{-tX_0}, e^{tX_0} f_0)$$

has minimal length along its path for $|t| \leq \pi/2$.

Proof. Following Durán, Mata Lorenzo, and Recht [11], we shall construct a smooth map from $\mathcal{R}$, in fact, from the connected component of (P_0, f_0) in $\mathcal{R}$ to the unit sphere of an appropriate Hilbert space. Let π_0 and ξ_0 be the cyclic representation and the cyclic vector in the Hilbert space $\mathcal{H}_0$ obtained by means of Proposition 4.4. Denote by $\bar{\delta}_0$ the natural extension of $\delta_{(P_0, f_0)}$, which is defined in $\mathcal{B}_{ah}(\mathcal{H})$, to the whole $\mathcal{B}(\mathcal{H})$, namely: $\bar{\delta}_0(A) = (AP_0 - P_0A, Af_0)$. Consider the closed subspace

$$\mathcal{S}_0 = \overline{\{\pi_0(A)\xi_0 : A \in \ker \bar{\delta}_0\}} \subset \mathcal{H}_0.$$

and the symmetry ρ_0 acting in $\mathcal{H}_0$, which equals the identity in $\mathcal{S}_0$ and minus the identity in $\mathcal{S}_0^\perp$. Let W be in the isotropy group $\mathcal{I}_{(P_0, f_0)}$. Then $\pi_0(W)$ commutes with ρ_0 , indeed, it leaves $\mathcal{S}_0$ invariant

$$\pi_0(W)\pi_0(A)\xi_0 = \pi_0(WA)\xi_0 \in \mathcal{S}_0,$$

because $WA \in \ker \bar{\delta}_0$. Consider the map, defined in the connected component $\mathcal{R}_0$ of (P_0, f_0) in $\mathcal{R}$, to the set of symmetries of $\mathcal{H}_0$ (denoted appropriately in [11] the Grassmann manifold $Gr(\mathcal{H}_0)$ of $\mathcal{H}_0$),

$$F_0 : \mathcal{R}_0 \rightarrow Gr(\mathcal{H}_0), \quad F_0(U \cdot (P_0, f_0)) = \pi_0(U)\rho_0\pi_0(U)^*.$$

F_0 is well defined: if $U_1 \cdot (P_0, f_0) = U_2 \cdot (P_0, f_0)$, then $U_2^*U_1 \in \mathcal{I}_{(P_0, f_0)}$. Then

$$\pi_0(U_2^*U_1)\rho_0 = \rho_0\pi_0(U_2^*U_1), \quad \text{i.e. } \pi_0(U_1)\rho_0\pi_0(U_1)^* = \pi_0(U_2)\rho_0\pi_0(U_2)^*.$$

Also F_0 is smooth: because of the manifold structure of $\mathcal{R}_0$ is final with respect to the smooth submersion $\rho = \rho_{(P_0, f_0)}$ (Proposition 3.2) it suffices to check that

$$(F \circ \rho)(U) = \pi_0(U)\rho_0\pi_0(U)^*$$

is smooth. This is apparent since π_0 is bounded and linear, the product and the adjoint are smooth in $\mathcal{B}(\mathcal{H}_0)$, and $Gr(\mathcal{H}_0)$ is an embedded submanifold of $\mathcal{B}(\mathcal{H}_0)$.

There is a natural map from $Gr(\mathcal{H}_0)$ to the unit sphere $\mathbb{S}(\mathcal{H}_0)$, by means of ξ_0 ,

$$\Pi_0 : Gr(\mathcal{H}_0) \rightarrow \mathbb{S}(\mathcal{H}_0), \quad \Pi_0(\rho) = \rho\xi_0.$$

The proof of the theorem proceeds with the following syllogism:

1. The maps $\frac{1}{2}F_0$ and Π_0 are length reducing.
2. The composition $\Pi_0 F_0$ exactly doubles the length of δ (i.e. length of $\Pi_0 F_0 \delta$ is 2 times the length of δ), and $\Pi_0 F_0 \delta$ is a minimal geodesic of $\mathbb{S}(\mathcal{H}_0)$.
3. Therefore, δ has minimal length along its path, provided that $|t| \leq \pi/2$.

Let us check these steps.

Step 1: Since Π_0 is the restriction of a linear map, if $\rho(t)$ is a curve in $Gr(\mathcal{H}_0)$ with $\rho(0) = \rho$ and $\dot{\rho}(0) = X$

$$d(\Pi_0)_\rho(X) = \left. \frac{d}{dt} \right|_{t=0} \Pi_0(\rho(t)) = X\xi_0,$$

and thus $\|d(\Pi_0)_\rho(X)\| = \|X\xi_0\| \leq \|X\|$. On the other hand, let $\gamma(t) = U(t) \cdot (P_0, f_0)$ be a smooth curve in $\mathcal{R}$, with $\gamma(0) = (P, f) = U \cdot (P_0, f_0)$, $\dot{\gamma}(0) = (X, v) = V$. Denote $u(t) = \pi_0(U(t))$ and $u = \pi_0(U)$. Then

$$\left. \frac{d}{dt} \right|_{t=0} F_0(\gamma(t)) = \left. \frac{d}{dt} \right|_{t=0} u(t)\rho_0 u^*(t) = \dot{u}(0)\rho_0 u^* + u\rho_0 \dot{u}^*(0) = u\{u^* \dot{u}(0)\rho_0 + \rho_0 \dot{u}^*(0)u\}u^*. \quad (9)$$

Since $0 = \dot{u}^* u = \dot{u}^* u + u^* \dot{u}$, and thus $\dot{u}^* u = -u^* \dot{u}$, (9) equals

$$u\{u^* \dot{u}\rho_0 - \rho_0 u^* \dot{u}\}u^*,$$

whose norm equals

$$\|u\{u^* \dot{u}\rho_0 - \rho_0 u^* \dot{u}\}u^*\| = \|u^* \dot{u}\rho_0 - \rho_0 u^* \dot{u}\|.$$

Since $\rho_0 = 2P_{\mathcal{S}_0} - 1$,

$$u^* \dot{u}\rho_0 - \rho_0 u^* \dot{u} = [u^* \dot{u}, \rho_0] = 2[u^* \dot{u}, P_{\mathcal{S}_0}],$$

and therefore this commutant is a co-diagonal matrix with respect to the decomposition $\mathcal{H}_0 = \mathcal{S}_0 \oplus \mathcal{S}_0^\perp$. Then its norm equals the norm of the first column $u^* \dot{u}P_{\mathcal{S}_0}$. Thus,

$$\|d(F_0)_{(P,f)}(V)\| = 2\|[u^* \dot{u}, P_{\mathcal{S}_0}]\| = 2\|u^* \dot{u}P_{\mathcal{S}_0}\| \leq 2\|u^* \dot{u}(0)\| = 2\|\pi_0(U^* \dot{U}(0))\| \leq 2\|\dot{U}(0)\|.$$

On the other hand, $(UP_0U^*, Uf_0) = \gamma(0) = (P, f)$ and

$$V = \dot{\gamma}(0) = (\dot{U}(0)P_0U^* + UP_0\dot{U}^*(0), \dot{U}(0)f_0) = (\dot{U}(0)U^*P + PU\dot{U}^*(0), \dot{U}(0)U^*f).$$

Again using that $U\dot{U}^*(0) = -\dot{U}(0)U^*$, one has that

$$V = (\dot{U}(0)U^*P - P\dot{U}(0)U^*, \dot{U}(0)U^*f) = \delta_{(P,f)}(\dot{U}(0)U^*),$$

i.e. $\dot{U}(0)U^* \in \mathcal{B}_{ah}(\mathcal{H})$ is a lifting for $V \in (T\mathcal{R})_{(P,f)}$.

We claim that any lifting Z of V is obtained in this fashion. Indeed, if $\delta_{(P,f)}(Z) = (ZP - PZ, Zf) = V$, consider $U(t) = e^{tZ}$ and $\gamma(t) = U(t) \cdot (P, f) \in \mathcal{R}$. Clearly $U(0) \cdot (P, f) = (P, f)$ and $\dot{U}(0)U^* = Z$.

It follows from this claim and inequality (9) that

$$\|d(F_0)_{(P,f)}(V)\| \leq 2\|Z\|$$

for any lifting Z of V . Then

$$\|d(F_0)_{(P,f)}(V)\| \leq 2|V|_{(P,f)}.$$

Step 2: Clearly, we have $\|d(\Pi_0 F_0)_{\Pi_0 F_0(P,f)}(X)\| \leq 2|V|_{(P,f)}$. Let us compute the image of $\delta(t) = e^{tX_0} \cdot (P_0, f_0)$ under $\Pi_0 F_0$:

$$\Pi_0 F_0(\delta(t)) = \pi_0(e^{tX_0})\rho_0\pi_0(e^{-tX_0})\xi_0 = e^{t\pi_0(X_0)}\rho_0 e^{-t\pi_0(X_0)}\xi_0.$$

Note that since $\pi_0(X_0^2)\xi_0 = -\|X_0\|^2\xi_0$, we have

$$\pi_0(X_0)^{2n}\xi_0 = \|X_0\|^{2n}\xi_0 \text{ and } \pi_0(X_0)^{2n+1}\xi_0 = (-1)^n\|X_0\|^{2n}\pi_0(X_0)\xi_0$$

for $n \geq 0$. Then $e^{-t\pi_0(X_0)}\xi_0 = \cos(t\|X_0\|)\xi_0 - \frac{1}{\|X_0\|} \sin(t\|X_0\|)\pi_0(X_0)\xi_0$. Note that since $\langle \pi(X_0)\xi_0, \pi_0(A)\xi_0 \rangle = 0$ for any $A \in \ker \bar{\delta}_0$, one has that $\pi_0(X_0)\xi_0 \in \mathcal{S}_0^\perp$ (whereas $\xi_0 \in \mathcal{S}_0$). Then $\rho_0\xi_0 = \xi_0$ and $\rho_0\pi_0(X_0)\xi_0 = -\pi_0(X_0)\xi_0$. Thus

$$\rho_0 e^{-t\pi_0(X_0)}\xi_0 = \cos(t\|X_0\|)\xi_0 + \frac{1}{\|X_0\|} \sin(t\|X_0\|)\pi_0(X_0)\xi_0 = e^{t\pi_0(X_0)}\xi_0,$$

and

$$\Pi_0 F_0 \delta(t) = e^{t\pi_0(X_0)} \rho_0 e^{-t\pi_0(X_0)} \xi_0 = e^{2t\pi_0(X_0)} \xi_0 = \cos(2t\|X_0\|)\xi_0 + \frac{1}{\|X_0\|} \sin(2t\|X_0\|)\pi_0(X_0)\xi_0,$$

which is a minimal geodesic of $\mathbb{S}(\mathcal{H}_0)$, up to $|2t| \leq \pi$, i.e. $|t| \leq \pi/2$. Finally, the length of $\Pi_0 F_0 \delta$ is $2\|\pi_0(X_0)\xi_0\| = 2\|X_0\|$, by the properties of π_0 and ξ_0 established in the previous lemma. The proof finishes comparing the lengths of the images in the unit sphere (using the mapping $\Pi_0 F_0$) of δ on any subinterval of $[-\pi/2, \pi/2]$ and any other curve in $\mathcal{R}$ joining the same endpoints. $\square$

Remark 4.7. Let us examine particular directions in $\mathcal{R}$, and compute minimal liftings (and thus minimal geodesics).

1. Let $V = (0, g) \in (T\mathcal{R})_{(P_0, f_0)}$. Then $P_0 g = g$, and $\langle f_0, g \rangle \in i\mathbb{R}$. Also note that

$$X_g = g \otimes f_0 - f_0 \otimes g + \langle f_0, g \rangle f_0 \otimes f_0$$

satisfies $X_g^* = -X_g$ and $X_g f_0 = g$. Since $g \in R(P_0)$, X_g commutes with P_0 :

$$X_g P_0 = g \otimes P_0 f_0 - f_0 \otimes P_0 g + \langle f_0, g \rangle f_0 \otimes P_0 f_0 = X_g = P_0 X_g.$$

Thus, X_g is a lifting of V , i.e. $\delta_{(P_0, f_0)}(X_g) = ([X_g, P_0], X_g f_0) = (0, g) = V$. To find minimal liftings, use the vector f_0 as the first vector of an orthonormal basis $\{f_i\}_{i=0}^\alpha$ of $R(P_0)$, where $\alpha = \dim R(P_0) \in [1, \infty]$. Note that if $e, e' \in \mathcal{H}$ are orthogonal to f_0 , then $\langle X_g e, e' \rangle = 0$. Thus, if $g = \sum_{i=0}^\alpha g_i f_i$, then the matrix of X_g in the decomposition $\mathcal{H} = (\langle f_0 \rangle \oplus (R(P_0) \ominus \langle f_0 \rangle)) \oplus N(P_0)$ is

$$X_g = \left(\begin{array}{cccc|c} ig_0 & -\bar{g}_1 & -\bar{g}_2 & \dots & 0 \\ g_1 & 0 & 0 & \dots & \\ g_2 & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \ddots & \\ \hline & & 0 & & 0 \end{array} \right)$$

with $g_0 \in \mathbb{R}$. Since the 1, 2, 2, 1 and 2, 2 block entries are zero, the second step stated in (6) has the trivial solution, so it remains to solve the best approximation problem in (5). According to [10, Corol. 1.3] all the solutions of this problem can be parametrized as

$$Y_Z = ig_0 T + \|g\| (1 - T)^{1/2} Z (I - T)^{1/2},$$

where Z is any anti-Hermitian operator and

$$T = (\|g\|^2 - |g_0|^2)^{-1} \begin{pmatrix} g_1 \\ g_2 \\ \vdots \end{pmatrix} \begin{pmatrix} \bar{g}_1 & \bar{g}_2 & \dots \end{pmatrix}.$$

We thus get that

$$X_Z = \left(\begin{array}{cccc|c} ig_0 & -\bar{g}_1 & -\bar{g}_2 & \dots & 0 \\ g_1 & & & & \\ g_2 & & Y_Z & & \\ \vdots & & & & \\ \hline & & 0 & & 0 \end{array} \right)$$

is a minimal lifting of V . Next note that the minimal curve $\delta_Z(t) = e^{tX_Z} f_0$ remains inside the unit sphere of $R(P_0)$:

$$\delta_Z(t) = (e^{tX_Z} P_0 e^{-tX_Z}, e^{tX_Z} f_0) = (P_0, e^{tX_Z} f_0).$$

If $Z = 0$, then the minimal lifting X_0 may be expressed as

$$X_0 = \gamma_0 \otimes f_0 - f_0 \otimes \gamma_0 + ig_0(f_0 \otimes f_0 + e \otimes e),$$

where $e = \gamma_0 / \|\gamma_0\|$. Since $X_1 = \gamma_0 \otimes f_0 - f_0 \otimes \gamma_0$ and $X_2 = ig_0(f_0 \otimes f_0 + e \otimes e)$ commute, a straightforward computation shows that

$$\begin{aligned} \delta_0(t) &= (P_0, e^{tX_0} f_0) = (P_0, e^{tX_1} e^{tX_2} f_0) \\ &= (P_0, e^{itg_0} e^{tX_1} f_0) = (P_0, e^{itg_0} (\cos(t\|\gamma_0\|)f_0 + \sin(t\|\gamma_0\|)e)). \end{aligned}$$

Clearly, this curve is a geodesic in the unit sphere if and only if $g_0 = 0$. Therefore our metric, given by minimal liftings, does not induce the usual metric when restricted to the sphere.

2. Let $V = (Z, 0) \in (T\mathcal{R})_{(P_0, f_0)}$. Then $Zf_0 = f_0$, with $Z^* = Z$ P_0 -codiagonal. Consider $X_Z = ZP_0 - P_0Z$. Then clearly $X_Z^* = -X_Z$ and $X_Z f_0 = ZP_0 f_0 - P_0 Z f_0 = f_0 - f_0 = 0$. Moreover,

$$X_Z P_0 - P_0 X_Z = (ZP_0 - P_0 Z)P_0 - P_0(ZP_0 - P_0 Z) = ZP_0 - 2P_0 Z P_0 + P_0 Z = Z,$$

because Z is P_0 -codiagonal. Thus $\delta_{(P_0, f_0)}(X_Z) = (Z, 0) = V$. Also it is clear that X_Z is a minimal lifting: it is codiagonal. Its norm coincides with the norm of Z . Finally, since $X_Z f_0 = 0$, the geodesic $\delta(t) = e^{tX_Z} f_0$ is

$$\delta(t) = (e^{tX_Z} P_0 e^{-tX_Z}, f_0),$$

where the first coordinate of the pair is a minimal geodesic of $\mathcal{P}(\mathcal{H})$ (alas a special one, which leaves the unit vector $f_0 \in R(P_0)$ fixed).

5 The restricted sphere bundle

In this section we shall consider the restricted version of the canonical bundle by using the restricted Grassmann manifold induced by a (fixed) decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Denote by P_+ the orthogonal projection onto $\mathcal{H}_+$, and by P_- the projection onto $\mathcal{H}_-$. The restricted Grassmann manifold is the space

$$\mathcal{P}_2^+(\mathcal{H}) = \{ P \in \mathcal{P}(\mathcal{H}) : P - P_+ \in \mathcal{B}_2(\mathcal{H}), j(P_+, P) = 0 \},$$

where $B_2(\mathcal{H})$ denotes the ideal of Hilbert-Schmidt operators and $j(P_+, P)$ is the index of the pair of projections (P_+, P) defined by $j(P_+, P) = \text{index}(PP_+ : R(P_+) \rightarrow R(P))$. It should be said that this space $\mathcal{P}_2^+(\mathcal{H})$ is in fact the *zero index* component of the restricted Grassmannian, or the component containing P_+ . In [1] the geodesic structure of $\mathcal{P}_2^+(\mathcal{H})$ was studied.

$\mathcal{P}_2^+(\mathcal{H})$ is a Hilbert-Riemann manifold, its tangent spaces are naturally embedded in $\mathcal{B}_2(\mathcal{H})_h$, the (real) Hilbert space of Hermitian Hilbert-Schmidt operators, endowed with the usual trace. The group $\mathcal{U}_2(\mathcal{H}) = \{U \in \mathcal{U}(\mathcal{H}) : U - 1 \in \mathcal{B}_2(\mathcal{H})\}$ acts transitively on $\mathcal{P}_2^+(\mathcal{H})$. The restricted version $\mathcal{R}_+^2$ of the space $\mathcal{R}$ is the set

$$\mathcal{R}_+^2 = \{ (P, f) \in \mathcal{R} : P \in \mathcal{P}_+^2(\mathcal{H}) \}.$$

With the same proof as in Proposition 3.2, restricting the action and the local sections, we obtain that $\mathcal{R}_+^2$ is an embeded, smooth (real analytic) submanifold of the Hilbert space $\mathcal{B}_2(\mathcal{H})_h \times \mathcal{H}$, and the action is transitive.

Thus $(T\mathcal{R}_+^2)_{(P_0, f_0)}$ has a natural inner product and Riemannian metric,

$$\langle (X, f), (Y, g) \rangle = \text{Tr}(XY) + \text{Re} \langle f, g \rangle, \quad (10)$$

that we shall call the *ambient* metric.

It is apparent that the tangent space $(T\mathcal{R}_+^2)_{(P_0, f_0)}$ at a given (P_0, f_0) is

$$\begin{aligned} (T\mathcal{R}_+^2)_{(P_0, f_0)} &= \{ (ZP_0 - P_0Z, Zf_0) : Z \in \mathcal{B}_2(\mathcal{H}), Z^* = -Z \} \\ &= \{ (X, f) \in \mathcal{B}_2(\mathcal{H})_h \times \mathcal{H} : X \text{ is } P_0 - \text{codiagonal and } \text{Re} \langle f, f_0 \rangle = 0 \} \\ &\subset \mathcal{B}_2(\mathcal{H})_h \times \mathcal{H}. \end{aligned}$$

It will be useful to compute explicitly the orthogonal projection

$$\Pi = \Pi_{(P_0, f_0)} : \mathcal{B}_2(\mathcal{H})_h \times \mathcal{H} \rightarrow (T\mathcal{R}_+^2)_{(P_0, f_0)} \subset \mathcal{B}_2(\mathcal{H})_h \times \mathcal{H}.$$

In the proof of Propostion 3.2, we obtained the formula (2) of the projection $\mathcal{E}$, defined in $\mathcal{B}(\mathcal{H}) \times \mathcal{H}$ onto the tangent space of the whole space $\mathcal{R}$ at (P_0, f_0) . Namely

$$\mathcal{E}(X, f) = (P_0^\perp X P_0 + P_0 X P_0^\perp, i\text{Im} \langle h, f_0 \rangle f_0 + (P_0 - (f_0 \otimes f_0))h + P_0^\perp X f_0).$$

It is apparent that if $X \in \mathcal{B}_2(\mathcal{H})_h$, then $\mathcal{E}(X, f)$ belongs to the tangent space of the restricted version $\mathcal{R}_+^2$, and thus $\mathcal{E}$ is a projection onto $(T\mathcal{R}_+^2)_{(P_0, f_0)}$. However $\mathcal{E}$ is not orthogonal. To compute the orthogonal projection Π we shall use the following known fact (see [14, Theorem 1.3]):

$$\Pi = \mathcal{E}(\mathcal{E} + \mathcal{E}^* - 1)^{-1}.$$

Proposition 5.1.

$$\Pi(X, f) = (\Pi_1(X, f), \Pi_2(X, f)) \quad (11)$$

where

$$\Pi_1(X, f) = P_0^\perp X P_0 + P_0 X P_0^\perp + \frac{1}{3}f_0 \otimes P_0^\perp f + \frac{1}{3}P_0^\perp f \otimes f_0 - \frac{1}{3}f_0 \otimes f_0 X P_0^\perp - \frac{1}{3}P_0^\perp X f_0 \otimes f_0, \quad (12)$$

and

$$\Pi_2(X, f) = i\text{Im} \langle f, f_0 \rangle f_0 + (P_0 - f_0 \otimes f_0)f + \frac{2}{3}P_0^\perp X f_0 + \frac{1}{3}P_0^\perp f. \quad (13)$$

Proof. Straightforwrd verification. $\square$

5.1 Levi-Civita connection of the ambient metric

From now on, for $(P_0, f_0) \in \mathcal{R}_2^+$, let us denote by $\Pi_{(P_0, f_0)}$ the projection Π defined in (11), (12), (13). We endow $\mathcal{R}_2^+$ with the Hilbert-Riemann metric given by the usual trace and real part of the inner product of $\mathcal{H}$:

Definition 5.2. Let $(X, f), (Y, g) \in (T\mathcal{R}_2^+)_{(P_0, f_0)}$. Then we define

$$\langle (X, f), (Y, g) \rangle_{(P_0, f_0)} = \text{Tr}(XY) + \text{Re} \langle f, g \rangle.$$

Remark 5.3. Let $(P, f) \in \mathcal{R}_2^+$ and $U \in \mathcal{U}_2(\mathcal{H})$. Then

$$\Pi_{U \cdot (P, f)}(X, g) = \Pi_{(P, f)}(U^* X U, U^* g).$$

This can be verified by means of routine computations, involving elementary facts, such as $U P_0^\perp U^* = (U P U^*)^\perp$, $U(f \otimes f)U^* = Uf \otimes Uf$, etc.

Given $(P_0, f_0) \in \mathcal{R}_2^+$, we still denote by $\rho_{(P_0, f_0)}$ the restriction of the former map $\rho_{(P_0, f_0)}$ to the group $\mathcal{U}_2(\mathcal{H})$, and by $\delta_{(P_0, f_0)}$ its differential at the identity,

$$\delta_{(P_0, f_0)} : \mathcal{B}_2(\mathcal{H})_{ah} \rightarrow T(\mathcal{R}_2^+)_{(P_0, f_0)} \subset \mathcal{B}_2(\mathcal{H})_h \times \mathcal{H}, \quad \delta_{(P_0, f_0)}(Z) = (ZP_0 - P_0Z, Zf_0),$$

where $\mathcal{B}_2(\mathcal{H})_{ah}$ denotes the space of anti-Hermitian Hilbert-Schmidt operators. The description of the nullspace of $\delta_{(P_0, f_0)}$ done in the proof of Proposition 3.2, is valid in this context, assuming that the elements lie in $\mathcal{B}_2(\mathcal{H})_{ah}$. Namely, $X \in N(\delta_{(P_0, f_0)})$ if $X \in \mathcal{B}_2(\mathcal{H})$, and it is of the form

$$X = \begin{pmatrix} X_{11} & 0 \\ 0 & X_{22} \end{pmatrix}$$

with X_{ii} anti-Hermitian and $X_{11}f_0 = 0$, and X_{11} can be written as a matrix in terms of the decomposition $R(P_0) = \langle f_0 \rangle \oplus \langle f_0 \rangle^\perp$,

$$X_{11} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & x'_{11} & x'_{12} & \cdots \\ 0 & x'_{21} & x'_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & X' \end{pmatrix},$$

where X' is an anti-Hermitian Hilbert-Schmidt operator acting in $R(P_0) \ominus \langle f_0 \rangle$. The intersection of $\mathcal{B}_2(\mathcal{H})$ with the supplement found in Proposition 3.2, presented as matrices in the decomposition

$$\mathcal{H} = \langle f_0 \rangle \oplus (R(P_0) \ominus \langle f_0 \rangle) \oplus N(P_0)$$

which are the elements of the form

$$\begin{pmatrix} it & \vec{g} & Y \\ -\vec{g}^t & 0 & \\ -Y^* & 0 & \end{pmatrix} = \begin{pmatrix} it & g & Y_1 \\ -g^t & 0 & Y_2 \\ -Y_1^* & -Y_2^* & 0 \end{pmatrix}, \quad (14)$$

with $t \in \mathbb{R}$. Denote this set of matrices by $\mathbb{H}_{(P_0, f_0)}$. A straightforward computation shows that

Proposition 5.4. *The space $\mathbb{H}_{(P_0, Q_0)}$ is the orthogonal complement for $N(\delta_{(P_0, f_0)})$, for the trace inner product in $\mathcal{B}_2(\mathcal{H})_{ah}$.*

Since the metric is defined by means of a metric in the ambient space, the covariant derivative consists of differentiating in the ambient space and projecting onto the tangent spaces. Therefore a curve δ is a geodesic if it satisfies the equation

$$\Pi_{\delta(t)}(\ddot{\delta}) = 0$$

for all t .

Remark 5.5. Let $Z \in \mathcal{B}_2(\mathcal{H})_{ah}$. Then $\gamma(t) = e^{tZ} \cdot (P_0, f_0)$ is a geodesic of the connection if and only if

$$\Pi_{(P_0, f_0)}(Z^2 P_0 - 2Z P_0 Z + P_0 Z^2, Z^2 f_0) = 0.$$

Indeed, $\ddot{\gamma}(t) = (e^{tZ} \{Z^2 P_0 - 2Z P_0 Z + P_0 Z^2\} e^{-tZ}, e^{tZ} Z^2 f_0) = e^{tZ} \cdot (Z^2 P_0 - 2Z P_0 Z + P_0 Z^2, Z^2 f_0)$, and thus, using Remark 5.3,

$$\begin{aligned} \Pi_{\gamma(t)}(\ddot{\gamma}(t)) &= \Pi_{e^{tZ} \cdot (P_0, f_0)}(e^{tZ} \cdot (Z^2 P_0 - 2Z P_0 Z + P_0 Z^2, Z^2 f_0)) \\ &= \Pi_{(P_0, f_0)}(Z^2 P_0 - 2Z P_0 Z + P_0 Z^2, Z^2 f_0). \end{aligned}$$

If one takes Z to be P_0 -codiagonal (and thus in $\mathbb{H}_{(P_0, f_0)}$), such that f_0 is not an eigenvector for Z^2 , then γ is not a geodesic.

5.2 Reductive connection

We shall consider another natural linear connection in $\mathcal{R}_2^+$, induced by the decomposition of the Banach-Lie algebra $\mathcal{B}_2(\mathcal{H})$ of $\mathcal{U}_2(\mathcal{H})$, induced by $(P_0, f_0) \in \mathcal{R}_2^+$: $\mathcal{B}_2(\mathcal{H})_{ah} = N(\delta_{(P_0, f_0)}) \oplus \mathbb{H}_{(P_0, f_0)}$.

Let us denote by $\mathcal{I}_2(P_0, f_0)$ the isotropy group of the restricted action

$$\mathcal{I}_2(P_0, f_0) = \{V \in \mathcal{U}_2(\mathcal{H}) : V \cdot (P_0, f_0) = (P_0, f_0)\}.$$

Note that if $V \in \mathcal{I}_2(P_0, f_0)$, then $Ad(V)(N(\delta_{(P_0, f_0)})) = N(\delta_{(P_0, f_0)})$. Indeed, let $Y \in N(\delta_{(P_0, f_0)})$, then $Y P_0 = P_0 Y$ and $Y f_0 = 0$. Since $V \in \mathcal{I}_2(P_0, f_0)$, $V P_0 = P_0 V$ and $V f_0 = f_0$ (thus, $V^* f_0 = f_0$). Then $V Y V^*$ also commutes with P_0 and $V Y V^* f_0 = V Y f_0 = 0$. On the other hand, $Ad(V)$ is an orthogonal transformation of $\mathcal{B}_2(\mathcal{H})$ (endowed with the (real part of the) trace inner product). Therefore $Ad(V)$ leaves $\mathbb{H}_{(P_0, f_0)} (= N(\delta_{(P_0, f_0)})^\perp)$ invariant.

Let us prove that the distribution

$$\mathcal{R}_2^+ \ni (P_0, f_0) \mapsto \mathbb{H}_{(P_0, f_0)} \subset \mathcal{B}_2(\mathcal{H})_{ah}$$

is *smooth*. In other words, if we denote by $\mathbf{P}_{(P_0, f_0)}$ the *ReTr*-orthogonal projection onto $\mathbb{H}_{(P_0, f_0)}$, we must prove that the mapping

$$\mathcal{R}_2^+ \ni (P_0, f_0) \mapsto \mathbf{P}_{(P_0, f_0)} \in \mathcal{B}_{\mathbb{R}}(\mathcal{B}_2(\mathcal{H})_{ah})$$

is smooth, where $\mathcal{B}_{\mathbb{R}}(\mathcal{B}_2(\mathcal{H})_{ah})$ denotes the Banach space of real linear bounded operators acting in $\mathcal{B}_2(\mathcal{H})_{ah}$ (endowed with the Hilbert-Schmidt norm).

Lemma 5.6. Let $\mathbf{Q} = \mathbf{P}_{(P_0, f_0)}^\perp = 1_{\mathcal{B}_2(\mathcal{H})_{ah}} - \mathbf{P}_{(P_0, f_0)}$. Then

$$\mathbf{Q}(X) = P_0 X P_0 + P_0^\perp X P_0^\perp + f_0 \otimes f_0 X f_0 \otimes f_0 - f_0 \otimes f_0 X P_0 - P_0 X f_0 \otimes f_0.$$

In particular, the mapping $\mathcal{R}_2^+ \rightarrow \mathcal{B}_{\mathbb{R}}(\mathcal{B}_2(\mathcal{H})_{ah})$, $(P_0, f_0) \mapsto \mathbf{P}_{(P_0, f_0)}$ is smooth.

Proof. Since $f_0 \in R(P_0)$, clearly $\mathbf{Q}(X)$ commutes with P_0 . Also it is clear that if X anti-Hermitian and Hilbert-Schmidt, then so is $\mathbf{Q}(X)$. Finally, note that $\mathbf{Q}(X)f_0 = 0$. With this formula, it is clear that the map $(P_0, f_0) \mapsto \mathbf{P}_{(P_0, f_0)}$ is smooth. $\square$

These facts imply:

Corollary 5.7. The distribution of supplements $\mathcal{R}_2^+ \ni (P, f) \mapsto \mathbb{H}_{(P, f)}$ defines a reductive structure in $\mathcal{R}_2^+$ (as a $\mathcal{U}_2(\mathcal{H})$ -homogeneous space).

Remark 5.8. For $(P_0, f_0) \in \mathcal{R}_2^+$, we will denote

$$\delta_0 = \delta_{(P_0, f_0)} = (\rho_{(P_0, f_0)})_* 1,$$

then the map $\delta_0|_{\mathbb{H}_{(P_0, f_0)}} : \mathbb{H}_{(P_0, f_0)} \rightarrow (T\mathcal{R}_2^+)_{(P_0, f_0)}$ is a linear isomorphism. Let us denote by $\kappa_{(P_0, f_0)}$ its inverse. The distribution $(P, f) \mapsto \kappa_{(P, f)}$ is usually called the 1-form of the reductive structure, and is used to define the invariants of the linear connection.

Accordingly, we define a Hilbert-Riemann metric which is coherent with this structure: we postulate that the 1-form to be isometric. That is,

Definition 5.9. If $(X, g) \in (T\mathcal{R}_2^+)_{(P_0, f_0)}$, then

$$|(X, g)|_{r, (P_0, f_0)}^2 = -\text{Tr} \left((\kappa_{(P_0, f_0)}(X, g))^2 \right),$$

i.e., $|(X, g)|_{r, (P_0, f_0)} = \|Z\|_2$, where $Z \in \mathbb{H}_{(P_0, f_0)}$ is the unique element such that $\delta_{(P_0, f_0)}(Z) = (X, g)$.

Note that we endowed $\mathcal{R}_2^+$ with the *quotient metric*, giving it the Riemannian metric such that the map $\rho_0 = \rho_{(P_0, f_0)} : \mathcal{U}_2(\mathcal{H}) \rightarrow \mathcal{R}_2^+$ is a *Riemannian submersion*. That is, ρ_0 is a smooth submersion such that for any $U \in \mathcal{U}_2(\mathcal{H})$, if $(P, f) = \rho(U)$, then $\delta = \delta_{(P, f)}$ is an isometry between $\mathbb{H}_{(P, f)}$ and $(T\mathcal{R}_2^+)_{(P, f)}$.

Remark 5.10. The metric defined above in Definition 5.9 coincides with the quotient metric, as defined in (3), if one replaces the operator norm by the Hilbert-Schmidt norm.

If $V \in (T\mathcal{R}_2^+)_{(P, f)}$, then $\kappa_{(P, f)}(V) \in \mathbb{H}_{(P, f)}$ is its unique *horizontal lift*; and if X is a smooth vector field in $\mathcal{R}_2^+$ then $X^h = \kappa \circ X : u \mapsto \kappa_{\rho(u)}(X_{\rho(u)})$ is a smooth vector field in $\mathcal{U}_2(\mathcal{H})$, the *horizontal lift* of X . For obvious reasons $N(\delta)$, the kernel of $\delta = \rho_*$ is also known as the *vertical space*, and vectors in that kernel are named *vertical vectors*.

Remark 5.11. It is apparent from the definition of quotient metric that if $V = \delta_0(Z) = (ZP_0 - P_0Z, Zf_0) \in (T\mathcal{R}_2^+)_{(P_0, f_0)}$ with Z a horizontal vector described as in (14), then

$$\begin{aligned} \|X\|_q^2 &= \|Z\|^2 = \text{Tr} \begin{pmatrix} t^2 + g \cdot \bar{g}^t + Y_1 Y_1^* & \dots & \dots \\ \dots & \bar{g}^t \cdot g + Y_2 Y_2^* & \dots \\ \dots & \dots & Y_1 Y_1^* + Y_2 Y_2^* \end{pmatrix} \\ &= t^2 + 2\|g\|^2 + 2\|Y_1\|_2^2 + 2\|Y_2\|_2^2. \end{aligned}$$

On the other hand, if we endow $\mathcal{R}_2^+$ with the *ambient* metric of Definition 5.2, we obtain

$$\begin{aligned}
\|X\|_a^2 &= \|\delta(Z)\|_2^2 = \|ZP_0 - P_0Z\|^2 + \|Zf_0\|^2 \\
&= \left\| \begin{pmatrix} 0 & 0 & Y_1 \\ 0 & 0 & Y_2 \\ Y_1^* & Y_2^* & 0 \end{pmatrix} \right\|_2^2 + \|(it \ g \ Y_1)\|^2 \\
&= \text{Tr} \begin{pmatrix} Y_1Y_1^* & \cdots & \cdots \\ \cdots & Y_2Y_2^* & \cdots \\ \cdots & \cdots & Y_1^*Y_1 + Y_2^*Y_2 \end{pmatrix} + t^2 + \|g\|^2 + \|Y_1\|^2 \\
&= t^2 + \|g\|^2 + 3\|Y_1\|_2^2 + 2\|Y_2\|_2^2.
\end{aligned}$$

Therefore

$$\frac{1}{\sqrt{2}}\|X\|_q \leq \|X\|_a \leq \sqrt{\frac{3}{2}}\|X\|_q$$

which makes both Riemannian metrics equivalent (the inequalities are sharp). Consequently, if d denotes the Riemannian distance given by taking the infima of the lengths of paths joining given endpoints, both distances are equivalent with constants

$$\frac{1}{\sqrt{2}}d_q(x, y) \leq d_a(x, y) \leq \sqrt{\frac{3}{2}}d_q(x, y) \quad (15)$$

for all $x, y \in \mathcal{R}_2^+$.

Corollary 5.12. *The distance induced by the Levi-Civita connection of the quotient metric induces in $\mathcal{R}_2^+$ the same ambient topology of its embedding in the Hilbert space $\mathcal{B}_2(\mathcal{H})_h \times \mathcal{H}$.*

Proof. Combine the equivalence of distances of the previous remark, with the fact that $\mathcal{R}_2^+$ is an embedded submanifold, and use that the Riemannian distance gives the manifold topology [15, Proposition 6.1]. $\square$

We collect below some known results we will use on Riemannian submersions:

Theorem 5.13. *Let $\rho : (U, g_0) \rightarrow (R, g)$ be a Riemannian submersion, let X, Y be smooth vector fields in (R, g) . Then*

1. *If Z_0 is a smooth vertical field in (U, g_0) , then*

$$[X, Y]^v := [X^h, Y^h] - [X, Y]^h \text{ and } [X^h, Z_0]$$

are smooth vertical vector fields in (U, g_0) (here $[\cdot, \cdot]$ denotes the usual Lie bracket of smooth vector fields).

2. *If ∇^0 is the Levi-Civita connection of (U, g_0) and ∇ is the Levi-Civita connection of (R, g) , then*

$$(\nabla_X Y)_{\rho(u)} = \delta(\nabla_{X^h}^0 Y^h)$$

where $\delta = \rho_$ and moreover*

$$\nabla_{X^h}^0 Y^h = (\nabla_X Y)^h + \frac{1}{2}[X, Y]^v.$$

3. Let Γ be a geodesic of (U, g_0) , with $\Gamma'(0)$ horizontal, then $\Gamma'(t)$ is horizontal for all t in the domain of Γ , and $\gamma = \rho(\Gamma)$ is a geodesic of (R, g) of the same length of Γ . Conversely, if γ is a geodesic of (R, g) with $\gamma(0) = \rho(u)$, there exists a unique horizontal lift Γ of γ such that $\Gamma(0) = u$ and Γ is a geodesic of (U, g_0) .
4. The exponential maps are thus related by $\text{Exp}^R(v) = \rho \circ \exp^U(v^h)$.
5. If (U, g_0) is complete (resp. geodesically complete) then (R, g) is complete (resp. geodesically complete).
6. If X, Y are orthonormal, the sectional curvatures are related by O'Neill's formula.

$$K^R(X, Y) = K^U(X^h, Y^h) + \frac{3}{4} \|[X, Y]^v\|^2.$$

7. In particular, if $R = U/I$ is a quotient of Lie groups and the metric g_0 on U is bi-invariant, then geodesics of (R, g) are given by $\rho(e^{tZ})$ where Z is an horizontal vector, and sectional curvature is positive and given by

$$K^R(v, w) = \frac{1}{4} \|[v, w]^h\|^2 + \|[v, w]^v\|^2,$$

if $v, w \in (TR)_{\rho(1)}$ and now $[\cdot, \cdot]$ denotes the usual Lie algebra bracket.

Proof. See for instance [12] Lemma 3.54, Proposition 3.55, Proposition 2.109, Theorem 3.61, Theorem 3.65. $\square$

Remark 5.14. If $Z(t)$ is a tangent field along a curve $\gamma(t) \in \mathcal{R}_2^+$, the covariant derivative $D_{\dot{\gamma}}Z$ can be computed using the formula in the second item above, and the formula for the covariant derivative in the unitary group

$$(\nabla_X^0 Y)_u = DY_u(X_u) + \frac{1}{2} [X_u Y_u^* u + u Y_u^* X_u].$$

Combining these formulas, we obtain

$$D_{\dot{\gamma}}Z = \delta_{\gamma} \left(\frac{d}{dt} \kappa_{\gamma}(Z) - \frac{1}{2} [\kappa_{\gamma}(\dot{\gamma}) \kappa_{\gamma}(Z) + \kappa_{\gamma}(Z) \kappa_{\gamma}(\dot{\gamma})] \right).$$

Remark 5.15. Since the geodesics of the unitary group are the one-parameter groups $t \mapsto e^{tZ}$, it follows from the fourth item above that geodesics of $\mathcal{R}_2^+$ are exactly the image through ρ of the one-parameter groups with horizontal speed, i.e. $D_{\dot{\gamma}}\dot{\gamma} = 0$ and $\gamma(0) = (P, f)$ if and only if $\gamma(t) = \rho(e^{tZ})$ with $Z \in \mathbb{H}_{(P, f)}$

The exponential map of $\mathcal{R}_2^+$ with the quotient metric is then given by

$$\text{Exp}(V) = \text{Exp}_{(P_0, f_0)}(V) = \rho(e^{\kappa_{(P_0, f_0)} V}),$$

or equivalently, if $V = (ZP_0 - P_0Z, Zf_0)$ for horizontal Z , then

$$\text{Exp} : V = (ZP_0 - P_0Z, Zf_0) \mapsto (e^Z P_0 e^{-Z}, e^Z f_0). \quad (16)$$

Since $\delta_0 = \delta|_{\mathbb{H}_{(P_0, f_0)}} : \mathbb{H}_{(P_0, f_0)} \rightarrow (\mathcal{R}_2^+)_{(P_0, f_0)}$ is an isometric isomorphism, we can equivalently study the map $E = \text{Exp} \circ \kappa = \rho \circ \exp : \mathbb{H}_{(P_0, f_0)} \rightarrow \mathcal{R}_2^+$, that is

$$E : Z \mapsto (e^Z P_0 e^{-Z}, e^Z f_0) = \rho(e^Z). \quad (17)$$

If $\exp_{*Z}(W)$ denotes the differential at Z of the Lie-group exponential, $e^Z = \sum_{n \geq 1} \frac{1}{n!} Z^n$, evaluated in W , then it is well-known that

$$\exp_{*Z}(W) = e^Z(F(adZ)W) = \int_0^1 e^{(1-s)Z} W e^{sZ} ds = (G(adZ)W)e^Z.$$

Here $F, G : \mathbb{C} \rightarrow \mathbb{C}$ denote the entire functions $F(\lambda) = \frac{1-e^{-\lambda}}{\lambda}$, $G(\lambda) = \frac{e^\lambda-1}{\lambda}$, note that $G(-\lambda) = -F(\lambda)$. Therefore

$$E_{*Z}(W) = (e^Z(F(adZ)W)P_0 e^{-Z} + e^Z P_0 (G(-adZ)W) e^{-Z}, e^Z(F(adZ)W)f_0),$$

and thus

$$e^{-Z} \cdot E_{*Z}(W) = ((F(adZ)W)P_0 - P_0(F(adZ)W), (F(adZ)W)f_0) = \delta_0(T(W)) \quad (18)$$

where we denote $T = T_Z = F(adZ) : \mathcal{B}_2(\mathcal{H})_{ah} \rightarrow \mathcal{B}_2(\mathcal{H})_{ah}$ for short. Note that T it is a bounded linear operator.

Remark 5.16. Since the roots of F are located at $\{\lambda = 2k\pi i, k \in \mathbb{Z} \neq 0\}$, and $\|adZ\| \leq 2\|Z\|$, it is well-know that as long as $\|Z\| < \pi$, T is an invertible operator. In what follows we need to refine this remark to obtain better control on the range of T acting on $\mathbb{H}$. Consider the real even function

$$g(t) = \frac{\sin(t)}{t} + \frac{\cos(t) - 1}{t^2},$$

extended as $g(0) = 1/2$ to make it continuous. Then the first root of g will be denoted by $r_0 > 0$, a straightforward computation indicates that $\frac{\pi}{2} < r_0 < \frac{2\pi}{3}$ and g is strictly decreasing in $(0, r_0)$ (and increasing in $(-r_0, 0)$ by symmetry).

Lemma 5.17. *For each $Z \in \mathbb{H}_{(P_0, f_0)}$, let $T = F(adZ)$. Then T is a normal contraction acting on $\mathcal{B}_2(\mathcal{H})_{ah}$, and moreover if $\|Z\| < \frac{r_0}{2}$, then $\|T - 1\| < 1$.*

Proof. From the definition of $adZ(W) = ZW - WZ$ and the trace inner product in the Hilbert space $\mathcal{B}_2(\mathcal{H})_{ah}$, it follows that adZ is anti-Hermitian for any anti-Hermitian Z . Therefore $T = F(adZ)$ is a normal operator; it is a contraction because

$$\|T\| = \|F(adZ)\| = \left\| \int_0^1 e^{s adZ} ds \right\| \leq \int_0^1 \|e^{s adZ}\| ds \leq 1.$$

Here, since $s adZ$ is anti-Hermitian, $e^{s adZ}$ is unitary. Now since T is normal, its norm can be computed using its spectral radius, and for $it \in \sigma(Z)$ we obtain

$$\begin{aligned} |F(it) - 1|^2 &= \left| \frac{1 - e^{-it}}{it} - 1 \right|^2 = \frac{1}{t^2} |1 - e^{-it} - it|^2 = \frac{1}{t^2} |1 - \cos t + i(\sin t - t)|^2 \\ &= \frac{1}{t^2} (1 + \cos^2 t - 2 \cos t + \sin^2 t + t^2 - 2t \sin t) \\ &= 1 + 2\left(\frac{1 - \cos t}{t^2} - \frac{\sin t}{t}\right) = 1 - 2g(t). \end{aligned} \quad (19)$$

This tells us that $\|F(adZ) - 1\| < 1$ when $|t| < r_0$. If $\|Z\| < r_0/2$ then $\|adZ\| < r_0$, then $\sigma(adZ) \subset (-ir_0, ir_0)$ and the conclusion of the lemma follows. $\square$

Corollary 5.18. *Let $(P, f) = U \cdot (P_0, f_0) \in \mathcal{R}_2^+$, let $\text{Exp}(P, f) : (T\mathcal{R}_2^+)_{(P, f)} \rightarrow \mathcal{R}_2^+$ be the Riemannian exponential of the quotient metric, given by*

$$\text{Exp}_{(P, f)}(V) = \rho_{(P, f)}(e^{\kappa_{(P, f)}} V).$$

Then its differential is an isomorphism in the ball of radius $r_0/2$ around $0 \in (T\mathcal{R}_2^+)_{(P, f)}$.

Proof. Since the action of the unitary group is an isometric isomorphism of the Riemannian manifold $\mathcal{R}_2^+$, it will suffice to prove the assertion for (P_0, f_0) . Since δ_0 is an isometric isomorphism, it will be more convenient to prove this assertion of the simplified exponential map $E : \mathbb{H}_{(P_0, f_0)} \rightarrow \mathcal{R}_2^+ \subset \mathcal{B}_2(\mathcal{H}) \times \mathcal{H}$ given by equation (17). We have to show that the differential of E is invertible in a certain ball; since the action of the unitary group is isometric, it suffices to prove that the map given in (18) is an isomorphism for $V = (X, g) = \delta_0(Z) = (ZP_0 - P_0Z, Zf_0)$ if $\|V\|_q = \|Z\| < r_0/2$ (here Z is the unique horizontal lift of V).

In what follows we denote $\mathbb{H} = \mathbb{H}_{(P_0, f_0)}$ and $\mathbf{P} = \mathbf{P}_{(P_0, f_0)}$ (the orthogonal projection onto the horizontal space $\mathbb{H}$), Let $A = T\mathbf{P} + (\mathbf{1} - \mathbf{P})$ which is a bounded operator acting on the Hilbert space $\mathcal{B}_2(\mathcal{H})_{ah}$ ($T = F(adX)$ as before). Then $\|A - \mathbf{1}\| = \|(T - \mathbf{1})\mathbf{P}\| \leq \|1 - T\| = c_0 < 1$ therefore A is invertible, which shows that $\mathcal{B}_2(\mathcal{H})_{ah} = T\mathbb{H} + \ker(\delta) = T\mathbb{H} + \mathbb{H}^\perp$. On the other hand if $T\mathbf{P}\xi = (\mathbf{1} - \mathbf{P})\eta$,

$$\|\mathbf{P}\xi\|^2 > \|(1 - T)\mathbf{P}\xi\|^2 = \|T\mathbf{P}\xi - \mathbf{P}\xi\|^2 = \|(\mathbf{1} - \mathbf{P})\eta - \mathbf{P}\xi\|^2 = \|(\mathbf{1} - \mathbf{P})\eta\|^2 + \|\mathbf{P}\xi\|^2,$$

i.e. $(\mathbf{1} - \mathbf{P})\eta = 0$, which proves that $T\mathbb{H} \cap \mathbb{H}^\perp = \{0\}$.

Therefore $\mathcal{B}_2(\mathcal{H})_{ah}$ is the direct sum of the kernel of δ_0 (the vertical space, $\mathbb{H}^\perp$) and $T\mathbb{H}$,

$$\mathcal{B}_2(\mathcal{H})_{ah} = T\mathbb{H} \dot{+} \mathbb{H}^\perp.$$

This implies that $\delta_0|_{T\mathbb{H}}$ is still an isomorphism onto $(T\mathcal{R}_2^+)_{(P_0, f_0)}$. By equation (18), this proves the claim that E_{*Z} is an isomorphism. $\square$

Recall the Riemannian distances d_q, d_a in $\mathcal{R}_2^+$ are defined as the infima of the lengths of the piecewise smooth paths joining given endpoints, when the curves are measured with the quotient metric (resp. the ambient metric). Regarding the minimality of geodesics, the injectivity radius, and the metric completeness, the following can be established.

Theorem 5.19. *Let $(P_0, f_0) \in \mathcal{R}_2^+$, endowed with the quotient metric given in Definition 5.9.*

1. *For any $(X, g) \in (T\mathcal{R}_2^+)_{(P_0, f_0)}$, let $Z \in \mathbb{H}_{(P_0, f_0)}$ the unique element such that $\delta_{(P_0, f_0)}(Z) = (X, g)$. Then*

$$\delta(t) = e^{tZ} \cdot (P_0, f_0),$$

which is the unique geodesic of the reductive connection which satisfies $\delta(0) = (P_0, f_0)$ and $\dot{\delta}(0) = (X, g)$, has minimal length along its path for t such that $|t| \leq \frac{\pi}{4\|Z\|_2}$. This geodesic is unique if $|t| < \frac{\pi}{4\|Z\|_2}$.

2. *The space $\mathcal{R}_2^+$ is complete with both its metrics d_q, d_a (quotient and ambient).*
3. *Let $(P_1, f_1) \in \mathcal{R}_2^+$ such that $d_q((P_1, f_1), (P_0, f_0)) < \pi/4$. Then there exists a unique geodesic δ of the reductive connection such that $\delta(0) = (P_0, f_0)$ and $\delta(1) = (P_1, f_1)$. This geodesic has minimal length (among smooth curves joining (P_0, f_0) and (P_1, f_1)).*

Proof. The first assertion is a consequence of the estimation for the injectivity radius of homogeneous spaces of the p -Schatten unitary groups given in [3, Theorem 4.9]. From that theorem also follows the uniqueness when $4\|Z\|_2|t| < \pi$.

Regarding the second assertion, it was also proven in [3, Theorem 5.5] that homogeneous spaces of the restricted unitary group are complete with the quotient metric. Therefore $(\mathcal{R}_2^+, d_q)$ is complete. Since both metrics are uniformly equivalent by (15), $(\mathcal{R}_2^+, d_a)$ is also complete.

Now note that the first assertion says that the Riemannian exponential is injective in the open ball of (quotient) radius $\pi/4$. Combining this with Corollary 5.18, it follows that the injectivity radius of the Riemannian exponential of $\mathcal{R}_2^+$ is at least $\pi/4$ (since $r_0/2 > \pi/4$). Therefore by the theory of Hilbert-Riemann manifolds (see for instance [15, Theorem 6.4]), the third assertion follows. $\square$

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