Non-local Thirring model with backward and umklapp interactions

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Abstract

We extend a non-local and non-covariant version of the Thirring model in order to describe a many-body system with backward and umklapp scattering processes. We express the vacuum to vacuum functional in terms of a non-trivial fermionic determinant. Using path-integral methods we find a bosonic representation for this determinant which allows us to obtain an effective action for the collective excitations of the system. By introducing a non-local version of the self-consistent harmonic approximation, we get an expression for the gap of the charge-density excitations as functional of arbitrary electron–electron potentials. As an example we also consider the case of a non-contact umklapp interaction. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

In recent years there has been much interest in the study of low-dimensional field theories. In particular they are useful to describe the behavior of strongly anisotropic physical systems in condensed matter, such as organic conductors [1], charge transfer salts [2] and quantum wires [3]. Probably the two most widely studied 1d systems are the Hubbard model [4] and the so-called "g-ology" model [5]. They are known to display the Luttinger liquid [6] behavior characterized by spin-charge separation and by non-universal

(interaction dependent) power-law correlation functions. In a recent series of papers [7,8] an alternative, field-theoretical approach was developed to consider this problem. In these works a non-local and non-covariant version of the Thirring model [9] was introduced, in which the fermionic densities and currents are coupled through bilocal, distance-dependent potentials. This non-local Thirring model (NLT) contains the Tomonaga–Luttinger (TL) model [10] as a particular case. Although it constitutes an elegant framework to analyze the 1d many-body problem, one serious limitation appears if one tries to make contact with real systems. Indeed, one has to recall that the building blocks of the NLT are the forwardscattering (fs) processes which are supposed to dominate the scene only in the low transferred momentum limit. This means that in its present form it can only provide a very crude description of the Luttinger liquid equilibrium and transport properties. In Ref. [11] we started to develop an improved version of the NLT in which larger momentum transfers are taken into account by including backscattering (bs) [12] and umklapp (us) [13]. This formulation led us to a highly non-trivial action whose physical content was very hard to extract. In the present paper we reconsider this problem from a different point of view which allowed us to get insight on the spectrum of the collective modes. By using a non-local version of the self-consistent harmonic approximation (SCHA) [14] we were able to obtain general expressions for the bosonic gaps as functionals of arbitrary forward, backward and umklapp scattering coupling functions. In Section 2 we present the model and review the steps that allow us to write a purely bosonic action describing the dynamics of collective excitations. In Section 3 we analyze the bosonic action by using a self-consistent harmonic approximation. We derive our main formal result, i.e., the general formula for the gap, valid for pure backscattering ($g_1 \neq 0$, $g_3 = 0$) and pure umklapp scattering ($g_3 \neq 0$, $g_1 = 0$). In Section 4, as an example, we study in some detail the case of a non-local umklapp interaction. In Section 5 we present our conclusions. In Appendix A we review the main points of the non local SCHA.

2. The model

In this section we begin the study of an extended version of the NLT which includes the contribution of forward, backward and umklapp scattering. Following the formulation proposed in [7] we shall attempt to describe these interactions by means of a fermionic $(1 + 1)$-dimensional Quantum Field Theory with Euclidean action given by

$$S = S_0 + S_{fs} + S_{bs} + S_{us}, \tag{2.1}$$

where

$$S_0 = \int d^2x \, \overline{\Psi} i \partial\!\!\!/ \Psi \tag{2.2}$$

is the unperturbed action associated to a linearized free dispersion relation. The contributions of the different scattering processes will be written as

$$S_{fs} = -\frac{g^2}{2} \int d^2x \, d^2y \, (\overline{\Psi}\gamma_\mu\Psi)(x) \, V_{(\mu)}(x, y) \, (\overline{\Psi}\gamma_\mu\Psi)(y) \tag{2.3}$$

and

$$S_{bs} + S_{us} = -\frac{g'^2}{2} \int d^2x \, d^2y \, (\overline{\Psi}\Gamma_\mu\Psi)(x) U_{(\mu)}(x, y)(\overline{\Psi}\Gamma_\mu\Psi)(y), \tag{2.4}$$

where the $\gamma'_\mu s$ are the usual two-dimensional Dirac matrices and $\Gamma_0 = 1$, $\Gamma_1 = \gamma_5$. Please keep in mind that no sum over repeated indices is implied when a subindex (μ) is involved. Let us also mention that the coupling potentials $V_{(\mu)}$ and $U_{(\mu)}$ are assumed to depend on the distance $|x - y|$ and can be expressed in terms of Solyom's "g-ology" [5] as

$$V_{(0)}(x, y) = \frac{1}{g^2}(g_2 + g_4)(x, y), \tag{2.5}$$

$$V_{(1)}(x, y) = \frac{1}{g^2}(g_2 - g_4)(x, y), \tag{2.6}$$

$$U_{(0)}(x, y) = \frac{1}{g'^2}(g_3 + g_1)(x, y), \tag{2.7}$$

$$U_{(1)}(x, y) = \frac{1}{g'^2}(g_3 - g_1)(x, y). \tag{2.8}$$

In the above equations g and g' are just numerical constants that could be set equal to one. We keep them to facilitate comparison of our results with those corresponding to the usual Thirring model. Indeed, this case is obtained by choosing $g' = 0$ and $V_{(0)}(x, y) = V_{(1)}(x, y) = \delta^2(x - y)$. On the other hand, the non-covariant limit $g' = 0$, $V_{(1)}(x, y) = 0$ gives one version ($(g_2 = g_4)$) of the TL model [10].

The terms in the action containing g_2 and g_4 represent forward scattering events, in which the associated momentum transfer is small. In the g_2 processes the two branches (left and right-moving particles) are coupled, whereas in the g_4 processes all four participating electrons belong to the same branch. On the other hand, g_1 and g_3 are related to scattering diagrams with larger momentum transfers of the order of $2k_F$ (bs) and $4k_F$ (us) respectively (this last contribution is important only if the band is half-filled). For simplicity, throughout this paper we will consider spinless electrons. The extension of our results to the spin-1/2 case with spin-flipping interactions, though not trivial, could be done by following the lines of Ref. [8].

Let us now turn to the treatment of the partition function. At this point we recall that in Ref. [7] we wrote the fs piece of the action in a localized way:

$$S_{fs} = -\frac{g^2}{2} \int d^2x \, J_\mu K_\mu, \tag{2.9}$$

where J_μ is the usual fermionic current, and K_μ is a new current defined as

$$K_\mu(x) = \int d^2y \, V_{(\mu)}(x, y) J_\mu(y). \tag{2.10}$$

Using a functional delta and introducing auxiliary bosonic fields in the path-integral representation of the partition function $\mathcal{Z}$, we were able to write (see [7] for details):

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}\overline{\Psi} \, \mathcal{D}\Psi \, \mathcal{D}\widetilde{A}_\mu \, \mathcal{D}\widetilde{B}_\mu$$

$$\times \exp\left\{-\int d^2x \left[\overline{\Psi} i \partial\!\!\!/ \Psi + \widetilde{A}_\mu \widetilde{B}_\mu + \frac{g}{\sqrt{2}}(\widetilde{A}_\mu J_\mu + \widetilde{B}_\mu K_\mu)\right]\right\}. \tag{2.11}$$

If we define

$$\overline{B}_\mu(x) = \int d^2y \, V_{(\mu)}(y,x) \widetilde{B}_\mu(y), \tag{2.12}$$

$$\widetilde{B}_\mu(x) = \int d^2y \, V_{(\mu)}^{-1}(y,x) \overline{B}_\mu(y), \tag{2.13}$$

with $V_{(\mu)}^{-1}(y,x)$ satisfying

$$\int d^2y \, V_{(\mu)}^{-1}(y,x) V_{(\mu)}(z,y) = \delta^2(x-z), \tag{2.14}$$

and change auxiliary variables in the form

$$A_\mu = \frac{1}{\sqrt{2}}(\widetilde{A}_\mu + \overline{B}_\mu), \tag{2.15}$$

$$B_\mu = \frac{1}{\sqrt{2}}(\widetilde{A}_\mu - \overline{B}_\mu), \tag{2.16}$$

we obtain

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}\overline{\Psi} \, \mathcal{D}\Psi \, \mathcal{D}A_\mu \, \mathcal{D}B_\mu e^{-S(A,B)-S_{bs}-S_{us}}$$
$$\times \exp\left[-\int d^2x \, \overline{\Psi}(i\partial\!\!\!/ - g\!\!\!/A)\Psi\right], \tag{2.17}$$

where

$$S(A,B) = \frac{1}{2} \int d^2x \, d^2y \, V_{(\mu)}^{-1}(x,y) \left[A_\mu(x)A_\mu(y) - B_\mu(x)B_\mu(y)\right]. \tag{2.18}$$

The Jacobian associated with the change $(\widetilde{A}, \widetilde{B}) \rightarrow (A, B)$ is field-independent and can then be absorbed in the normalization constant $\mathcal{N}$. Moreover, we see that the B-field is completely decoupled from both the A-field and the fermion field. Keeping this in mind, it is instructive to try to recover the partition function corresponding to the usual covariant Thirring model ($V_{(0)}^{-1}(y,x) = V_{(1)}^{-1}(x,y) = \delta^2(x-y)$, and $g' = 0$), starting from (2.17). In doing so one readily discovers that B_μ describes a negative-metric state whose contribution must be factorized and absorbed in $\mathcal{N}$ in order to get a sensible answer for $\mathcal{Z}$. This procedure parallels, in the path-integral framework, the operator approach of Klaiber [9], which precludes the use of an indefinite-metric Hilbert space. Consequently, from now on we shall only consider the A contribution.

At this stage we see that, when bs and us processes are disregarded, the procedure we have just sketched allows us to express $\mathcal{Z}$ in terms of a fermionic determinant. Now we will show that this goal can also be achieved when the larger momentum transfers are taken into account. To this end we write:

$$S_{bs} + S_{us} = -\frac{g'^2}{2} \int d^2x \, L_\mu M_\mu, \tag{2.19}$$

where L_μ and M_μ are fermionic bilinears defined as

$$L_\mu(x) = \overline{\Psi}(x)\Gamma_\mu\Psi(x), \tag{2.20}$$

$$M_\mu(x) = \int d^2y\, U_{(\mu)}(x,y)L_\mu(y). \tag{2.21}$$

Thus it is evident that we can follow the same prescriptions as above, with L_μ and M_μ playing the same roles as J_μ and K_μ, respectively. After the elimination of a new negative metric state whose decoupled partition function is absorbed in the normalization factor, as before, one obtains

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}A_\mu\, \mathcal{D}C_\mu \det(i\not{\partial} - g\not{A} - g'\Gamma_\mu C_\mu)\exp\bigl(-S[A] - S[C]\bigr), \tag{2.22}$$

where

$$S[A_\mu] = \frac{1}{2}\int d^2x\, d^2y\, A_\mu(x)V_{(\mu)}^{-1}A_\mu(y),$$

$$S[C_\mu] = \frac{1}{2}\int d^2x\, d^2y\, C_\mu(x)U_{(\mu)}^{-1}C_\mu(y) \tag{2.23}$$

and

$$\int d^2y\, U_{(\mu)}^{-1}(y,x)U_{(\mu)}(z,y) = \delta^2(x-z). \tag{2.24}$$

Then we have been able to express $\mathcal{Z}$ in terms of a fermionic determinant. Let us stress, however, that this determinant is a highly non-trivial one. Indeed, the term in g' is not only a massive-like term (in the sense that it is diagonal in the Dirac matrices space) but it also depends on the auxiliary field $C_\mu(x)$. As shown in [11] one can combine a chiral change in the fermionic path-integral measure with a formal expansion in g' in order to get a bosonic representation for the fermionic determinant. Let us start by performing the following transformation:

$$\Psi(x) = e^{g[\gamma_5\Phi(x) - i\eta(x)]}\chi(x), \tag{2.25}$$

$$\overline{\Psi}(x) = \bar{\chi}(x)\, e^{g[\gamma_5\Phi(x) + i\eta(x)]}, \tag{2.26}$$

$$\mathcal{D}\overline{\Psi}\,\mathcal{D}\Psi = J[\Phi,\eta]\,\mathcal{D}\bar{\chi}\,\mathcal{D}\chi, \tag{2.27}$$

where Φ and η are scalar fields and $J[\Phi,\eta]$ is the Jacobian of the transformation. As it is well known, the above transformation permits to decouple the field A_μ from the fermionic fields if one writes

$$A_\mu(x) = \partial_\mu\eta(x) + \epsilon_{\mu\nu}\partial_\nu\Phi(x) \tag{2.28}$$

which can also be considered as a bosonic change of variables with trivial (field independent) Jacobian. As a result we find

$$\det(i\not{\partial} - g\not{A} - g'\Gamma_\mu C_\mu) = J[\Phi,\eta]\det\bigl(i\not{\partial} - g'e^{2g\gamma_5\Phi}\Gamma_\mu C_\mu\bigr). \tag{2.29}$$

After a suitable regularization the fermionic Jacobian reads [15]

$$
J[\Phi, \eta] = \exp \frac{g^2}{2\pi}
$$
$$
\times \int \frac{d^2 p}{(2\pi)^2} \left[-p_1^2 \Phi(p)\Phi(-p) + p_1^2 \eta(p)\eta(-p) - 2 p_0 p_1 \Phi(p)\eta(-p) \right].
$$
(2.30)

The vacuum to vacuum functional is then expressed as

$$
\mathcal{Z} = \mathcal{N}' \int \mathcal{D}\Phi \, \mathcal{D}\eta \, \mathcal{D}C_\mu \, e^{-(S[\Phi,\eta]+S[C_\mu])} J[\Phi, \eta] \det\!\big(i\slashed{\partial} - g' e^{2g\gamma_5 \Phi} \Gamma_\mu C_\mu\big), \quad (2.31)
$$

where $S[\Phi, \eta]$ arises when one inserts (2.28) in $S[A_\mu]$ (see Eq. (2.23)):

$$
S[\Phi, \eta] = \int \frac{d^2 p}{(2\pi)^2} \big[\Phi(p)\Phi(-p)A(p) + \eta(p)\eta(-p)B(p) + \Phi(p)\eta(-p)C(p) \big]
$$
(2.32)

with

$$
A(p) = \frac{1}{2}\left[p_0^2 \widehat{V}_{(1)}^{-1}(p) + p_1^2 \left(\widehat{V}_{(0)}^{-1}(p) + \frac{g^2}{\pi} \right) \right],
$$
(2.33)

$$
B(p) = \frac{1}{2}\left[p_0^2 \widehat{V}_{(0)}^{-1}(p) + p_1^2 \left(\widehat{V}_{(1)}^{-1}(p) - \frac{g^2}{\pi} \right) \right],
$$
(2.34)

$$
C(p) = p_0 p_1 \left(\widehat{V}_{(0)}^{-1}(p) - \widehat{V}_{(1)}^{-1}(p) + \frac{g^2}{\pi} \right).
$$
(2.35)

The fermionic determinant in the above expression can be analyzed in terms of a formal perturbative expansion. Indeed, taking g' as perturbative parameter, and using the fermionic fields χ and $\bar{\chi}$ defined in (2.26) one can write

$$
\mathcal{Z}_F = \sum_{n=0}^{\infty} \frac{g'^n}{n!} \left\langle \prod_{j=1}^{n} \int d^2 x_j \, \bar{\chi}(x_j) \mathsf{C}(x_j) \chi(x_j) \right\rangle_0,
$$
(2.36)

where, for later convenience we have defined

$$
\mathcal{Z}_F = \det\!\big(i\slashed{\partial} - g' e^{2g\gamma_5 \Phi} \Gamma_\mu C_\mu\big)
$$
(2.37)

and

$$
\mathsf{C} = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix}
$$
(2.38)

with

$$
\begin{cases} C_+ = (C_0 + C_1)\, e^{2g\Phi(x)}, \\ C_- = (C_0 - C_1)\, e^{-2g\Phi(x)}. \end{cases}
$$
(2.39)

By carefully analyzing each term in the series we found a selection rule quite similar to the one obtained in the path-integral treatment of $(1+1)$ massive fermions with local [16,17] and non-local interactions [18]. Indeed, due to the fact that in (2.36) $\langle \ \rangle_0$ means

v.e.v. with respect to free massless fermions, the v.e.v.'s corresponding to $j = 2k + 1$ are zero. Thus, we obtain

$$
\mathcal{Z}_F = \sum_{k=0}^{\infty} \frac{(g'c\rho)^{2k}}{(k!)^2 (2\pi)^{2k}} \int \prod_{i=1}^{k} d^2x_i\, d^2y_i \prod_{i=1}^{k} \big[C_0(x_i) + C_1(x_i) \big] \big[C_0(y_i) - C_1(y_i) \big]
$$
$$
\times \exp 2g \sum_{i=1}^{k} \big[\Phi(x_i) - \Phi(y_i) \big] \frac{\prod_{i>j}^{k} (c\rho)^4 |x_i - x_j|^2 |y_i - y_j|^2}{\prod_{i,j}^{k} (c\rho)^2 |x_i - y_j|^2}, \tag{2.40}
$$

where ρ is a normal ordering parameter and c is related to Euler's constant.

In order to obtain a bosonic description of the present problem we shall now propose the following bosonic Lagrangian density

$$
\mathcal{L}_B = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{\alpha_0}{2\beta^2}\big(m_+ e^{i\beta\varphi} + m_- e^{-i\beta\varphi} \big) \tag{2.41}
$$

with β, $m_+(x)$ and $m_-(x)$ to be determined. The quantity α_0 is just a constant that we include to facilitate comparison of our procedure with previous works on local bosonization [16,17]. Please notice that for $m_+ = m_- = 1$ this model coincides with the well-known sine-Gordon model that can be used to describe a neutral Coulomb gas. In this context α_0/β^2 is nothing but the corresponding fugacity [19]. We shall now consider the partition function

$$
\mathcal{Z}_B = \int \mathcal{D}\varphi\, e^{-\int d^2x\, \mathcal{L}_B} \tag{2.42}
$$

and perform a formal expansion taking the fugacity as perturbative parameter. It is quite straightforward to extend the analysis of each term, already performed for $m_+ = m_- = 1$, to the present case in which these objects are neither equal nor necessarily constants. The result is

$$
\mathcal{Z}_B = \sum_{l=1}^{\infty} \frac{1}{(l!)^2} \left(\frac{\alpha_0}{2\beta^2} \right)^{2l} \int \left(\prod_{i=1}^{l} d^2x_i\, d^2y_i \right) \left(\prod_{i=1}^{l} m_+(x_i)\, m_-(y_i) \right) \left(\frac{\rho}{\Lambda} \right)^{2l\frac{\beta^2}{4\pi}}
$$
$$
\times \frac{\prod_{i>j}^{l} [(c\rho)^2 |x_i - x_j| |y_i - y_j|]^{\beta^2/2\pi}}{\prod_{i,j}^{l} [(c\rho) |x_i - y_j|]^{\beta^2/2\pi}}. \tag{2.43}
$$

Comparing this result with Eq. (2.40), we see that both series coincide if the following identities hold:

$$
\beta = \pm 2\sqrt{\pi}, \qquad \frac{\alpha_0}{\beta^2} = \frac{g'\Lambda c}{\pi} \tag{2.44}
$$

and

$$
m_+(x_i) = \big(C_0(x_i) + C_1(x_i) \big) e^{2g\Phi(x_i)},
$$
$$
m_-(y_i) = \big(C_0(y_i) - C_1(y_i) \big) e^{-2g\Phi(y_i)}. \tag{2.45}
$$

Thus we have found a bosonic representation for the fermionic determinant (2.40). This is given by (2.42) together with the identities (2.44) and (2.45). Let us emphasize

that Eqs. (2.44) are completely analogous to the bosonization formulae first obtained by Coleman [16] whereas Eqs. (2.45) constitute a new result, specially connected to the present problem.

3. Derivation of gap equations through self consistent harmonic approximation

Inserting (2.42) in (2.31) we can express the partition function of this system in terms of five scalars: Φ, η, C_0, C_1 and φ. We are thus led to a completely bosonized action S_{bos}:

$$
\begin{aligned}
S_{\text{bos}} = \int \frac{d^2 p}{(2\pi)^2} &\left[\Phi(p)\Phi(-p)A(p) + \eta(p)\eta(-p)B(p) \right. \\
&\left. + \Phi(p)\eta(-p)C(p) + \varphi(p)\varphi(-p)\frac{p^2}{2} \right] \\
&+ \frac{1}{2} \int d^2x\, d^2y\, C_\mu(x) U_{(\mu)}^{-1}(x,y) C_\mu(y) \\
&+ \frac{g'\Lambda c}{\pi} \int d^2x \left[C_{(0)}(x) f_0(x) + i C_{(1)}(x) f_1(x) \right],
\end{aligned}
\tag{3.1}
$$

where

$$
f_0(x) = \cos\big((\sqrt{4\pi}\,\varphi - 2ig\Phi)(x)\big), \tag{3.2}
$$
$$
f_1(x) = \sin\big((\sqrt{4\pi}\,\varphi - 2ig\Phi)(x)\big). \tag{3.3}
$$

Since the integrals in C_0 and C_1 are quadratic these fields are easily integrated out and one gets

$$
\mathcal{Z} = \mathcal{N} \int \mathcal{D}\Phi\, \mathcal{D}\eta\, \mathcal{D}\varphi\, e^{-S_{\text{eff}}[\Phi,\eta,\varphi]} \tag{3.4}
$$

with

$$
S_{\text{eff}}[\Phi,\eta,\varphi] = S_0 + S_{\text{int}}, \tag{3.5}
$$

where

$$
\begin{aligned}
S_0 = \int \frac{d^2 p}{(2\pi)^2} &\left[\Phi(p)\Phi(-p)A(p) + \eta(p)\eta(-p)B(p) \right. \\
&\left. + \Phi(p)\eta(-p)C(p) + \varphi(p)\varphi(-p)\frac{p^2}{2} \right],
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
S_{\text{int}} = &-\frac{(\Lambda c)^2}{2\pi^2} \int d^2x\, d^2y\, g_1(x,y) \cos\big[\sqrt{4\pi}\,(\varphi(x) - \varphi(y)) - 2ig(\Phi(x) - \Phi(y))\big] \\
&-\frac{(\Lambda c)^2}{2\pi^2} \int d^2x\, d^2y\, g_3(x,y) \cos\big[\sqrt{4\pi}\,(\varphi(x) + \varphi(y)) - 2ig(\Phi(x) + \Phi(y))\big].
\end{aligned}
\tag{3.7}
$$

It is now convenient to diagonalize the quadratic part of the effective action by introducing the fields ζ, χ and ξ:

$$\Phi = \frac{i\zeta}{\tilde{g}} + \frac{2i\tilde{g}Bp^2}{\Delta + 2B\tilde{g}^2 p^2}\xi, \tag{3.8}$$

$$\eta = \frac{-iC}{2B\tilde{g}}\zeta - \frac{i\tilde{g}Cp^2}{\Delta + 2B\tilde{g}^2 p^2}\xi + \frac{1}{g}\chi, \tag{3.9}$$

$$\varphi = -\zeta + \frac{\Delta}{\Delta + 2B\tilde{g}^2 p^2}\xi, \tag{3.10}$$

where we have defined $\tilde{g}^2 = g^2/\pi$ and $\Delta(p) = C(p)^2 - 4A(p)B(p)$. We then obtain

$$S_0 = \frac{1}{2}\int \frac{d^2 p}{(2\pi)^2}\left[\zeta(p)\left(p^2 + \frac{\Delta}{2B\tilde{g}^2}\right)\zeta(-p) + \chi(p)\frac{2B}{\tilde{g}^2}\chi(-p)\right.$$
$$\left. + \xi(p)\frac{p^2\Delta}{\Delta + 2B\tilde{g}^2 p^2}\xi(-p)\right], \tag{3.11}$$

$$S_{\text{int}} = -\frac{(\Lambda c)^2}{2\pi^2}\int d^2 x\, d^2 y\, g_1(x, y)\cos\sqrt{4\pi}\left[\xi(x) - \xi(y)\right]$$
$$- \frac{(\Lambda c)^2}{2\pi^2}\int d^2 x\, d^2 y\, g_3(x, y)\cos\sqrt{4\pi}\left[\xi(x) + \xi(y)\right]. \tag{3.12}$$

One can see that the ζ and χ fields become completely decoupled from ξ. Moreover, it becomes apparent that the ξ-dependent piece of the action S_{int} is the only one containing relevant contributions (i.e., gapped modes)

$$S[\xi] = \int \frac{d^2 p}{(2\pi)^2}\xi(p)\frac{F(p)}{2}\xi(-p)$$
$$- \frac{(\Lambda c)^2}{2\pi^2}\int d^2 x\, d^2 y\, g_1(x, y)\cos\sqrt{4\pi}\left[\xi(x) - \xi(y)\right]$$
$$- \frac{(\Lambda c)^2}{2\pi^2}\int d^2 x\, d^2 y\, g_3(x, y)\cos\sqrt{4\pi}\left[\xi(x) + \xi(y)\right] \tag{3.13}$$

with

$$F(p) = \frac{1}{Kv}\left(p_0^2 + v^2 p_1^2\right), \tag{3.14}$$

$$K = \sqrt{\frac{1 + g_4/\pi - g_2/\pi}{1 + g_4/\pi + g_2/\pi}}, \tag{3.15}$$

$$v = \sqrt{\left(1 + \frac{g_4}{\pi} + \frac{g_2}{\pi}\right)\left(1 + \frac{g_4}{\pi} - \frac{g_2}{\pi}\right)}, \tag{3.16}$$

where we have expressed all formulae in terms of Solyom's g coupling functions. In the local case, a renormalizaton group (RG) analysis shows that the "stiffness constant", K has to be lower than 0.5 in order to have a relevant cosine interaction, i.e., to have a gap in the spectrum.

Eqs. (3.11), (3.12) and (3.13) constitute our first non-trivial result. Even though this action could be obtained by standard operational methods, as far as we know it has not

been explicitly derived before. Unfortunately, even if we succeeded in simplifying the original action, $S[\xi]$ is not yet soluble, except for the local case, where it is the well-known integrable sine-Gordon theory. In order to analyze the physical content of our model, we introduce a non-local version of the self-consistent harmonic approximation [14]. Basically, this amounts to replacing the so-called *true* action (3.13) by a *trial* action in which the cosine terms are approximated as

$$-\frac{(\Lambda c)^2}{2\pi^2}g_{1,3}(x,y)\cos\sqrt{4\pi}\big[\xi(x)\pm\xi(y)\big] \longrightarrow \frac{\Omega_{1,3}(x,y)}{2}\xi(x)\xi(y), \qquad (3.17)$$

where the functions Ω_i of the trial action can be variationally determined (see Appendix A for details). We will consider two cases: pure backscattering ($g_1 \neq 0$, $g_3 = 0$) and pure umklapp scattering ($g_3 \neq 0$, $g_1 = 0$). Once this is done, it is straightforward to obtain the charge spectrum. Indeed, going to momentum space, and back to real frequencies, $p_0 = i\omega$, $p_1 = k$, the following equations are obtained:

$$Kv\Omega_i - \omega^2 + v^2 k^2 = 0. \qquad (3.18)$$

The gap equations satisfied by Ω_i are

$$\Omega_{1,3}(p) = \frac{4\Lambda^2 c^2}{\pi} \int d^2x \, g_{1,3}(x) e^{-4\pi[I_1(x)\mp I_1(0)]}\big(e^{ipx} \mp 1\big), \qquad (3.19)$$

where

$$I_1(x) = \int \frac{d^2p}{(2\pi)^2} \frac{e^{-ipx}}{F(p) + \Omega_i(p)}. \qquad (3.20)$$

Eq. (3.19) is our main formal result. It gives, within the SCHA, a closed expression for the gap as functional of any arbitrary strength of the interactions (the $g_i's$). Let us study, as examples, the cases where umklapp and backscattering are local interactions (i.e., $g_{1\atop 3}(x) = g_{1\atop 3}\delta^2(x)$), but keeping arbitrary fs interactions. It is easy to see that the pure backscattering process is gapless ($\Omega_1 = 0$) and that the umklapp interaction opens a gap in the charge density spectrum ($\Omega_3 = $ constant), as expected [12,13]. For this last case we obtain

$$\Omega_3 = \frac{8\Lambda^2 c^2}{\pi}g_3 e^{-8\pi I_1(0)}. \qquad (3.21)$$

As Ω_3 is a constant, i.e., it does not depend on p we can calculate the propagator in terms of Ω_3, and the equation then reduces to an algebraic equation for Ω_3. If we consider also contact fs potentials (i.e., $g_{2,4}(x) = g_{2,4}\delta^2(x)$), the integral $I_1(0)$ can be explicitly performed. After a suitable regularization one obtains:

$$\Omega_3 = \frac{4\Lambda^2}{K}\left(\frac{2c^2 K}{\pi}g_3\right)^{1/(1-2K)}. \qquad (3.22)$$

An equation completely analogous to (3.21) was found in [8] by working directly with a local coupling associated to spin-flipping interactions in that case. This provides a consistency check for our computation. Of course, having derived a general formula for the gap, like (3.19), it is interesting to analyze the effect of a non-contact interaction. We present this study in the next section.

4. Gap of charge-density modes for a non-local umklapp coupling

The purpose of this section is to find a solution to Eq. (3.19) for a pure umklapp potential of the form

$$g_3(p) = \lambda_0 + \frac{\epsilon}{\Lambda^2} p_1^2 \tag{4.1}$$

being $\epsilon \ll 1$. It consists of the local potential plus a small non-local correction. We start by writing the potential in coordinate space

$$g_3(x) = \lambda_0 \delta^2(x) - \frac{\epsilon}{\Lambda^2} \partial_1^2 \delta^2(x), \tag{4.2}$$

where $\partial_1^2 \delta^2(x)$ is the second derivative of the delta function with respect to x_1 only. By replacing this function in (3.19) and integrating we obtain

$$\Omega(p) = \frac{4\Lambda^2 c^2}{\pi} \left[2\lambda_0 + 8\pi \frac{\epsilon}{\Lambda^2} \partial_1^2 I_1(0) \right] e^{-8\pi I_1(0)} + \frac{4c^2 \epsilon}{\pi} e^{-8\pi I_1(0)} p_1^2, \tag{4.3}$$

where we have dropped the subindex 3 in $\Omega(p)$. We observe that Ω retains the same structure as $g_3(p)$, i.e., it is of the form $\Omega(p) = \mu^2 + g p_1^2$, where μ^2 and g are two parameters to be determined in terms of λ_0 and ϵ. The fact that Ω preserves its simple form allows to calculate the propagator, as occurs in the local case, and in consequence, to transform the equation into an algebraic one, easier to deal with. Introducing the adimensional quantity $x = \mu^2 K v / 4\Lambda^2 (v^2 + Kgv)$, and considering the case $\epsilon \ll 1$, we get the following pair of equations to be solved for g and x:

$$x = \frac{2c^2 K}{\pi} x^{2K} \left[\lambda_0 - \epsilon K (2x \ln ex + 1) \right], \tag{4.4}$$

$$g = \frac{4c^2 \epsilon}{\pi} x^{2K}. \tag{4.5}$$

It is straightforward to obtain plots of λ_0 as a function of x for fixed K and ϵ. These are shown in Figs. 1 and 2 for $K = 0.2$ and $K = 0.4$ respectively and different values of ϵ. We have taken $g_4 = 0$ in both cases. We observe that increasing the value of K leads to a bigger value for the gap (for fixed ϵ, x increases with K). On the other hand, for fixed K, and a given value of λ_0, the gap grows as ϵ decreases.

Alternatively, we can obtain an approximate analytical solution to the equations by employing a perturbative approach. Writing $x = x_0 + \delta x$, where $x_0 = (2c^2 K \lambda_0 / \pi)^{1/(1-2K)}$ is the solution for the $\epsilon = 0$ case, we find

$$\delta x = \frac{-2\epsilon x_0 (2x_0 \ln ex_0 + 1)}{\lambda_0 (1 - 2K)}, \tag{4.6}$$

$$g = \frac{2\epsilon x_0}{K \lambda_0}. \tag{4.7}$$

The first equation gives the correction to the gap of charge-density modes due to non-contact umklapp interactions. If one introduces a small non-local perturbation, it is enough
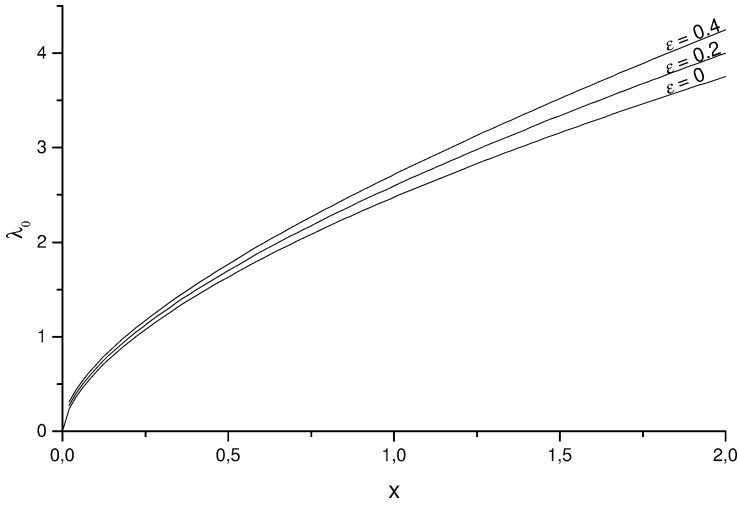
Fig. 1. λ_0 as a function of x for fixed ϵ and $K = 0.2$.
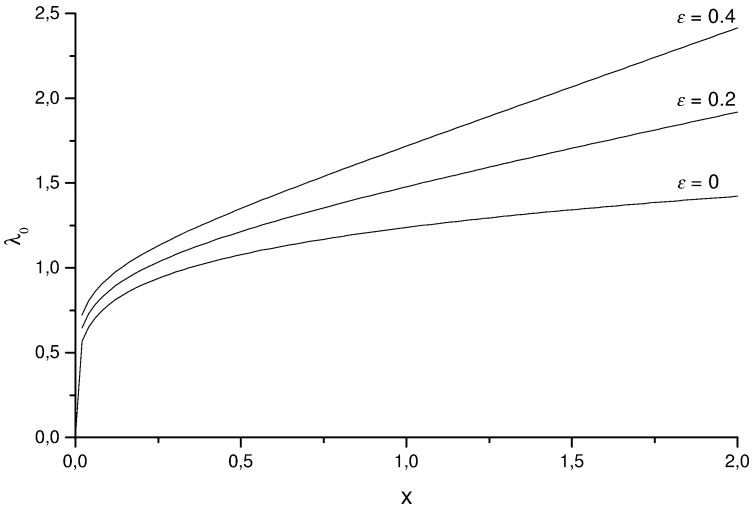

Fig. 2. λ_0 as a function of x for fixed ϵ and $K = 0.4$.

to modify the value of the gap. The numerator in Eq. (4.6) is always negative and then, since $K < 0.5$, this gives a gap which is decreasing with ϵ, a result consistent with the numerical approach. The second equation gives the correction to the kinetic part of the spectrum. As a last comment, we should mention that the computation of the gap can be also performed by using the RG approach, following for instance, the lines of Ref. [20]. The corresponding results are qualitatively consistent with ours, although a precise comparison is beyond the scope of this work.

5. Final result and next steps

In this paper we have presented a non-local and non-covariant extension of the Thirring model which can be used to describe a 1d many-body system when not only forward but also backward and umklapp scattering is considered. In this sense our results improve previous formulations reported in [7] and [11]. We were able to write the vacuum to vacuum functional of this model in terms of a non-trivial fermionic determinant. We obtained a bosonic representation for this determinant which led us to an effective action associated to the elementary collective excitations of the system. By employing the self-consistent harmonic approximation we found a closed expression for the gap of the charge-density spectrum as functional of arbitrary (not necessarily local) two-body potentials. This equation is valid for purely backward ($g_1 \neq 0$, $g_3 = 0$) or pure umklapp scattering ($g_3 \neq 0$, $g_1 = 0$). As an application of this formula we studied the effect of a non-local umklapp potential of the form $g_3(p) = \lambda_0 + (\epsilon/\Lambda^2)p_1^2$ for ϵ small. Besides its illustrative purpose, the result obtained in this example is interesting in itself because most of the previous investigations involving umklapp scattering do not consider non-local effects [21].

Acknowledgements

Appendix A. Details of the SCHA method

We shall give an sketch of the SCHA method. One usually starts from a partition function

$$Z_{\text{true}} = \int \mathcal{D}\mu \, e^{-S_{\text{true}}}, \tag{A.1}$$

where $\mathcal{D}\mu$ is a generic integration measure. An elementary manipulation leads to

$$Z_{\text{true}} = \frac{\int \mathcal{D}\mu \, e^{-(S_{\text{true}} - S_{\text{trial}})} \, e^{-S_{\text{trial}}}}{\int \mathcal{D}\mu \, e^{-S_{\text{trial}}}} \int \mathcal{D}\mu \, e^{-S_{\text{trial}}} = Z_{\text{trial}} \langle e^{-(S_{\text{true}} - S_{\text{trial}})} \rangle_{\text{trial}} \tag{A.2}$$

for any trial action S_{trial}. Now, by means of the property

$$\langle e^{-f} \rangle \geqslant e^{-\langle f \rangle}, \tag{A.3}$$

for f real, and taking natural logarithm in Eq. (A.2), we obtain Feynman's inequality [22]

$$\ln Z_{\text{true}} \geqslant \ln Z_{\text{trial}} - \langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}}. \tag{A.4}$$

We shall study a very general class of true bosonic actions of non-local sine Gordon type, given by

$$S_{\text{true}} = \int \frac{d^2 p}{(2\pi)^2} \, \varphi(p) \frac{F(p)}{2} \varphi(p) - \int d^2 x \, d^2 y \, \frac{\alpha(x - y)}{b^2} \cos b \, \frac{\varphi(x) + \varphi(y)}{2}, \tag{A.5}$$

where $F(p)$ and $\alpha(x, y)$ are arbitrary functions. The usual local sine Gordon theory is obtained by taking $F(p) = p^2$ and $\alpha(x, y) = \alpha_0 \delta^{(2)}(x - y)$ whereas (A.5) reduces to our model (3.13) without backscattering interactions (i.e., $g_1 = 0$) by taking

$$F(p) = \frac{1}{K v} \left(p_0^2 + v^2 p_1^2 \right),$$ (A.6)

$$b^2 = 16\pi,$$ (A.7)

$$\frac{\alpha(x, y)}{b^2} = \frac{(\Lambda c)^2}{2\pi^2} g_3(x, y).$$ (A.8)

The trial action we will consider is a quadratic one:

$$S_{\text{trial}} = \int \frac{d^2 p}{(2\pi)^2} \, \varphi(p) \frac{F(p)}{2} \varphi(p) - \int d^2 x \, d^2 y \, \frac{\Omega(x - y)}{2} \varphi(x) \varphi(y),$$ (A.9)

where the function Ω can be determined by maximizing the right-hand side of Eq. (A.4). In order to achieve this goal we first write

$$\ln \mathcal{Z}_{\text{trial}} = \ln \int \mathcal{D}\varphi \exp\left[-\frac{1}{2} \int d^2 x \, \varphi(x) (\widehat{A}\varphi)(x) \right]$$ (A.10)

$$= \ln(\det \widehat{A})^{-1/2} + \text{const}$$ (A.11)

$$= -\frac{1}{2} \operatorname{tr} \ln \widehat{A} + \text{const},$$ (A.12)

where the operator $\widehat{A}$ is defined, in Fourier space, by

$$(\widehat{A}\varphi)(p) = \left[F(p) + \Omega(p) \right] \varphi(p)$$ (A.13)

being $\Omega(p)$ is the Fourier transform of $\Omega(z)$. It is then easy to get

$$\operatorname{tr} \ln \widehat{A} = \mathcal{V} \int \frac{d^2 p}{(2\pi)^2} \ln\left[F(p) + \Omega(p) \right] \equiv \mathcal{V} I_0[\Omega],$$ (A.14)

where $\mathcal{V}$ is the volume (infinite) of the whole space $\int d^2 x$. On the other hand, it is straightforward to compute $\langle S_{\text{true}} - S_{\text{trial}} \rangle$, by following, for instance, the steps explained in Ref. [18]. The result is

$$-\langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}} = \mathcal{V} \int d^2 x \left[\frac{\alpha(x)}{b^2} e^{-\frac{1}{4} b^2 [I_1(x) + I_1(0)]} + \frac{\Omega(x)}{2} I_1(x) \right],$$ (A.15)

where $I_1(x)$ is a functional of Ω given by the propagator of the trial action

$$I_1(x) = \int \frac{d^2 p}{(2\pi)^2} \frac{e^{-ipx}}{F(p) + \Omega(p)}$$ (A.16)

and we have made use of the translational invariance of α and Ω. Finally we can gather all the terms, and write them as

$$\ln \mathcal{Z}_{\text{trial}} - \langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}} = \frac{\mathcal{V}}{2} \int d^2 x \frac{\alpha(x)}{b^2} e^{-\frac{1}{4} b^2 [I_1(x) + I_1(0)]}$$

$$+ \frac{\mathcal{V}}{2} \int \frac{d^2 p}{(2\pi)^2} \frac{\Omega(-p)}{F(p) + \Omega(p)} - \frac{\mathcal{V}}{2} I_0[\Omega] + \text{const},$$ (A.17)

where we have turned the last term of Eq. (A.15) into Fourier space. Now extremizing expression (A.17) with respect to Ω by functional derivation with respect to $\Omega(k)$, we finally obtain the gap equation

$$\Omega(p) = \frac{1}{2} \int d^2x \, \alpha(x) e^{-\frac{1}{4}b^2[I_1(x)+I_1(0)]} \big(e^{-ipx} + 1\big). \tag{A.18}$$

If instead of considering only umklapp interactions we consider only backscattering interactions ($g_3 = 0$) then we have to change $\varphi(x) + \varphi(y) \rightarrow \varphi(x) - \varphi(y)$ into the cosine of Eq. (A.5), and the resulting gap equation is

$$\Omega(p) = \frac{1}{2} \int d^2x \, \alpha(x) e^{-\frac{1}{4}ob^2[I_1(x)-I_1(0)]} \big(e^{-ipx} - 1\big). \tag{A.19}$$

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