On the lattice structure of probability spaces in quantum mechanics

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Abstract

Let $\mathcal{C}$ be the set of all possible quantum states. We study the convex subsets of $\mathcal{C}$ with attention focused on the lattice theoretical structure of these convex subsets and, as a result, find a framework capable of unifying several aspects of quantum mechanics, including entanglement and Jaynes' Max-Ent principle. We also encounter links with entanglement witnesses, which leads to a new separability criteria expressed in lattice language. We also provide an extension of a separability criteria based on convex polytopes to the infinite dimensional case and show that it reveals interesting facets concerning the geometrical structure of the convex subsets. It is seen that the above mentioned framework is also capable of generalization to any statistical theory via the so-called convex operational models' approach. In particular, we show how to extend the geometrical structure underlying entanglement to any statistical model, an extension which may be useful for studying correlations in different generalizations of quantum mechanics.

Key words: entanglement-quantum information-convex sets -MaxEnt approach

1 Introduction

In this work we will tackle the entanglement phenomenon from a special viewpoint, that of regarding quantum states as "probability measures" [see for example [1]], which leads us to discuss convex sets of probability measures. Quantum probabilities are of a very different nature than that of classical ones. After setting preliminary mathematical notions and notations in section 2 (which may be optionally complemented with appendix A), we shortly review the differences and definitions between classical and quantal probabilities in section 3. Preliminary matters may be skipped by the reader familiarized with the quantum formalism in infinite dimensions.

The existence of probability models of a very different nature, of which the classical and the quantum instances are just two particularly important examples of a wider family, was one of the motivations for the study of the so called operational or convex approach (COM), one of the protagonists of our present discourse. Refs. [2, 3, 4, 5, 6, 7] deal with the subject of COMs, for which states (understood as probability measures) and their convex structure play a key role, while other related quantities emerge in rather natural fashion. Note that there exist generalizations of quantum mechanics, including non-linear versions, that are axiomatized using the convex structure of the set of states (see [8], [9], and [10]). The approach treats in geometrical fashion the statistical theory of systems, which includes quantum and classical mechanics (and several other theories as well). In these generalized probabilistic models, generalized observables are used. In the particular case of quantum theory, one encounters the important notion of Positive Operator Valued Measures (POVM's). We will review the COM approach as well as POVM's in section 4. In section 5 we will revisit the formal structure and associated definitions of entanglement including the infinite dimensional case.

Lattices are other main character in our present discourse. They have been studied in the context of quantum mechanics since the seminal paper of von Newmann [11] and characterize the structure of the subspaces of the Hilbert space of a quantum system. The paper of reference [11] has motivated several investigations in logics, philosophy [12], foundations of physics, and algebraic logic. In the particular case of the foundations of quantum mechanics, several lines of investigation have been developed. It is difficult to list all of them. We just cite here [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. For a complete bibliography see for example [25], [26], and [27].

Part of the study of composite quantum systems was developed in [28, 29, 30]. As we will show in sections 6 and 7 of this work, the convex set of quantum states is endowed with canonical lattice structures, which allow us to disclose a new structural feature of quantum mechanics that, in particular, allows for an extension to the infinite dimensional case of the standpoint developed in [31, 32]. The reader non familiarized with lattice theory may find it useful to take glance at appendix B.

Our new lattice structures are not only mathematical curiosities. Instead, they are the key factor for achieving a unifying viewpoint regarding several constructs linked to the geometrical properties of the quantum set of states. As a first example of their power, we will use this approach to provide a generalization and reformulation of the celebrated Jaynes Max-Ent principle [33, 34] to arbitrary COM's in section 7.1 (see also [35] for more developments), thus displaying an interesting convergence between lattice theory and COM approaches.

In section 8 we restrict ourselves to the finite dimensional case to study a particular reformulation of entanglement witnesses in lattice theoretical terms, which provides new proofs of known results. Surprisingly enough, these new proofs serve i) as the source of new abstract entanglement criteria, which can be expressed in lattice "format" and ii) to study the volume of the space of separable states, a theme to be tackled elsewhere. We added appendix C for refreshing mathematical notions indispensable in this respect.

We return in section 9 to the infinite dimensional case to discuss the problem of characterizing entanglement, both from the geometrical and algebraic viewpoints. Emphasis will be put on maps that can be defined between the lattice of the system and its subsystems. We will extend to infinite dimension a recently advanced, abstract entanglement criterium [36] and study some consequences thereof. In particular, we will underline an interesting unifying characteristic of entanglement that is known to hold for pure states and reads

A *pure* state is separable$\Longleftrightarrow$ it is a product state ($\Longleftrightarrow$ the entropy of its reduced states is minimal)

It is also well known that no such a simple statement is valid for mixed states. Using both the lattice theoretical approach and our criteria we will show that it is possible to suitably generalize the above mentioned assertion from pure states to arbitrary states, thus leaving the pure instance as a particular case. In order to do so, we will introduce first the notion of informational invariant (advanced in [36]). This concept

- is advantageously cast in purely geometrical terms and holds for the infinite dimensional case as well, uncovering non trivial geometric and algebraic properties, and
- also provides us with a simple, unifying abstract framework to characterize separability properties of arbitrary states (not only pure ones).

Furthermore, we will extend in section 10 some of our results to any probabilistic model via the COM approach. In particular, we discuss how the geometrical structure found for the quantum case in section 9 can be extended via the COM approach to any probabilistic model. Such generalization is due to the purely geometrical nature of our criteria, and may be useful to define entanglement for theories more general than that of quantum mechanics (for example, semiclassical models or non-linear versions of quantum mechanics). Finally, in section 11 some conclusions are drawn.

2 Preliminaries

For a Hilbert space $\mathcal{H}$ of dimension $N > 2$ the set of pure states forms a $(2N-2)$-dimensional manifold, of measure zero, in the $(N^2-2)-$dimensional boundary $\partial\mathcal{C}_N$ of the set $\mathcal{C}_N$ of density matrices. The set of mixed quantum states $\mathcal{C}_N$ consists of Hermitian, positive matrices of size N, normalized by the trace condition, that is

$$\mathcal{C}_N = \{\rho : \rho = \rho^{\dagger};\ \ \rho \geq 0;\ \ tr(\rho) = 1;\ \ dim(\rho) = N\}. \tag{1}$$

It can be shown for finite dimensional bipartite states that there exist always a non-zero measure μ_s in the neighborhood of separable states containing maximum uncertainty ones. μ_s tends to zero as the dimension tends to infinity. Finally, for an infinitely dimensional Hilbert space almost all states are entangled (i. e., separable states are *never dense*) [37, 38].

2.1 Notation

Let us fix the notation to be employed here. $\mathcal{P}(\mathcal{H})$ will denote the set of all closed subspaces of a Hilbert space $\mathcal{H}$ (arbitrary dimension), which are in a one to one correspondence with the projection operators. Because of this one to one link, one usually employs the notions of "closed subspace" and "projector" in interchangeable fashion. An important construct is $\mathcal{A}$, the set of bounded Hermitian operators on $\mathcal{H}$, while the bounded operators on $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$. The projective Hilbert space $\mathbf{CP}(\mathcal{H})$ of a complex Hilbert space $\mathcal{H}$ is the set of equivalence classes of vectors v in $\mathcal{H}$, with $v \neq 0$, given by $v \sim w$ when $v = \lambda w$, with λ a non-zero scalar. Here the equivalence classes for $\sim$ are also called projective rays. A trace class operator is a compact one for which a finite trace may be defined (independently of the choice of basis). We will appeal below to the set $\mathcal{C}$ containing all positive, hermitian, and trace-class (normalized to unity) operators in $\mathcal{B}(\mathcal{H})$.
Let us remind the reader that a lattice $\mathcal{L}$ is a partially ordered set (also called a poset) in which any two elements a and b have a unique supremum (the elements' least upper bound "$a \vee b$"; called their join) and an infimum (greatest lower bound "$a \wedge b$"; called their meet). Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the

two definitions are equivalent, lattice theory draws on both order theory and universal algebra. For additional details, see Appendix B.

Let $\mathcal{H}$ be a separable Hilbert space of arbitrary dimension representing a quantum system. As stated above, the bounded operators on $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$. There are many topologies and relevant subsets of $\mathcal{B}(\mathcal{H})$. In the literature, $\mathcal{A}$ has denoted different subsets of $\mathcal{B}(\mathcal{H})$. While mainly it denotes the Hermitian operators, in some works it denotes the Hilbert-Schmidt operators [39]. In this work we will use the following notation

$$\mathcal{A} = \{T \in \mathcal{B}(\mathcal{H}) : T^\dagger = T\}$$

Suppose that T is a compact operator such that

$$\sum_{i \in I} \langle v_i | T v_i \rangle < \infty \tag{2}$$

for all orthonormal basis $\{|v_i\rangle\}_{i \in I}$. Then, the map $\mathrm{tr}(\cdot)$ defined as

$$\mathrm{tr}(T) = \sum_{i \in I} \langle v_i | T v_i \rangle \tag{3}$$

is independent of the choice of basis. The set of Hilbert Schmidt operators will be denoted by $\mathcal{B}_2(\mathcal{H})$ and are defined by

$$\mathcal{B}_2 = \{T \in \mathcal{B}(\mathcal{H}) : \mathrm{tr}(T^2) < \infty\}.$$

The space $\mathcal{B}_2$ endowed with the inner product $\langle T_1, T_2 \rangle = \mathrm{tr}(T_2^\dagger T_1)$ is a Hilbert space. For $T \in \mathcal{B}(\mathcal{H})$ the absolute value of T is defined by $|T| = (T^\dagger T)^{1/2}$. We can also consider the subspace formed by the trace class operator, defined by

$$\mathcal{B}_1 = \{T \in \mathcal{B}(\mathcal{H}) : |T|^{1/2} \in \mathcal{B}_2(\mathcal{H})\}.$$

It can be shown that the following statements are equivalent:

1. $T \in \mathcal{B}_1$.
2. $T = AB$ for $A, B \in \mathcal{B}_2(\mathcal{H})$.
3. $|T| \in \mathcal{B}_1(\mathcal{H})$.
4. $\mathrm{tr}(|T|) < \infty$.

The space of trace class operators is a Banach space endowed with the norm $\|T\| = \mathrm{tr}(|T|)$. Notice that in $\mathcal{B}_2(\mathcal{H}) \cap \mathcal{A}$, the norm induced by the inner product is given by $\|T\| = \mathrm{tr}(T^\dagger T)^{1/2} = \mathrm{tr}(T^2)^{1/2}$ and it coincides with the ℓ^2 norm of the eigenvalues while in $\mathcal{B}_1(\mathcal{H}) \cap \mathcal{A}$ the norm coincides with the ℓ^1 norm of the eigenvalues. Thus, $\mathcal{B}_2 \subset \mathcal{B}_1$ in $\mathcal{A}$.
Coming back to the closed subspaces of $\mathcal{H}$ which are in a one to one correspondence with the projection operators, an operator $P \in \mathcal{B}(\mathcal{H})$ is said to be a projector if it satisfies

$$P^2 = P \tag{4}$$

and

$$P = P^\dagger \tag{5}$$

2.2 Elementary measurements and projection operators

A projection operator represents an elementary measurement given by a yes-no experiment, i.e., a test in which we get the answer "yes" or the answer "no". If $\mathcal{R}$ is the real line, let $B(\mathcal{R})$ be the family of subsets of $\mathcal{R}$ such that

- 1 - The family is closed under set theoretical complements.
- 2 - The family is closed under denumerable unions.
- 3 - The family includes all open intervals.

The elements of $B(\mathcal{R})$ will be called the *Borel subsets* of $\mathcal{R}$ [40]. A projection valued measure (PVM) M, is a mapping

$$M : B(\mathcal{R}) \to \mathcal{P}(\mathcal{H}) \tag{6a}$$

such that

$$M(0) = 0 \tag{6b}$$

$$M(\mathcal{R}) = \mathbf{1} \tag{6c}$$

$$M(\cup_j(B_j)) = \sum_j M(B_j), \tag{6d}$$

for any disjoint family B_j. Also,

$$M(B^c) = \mathbf{1} - M(B) = (M(B))^{\perp} \tag{6e}$$

All operators representing observables may be expressed in terms of projection operators (and so by elementary measurements) via the spectral decomposition theorem, which asserts that the set of spectral measurements may be put in a bijective correspondence with the set $\mathcal{A}$ of Hermitian operators of $\mathcal{H}$. A list of set-theory concepts used in this work can be found in Appendix A Elementary (sharp) tests in quantum mechanics are represented by projection operators that form the well known von Newmann's lattice $\mathcal{L}_{vN}$, an orthomodular one (see Appendix B). The Born-rule implies that probabilities in quantum mechanics are linked to measures over the von Newmann's lattice. Using Gleason's theorem, it is possible to link in a bijective way density matrixes and non-kolmogorovian probability measures (more on this below).

3 Quantum vs. classical probabilities

The reader is advised to consult the Appendixes regarding some mathematics concepts appealed to below. It is a well known fact, since the 30's, that a quantum system represented by a Hilbert space $\mathcal{H}$ is associated to a lattice formed by all its closed subspaces $\mathcal{L}_{v\mathcal{N}}(\mathcal{H}) =< \mathcal{P}(\mathcal{H}),\ \cap,\ \oplus,\ \neg,\ 0,\ 1 >$, where 0 is the empty set $\emptyset$, 1 the total space $\mathcal{H}$, $\cap$ the intersection, $\oplus$ the closure of the sum, and $\neg(\mathcal{S})$ the orthogonal complement of a subspace $\mathcal{S}$ [41]. This is the Hilbert lattice, named "Quantum Logic" by Birkhoff and von Neumann [11]. We will refer to this lattice as $\mathcal{L}_{v\mathcal{N}}$, the 'von Neumann lattice'. Thus, the set of elementary yes-no tests has an orthomodular lattice structure, which is itself non-boolean, modular in the finite dimensional case, and never modular in the infinite one. We will relate below elementary tests to quantum probability spaces and study their lattice structure.

Given a set Ω, let us consider a σ-algebra (see Appendixes) Σ of Ω. Then, a probability measure will be given by a function μ such that

$$\mu : \Sigma \rightarrow [0, 1] \tag{7a}$$

which satisfies

$$\mu(\emptyset) = 0 \tag{7b}$$

$$\mu(A^c) = 1 - \mu(A), \tag{7c}$$

where $(\ldots)^c$ means set-theoretical-complement and for any pairwise disjoint denumerable family $\{A_i\}_{i\in I}$

$$\mu(\bigcup_{i\in I} A_i) = \sum_i \mu(A_i) \tag{7d}$$

where conditions (7) are the well known axioms of Kolmogorov.
In the formulation of both classical and quantum probabilities, states can be regarded as representing consistent probability assignments [42]. In the quantum mechanics instance *this "states as mappings" visualization* is achieved via a function [41]

$$s : \mathcal{P}(\mathcal{H}) \rightarrow [0; 1] \tag{8a}$$

such that:

$$s(\mathbf{0}) = 0 \ (\mathbf{0} \text{ is the null subspace}). \tag{8b}$$

$$s(P^{\perp}) = 1 - s(P), \tag{8c}$$

and, for a denumerable and orthogonal family of projections

$$P_j,\ s(\sum_j P_j) = \sum_j s(P_j). \tag{8d}$$

The above equation defines a probability, but in fact, not a classical one, because classical probability axioms obey the Kolmogorov's axioms of equation (7). The main difference comes from the fact that the σ-algebra in (7) is boolean, while $\mathcal{P}(\mathcal{H})$ is not. Thus, quantum probabilities are also called non-kolmogorovian (or non-boolean) probability measures. The crucial fact is that, in the quantum case, we do not have a σ-algebra, but an orthomodular lattice of projections. Most importantly, Gleason's theorem [43, 44] asserts that if $dim(\mathcal{H}) \geq 3$, then:

> The set of all measures of the form (8a) can be put into one to one correspondence with the set $\mathcal{C}$ formed by all positive, hermitian and trace-class (normalized to unity) operators in $\mathcal{B}(\mathcal{H})$

More generally, consider a C^*-algebra $\mathbf{A}$. The prototypical example of a such an algebra is the algebra $\mathcal{B}(\mathcal{H})$ of bounded (equivalently continuous) linear operators defined on a complex Hilbert space. In general, a state φ will be a positive linear functional of norm equal to unity. If the algebra has a unit (as is the case for $\mathcal{B}(\mathcal{H})$), states will be given by the intersection of the closed affine hyperplane $\varphi(\mathbf{1}) = 1$ and the set of positive linear forms on $\mathbf{A}$ of norm ≤ 1 (which is compact in the topology of pointwise convergence) and then the concomitant extension of the set $\mathcal{C}$ will be a convex and compact space. If $P \in \mathcal{P}(\mathcal{H})$ the correspondence between $\rho \in \mathcal{C}$ and its induced probability measure is given by

$$s_\rho(P) = \operatorname{tr}(\rho P) \tag{9}$$

Equation (9) is essentially Born's rule. Any $\rho \in \mathcal{C}$ may be written as

$$\rho = \sum_i p_i P_{\psi_i} \tag{10}$$

where the P_{ψ_i} are one dimensional projection operators on the rays (subspaces of dimension one) generated by the vectors ψ_i and $\sum_i p_i = 1$ ($p_i \geq 0$). Thus, it is clear that $\mathcal{C}$ is a convex set. If the sum in (10) is finite, then ρ is said to be of finite range. It is important to remark that in the infinite dimensional case, the sum in (10) may be infinite in a non-trivial sense. $\mathcal{C}$ is then a set formed of non-boolean probability measures. That $\mathcal{C}$ is a closed convex set can also be seen by the fact that if we define the half-planes

$$H_x = \{\rho \in \mathcal{A} \,|\, x^\dagger \rho x < 0\} \tag{11a}$$

$$H_x^+ = \{\rho \in \mathcal{A} \,|\, x^\dagger \rho x \geq 0\} \tag{11b}$$

then

$$\mathcal{C} = \bigcap_{x \in \mathcal{H}} H_x^+ \cap \{\rho \,|\, \operatorname{tr}(\rho) = 1\} \tag{12}$$

If we consider now $\mathcal{L}_\mathcal{C}$ as the set of all convex subsets of $\mathcal{C}$, that is

Definition 3.1. $\mathcal{L}_\mathcal{C} = \{C \subseteq \mathcal{C} \mid C \textit{ is convex}\}$

then any element of $\mathcal{L}_\mathcal{C}$ will be itself a "probability space", in the sense that it is a set of non boolean probability measures closed under convex combinations (not to be confused with the usual mathematical notion of sample space). We will show that $\mathcal{L}_\mathcal{C}$ is endowed with a canonical lattice structure in Section 7.

A general (pure) state can be written as (using Dirac's notation):

$$\rho = |\psi\rangle\langle\psi| \tag{13}$$

We will denote the set of all pure states by

$$P(\mathcal{C}) := \{\rho \in \mathcal{C} \,|\, \rho^2 = \rho\} \tag{14}$$

This set is in correspondence with the rays of $\mathcal{H}$ via the association:

$$\mathcal{F} : \mathcal{CP}(\mathcal{H}) \to \mathcal{C}, \quad [|\psi\rangle] \mapsto |\psi\rangle\langle\psi| \tag{15}$$

where $\mathcal{CP}(\mathcal{H})$ is the projective space of $\mathcal{H}$, and $[|\psi\rangle]$ is the class defined by the vector $|\psi\rangle$ ($|\varphi\rangle \sim |\psi\rangle \longleftrightarrow |\varphi\rangle = \lambda|\psi\rangle$, $\lambda \neq 0$). If M represents an observable, its mean value $\langle M\rangle$ is given by

$$\operatorname{tr}(\rho M) = \langle M\rangle \tag{16}$$

Notice that the set of positive operators forms a cone, and that the set of trace class operators (of trace one) forms an hyperplane. Thus, $\mathcal{C}$ is the intersection of a cone and an hyperplane embedded in $\mathcal{A}$. This structure (or geometrical convex setting) is susceptible of considerable generalization (see [2, 3, 4] for an excellent overview, partly reproduced in this work for the sake of completeness).

4 Quantal effects and convex operational approach (COM)

In modelling probabilistic operational theories one associates to any probabilistic system a triplet (X, Σ, p), where

1. Σ represents the set of states of the system,
2. X is the set of possible measurement outcomes, and
3. $p : X \times \Sigma \mapsto [0, 1]$ assigns to each outcome $x \in X$ and state $s \in \Sigma$ a probability $p(x, s)$ of x to occur if the system is in the state s.
4. If we fix s we obtain the mapping $s \mapsto p(\cdot, s)$ from $\Sigma \to [0, 1]^X$.

Note that

- This again identifies all the states of Σ with maps.
- Considering their closed convex hull, we obtain the set Ω of possible probabilistic mixtures (represented mathematically by convex combinations) of states in Σ.
- In this way one also obtains, for any outcome $x \in X$, an affine evaluation-functional $f_x : \Omega \to [0, 1]$, given by $f_x(\alpha) = \alpha(x)$ for all $\alpha \in \Omega$.
- More generally, any affine functional $f : \Omega \to [0, 1]$ may be regarded as representing a measurement outcome and thus use $f(\alpha)$ to represent the probability for that outcome in state α.

For the special case of quantum mechanics, the set of all affine functionals so-defined are called effects. They form an algebra (known as the *effect algebra*) and represent generalized measurements (unsharp, as opposed to sharp measures defined by projection valued measures). The specifical form of an effect in quantum mechanics is as follows. A generalized observable or *positive operator valued measure* (POVM) [45, 46, 47] will be represented by a mapping

$$E : B(\mathcal{R}) \to \mathcal{B}(\mathcal{H}) \tag{17a}$$

such that

$$E(\mathcal{R}) = \mathbf{1} \tag{17b}$$

$$E(B) \geq 0, \text{ for any } B \in B(\mathcal{R}) \tag{17c}$$

$$E(\cup_j(B_j)) = \sum_j E(B_j), \text{ for any disjoint familly } B_j \tag{17d}$$

The first condition means that E is normalized to unity, the second one that E maps any Borel set B to a positive operator, and the third one that E is σ-additive with respect to the weak operator topology. In this way, a generalized POVM can be used to define a family of affine functionals on the state space $\mathcal{C}$ (which corresponds to Ω in the general probabilistic setting) of quantum mechanics as follows

$$E(B) : \mathcal{C} \to [0, 1] \tag{18a}$$

$$\rho \mapsto \mathrm{tr}(E\rho) \tag{18b}$$

Positive operators $E(B)$ which satisfy $0 \leq E \leq \mathbf{1}$ are called effects (which form an *effect algebra* [7, 48]). Let us denote by $\mathrm{E}(\mathcal{H})$ the set of all effects.

Indeed, a POVM is a measure whose values are non-negative self-adjoint operators on a Hilbert space. It is the most general formulation of a measurement in the theory of quantum physics.

A rough analogy would consider that a POVM is to a projective measurement what a density matrix is to a pure state. Density matrices can describe part of a larger system that is in a pure state (purification of quantum state); analogously, POVMs on a physical system can describe the effect of a projective measurement performed on a larger system. Another, slightly different way to define them is as follows:
Let (X, M) be measurable space; i.e., M is a $\sigma-$algebra of subsets of X. A POVM is a function F defined on M whose values are bounded non-negative self-adjoint operators on a Hilbert space $\mathcal{H}$ such that $F(X) = I_H$ (identity) and for every i) $\xi \in \mathcal{H}$ and ii) projector $P = |\psi\rangle\langle\psi|;\ |\psi\rangle \in \mathcal{H}$, $P \rightarrow \langle F(P)\xi|\xi\rangle$ is a non-negative countably additive measure on M. This definition should be contrasted with that for the projection-valued measure, which is very similar, except that, in the projection-valued measure, the Fs are required to be projection operators.

4.1 Convex operational approach

Returning now to the general model of probability states we may, as explained in the Appendix, consider the convex set Ω as the basis of a positive cone $V_+(\Omega)$ of the linear space $V(\Omega)$. Thus, every affine linear functional can be extended to a linear functional in $V(\Omega)^*$ (the dual linear space). It can be shown that there is a unique unity functional such that $u_\Omega(\alpha) = 1$ for all $\alpha \in \Omega$ (in quantum mechanics, this unit functional is the trace function). The general operational or convex approach may be viewed as the triplet $(A, A^\sharp, u_A)$, where

1. A is a normed space endowed with
2. a strictly positive linear functional u_A, and
3. $A^\sharp$ is a weak-$*$ dense subspace of A^* ordered by a chosen regular cone $A^\sharp_+ \subseteq A^*_+$ containing u_A.
4. Effects will be given by functionals f in $A^\sharp_+$ such that $f \leq u_A$.

As viewed from the COM's standpoint, one of the characteristic features which distinguish classical from quantum mechanics is the fact that in the classical instance any non-pure state has a unique convex decomposition in pure states, while this is no longer true in the quantum case. The forthcoming section reviews the principal features of the convex set of states and relates them to entanglement. This will be useful for studying canonical maps defined between the probability spaces of a composite system and those of its subsystems. Afterwards, in subsequent sections, *we will i) show that $\mathcal{L}_\mathcal{C}$ is endowed with a canonical lattice theoretical structure, ii) find its main features, and iii) relate them to quantum entanglement and positive maps.* Of course, some of these results can easily be extended to the general setting of convex operational models.

5 Entanglement and the convex set of states: an overview

Consider composite quantal systems S of subsystems S_1 and S_2, with associated separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. The pure states are given by rays in the tensor product space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. It is not true that any pure state of S factorizes after the interaction into pure states of the subsystems, a situation very different to that of classical mechanics, where for state spaces Γ_1 and Γ_2 we assign $\Gamma = \Gamma_1 \times \Gamma_2$ for their composition. This is an absolutely central issue.

Let us now briefly review the quantal relationship between the $S-$states and the states of the subsystems. Consider for simplicity the bipartite case with systems S_1 and S_2. If $\{|x_i^{(1)}\rangle\}$ and $\{|x_i^{(2)}\rangle\}$ are the corresponding orthonormal basis of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then the set $\{|x_i^{(1)}\rangle \otimes |x_j^{(2)}\rangle\}$ constitutes an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$.
For observables of the form $A_1 \otimes \mathbf{1}_2$ and $\mathbf{1}_1 \otimes A_2$ (with $\mathbf{1}_1$ and $\mathbf{1}_2$ the identity operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively), then reduced state operators ρ_1 and ρ_2 can be defined for systems S_1 and S_2 such that

$$\mathrm{tr}(\rho A_1 \otimes \mathbf{1}_2) = \mathrm{tr}(\rho_1 A_1) \tag{19}$$

with an analogous equation for ρ_2. This assignation of reduced states is done via *partial traces maps*. It is of interest for us to study how to define these maps in detail. In order to do so and given a state $\rho \in \mathcal{C}$, consider the functional

$$\begin{aligned} F_\rho : \mathcal{A}(\mathcal{H}_1) \otimes \mathbf{1}_2 &\longrightarrow \mathcal{R} \\ A \otimes \mathbf{1}_2 &\mapsto \mathrm{tr}(\rho A \otimes \mathbf{1}) \end{aligned} \tag{20}$$

Clearly F_ρ induces a map

$$\begin{aligned} f_\rho : \mathcal{A}(\mathcal{H}_1) &\longrightarrow \mathcal{R} \\ A &\mapsto \mathrm{tr}(\rho A \otimes \mathbf{1}) \end{aligned} \tag{21}$$

which specifies a state as defined in (8a) when restricted to projections in $\mathcal{A}(\mathcal{H}_1)$. Thus, using Gleason's theorem, there exists ρ_1 such that $f_\rho(A) = \mathrm{tr}(\rho_1 A)$ (where now the trace is taken on Hilbert space $\mathcal{H}_1$). We have thus shown that for any $\rho \in \mathcal{C}$ there exists $\rho_1 \in \mathcal{C}_1$ such that for any $A_1 \in \mathcal{A}(\mathcal{H}_1)$ Equation (19) is satisfied.
This assignment is the one which allows to define the partial trace $\mathrm{tr}_2(\cdot)$ given by equation (22) below. An analogous reasoning leads to $\mathrm{tr}_1(\cdot)$. Accordingly, we can consider the maps:

$$\mathrm{tr}_i : \mathcal{C} \longrightarrow \mathcal{C}_j \tag{22a}$$

$$\rho \mapsto \mathrm{tr}_i(\rho) = \rho_j \tag{22b}$$

In [37, 38] it is shown that these maps are continuous and onto. *Thus, partial traces defined in equation (22) are continuous and onto, a fact that will be used in section 9.*
Operators of the form $A_1 \otimes \mathbf{1}_2$ and $\mathbf{1}_1 \otimes A_2$ represent magnitudes related to S_1 and S_2, respectively. When S is in a product state $|\varphi_1\rangle \otimes |\varphi_2\rangle$, the mean value of the product operator $A_1 \otimes A_2$ will be

$$\mathrm{tr}(|\varphi_1\rangle \otimes |\varphi_2\rangle\langle\varphi_1| \otimes \langle\varphi_2|A_1 \otimes A_2) = \langle A_1\rangle\langle A_2\rangle \tag{23}$$

reproducing statistical independence. Separable states are defined (see [49]) as those states of $\mathcal{C}$ which can be approximated by a succession of states written as a convex combination of product states of the form

$$\rho_{Sep} = \sum_{i,j} \lambda_{ij}\rho_i^{(1)} \otimes \rho_j^{(2)} \tag{24}$$

where $\rho_i^{(1)} \in \mathcal{C}_1$ and $\rho_j^{(2)} \in \mathcal{C}_2$, $\sum_{i,j}\lambda_{ij} = 1$ and $\lambda_{ij} \geq 0$. Thus, the set $\mathcal{S}(\mathcal{H})$ of separable states is defined as the convex hull of all product states closed in the trace norm topology (i.e., the trace norm topology given by $\|\rho\|_{\mathrm{tr}} = \mathrm{tr}((\rho^\dagger\rho)^{\frac{1}{2}})$. Accordingly,

Definition 5.1. $\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{C} \mid \rho \textit{ is separable}\}$

There exist very many non-separable states in $\mathcal{C}$, called entangled ones [50]. They form a set we will call $\mathcal{E}(\mathcal{H})$, i.e.,

Definition 5.2. $\mathcal{E}(\mathcal{H}) = \mathcal{C}\backslash\mathcal{S}(\mathcal{H})$

Alike entangled states, separable states are reproducible by local classical devices (but this by no means implies that they *are* classical). It is a fact that, equipped with the trace-norm distance, $\mathcal{S}(\mathcal{H})$ is a complete separable metric space. Also that a sequence of quantum states converging to a state in the weak operator topology converges to it as well in the trace norm. Stated in a more technical form, a state in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is called separable if it is in the convex closure of the set of all product states in $\mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.
Estimating the volume of $\mathcal{S}(\mathcal{H})$ is of great interest (see –among others–[39], [51] and [52]). For pure states we have at hand the superposition principle

Principle 5.3. *Superposition Principle. If $|\psi_1\rangle$ and $|\psi_1\rangle$ are physical states, then $\alpha|\psi_1\rangle + \beta|\psi_1\rangle$ ($|\alpha|^2 + |\beta|^2 = 1$) will be a physical state too.*

Additionally, when we include mixtures we have

Principle 5.4. *Mixing Principle. If ρ and ρ' are physical states, then $\alpha\rho + \beta\rho'$ ($\alpha + \beta = 1$, $\alpha, \beta \geq 0$) will be a physical state too.*

There exist many studies which concentrate on mixtures. For example, this is the case for works on quantum decoherence [53, 54, 55], quantum information processing, or generalizations of quantum mechanics which emphasize its convex nature (not necessarily equivalent to "Hilbertian" QM). The set of interest in such studies is $\mathcal{C}$ and not the lattice of projections. Consequently, it seems adequate to consider in some detail the notion of structures, including improper mixtures [31, 56, 57, 58] and pure states, acknowledging their important place in the physical "discourse", as anticipated in the Introduction, in the hope that such structures will provide a natural framework for studying foundational issues.
A good question to ask is under which conditions a given state may be decomposed in terms of other states, separable states being a particular case of decomposing a given state in terms of product states. This is the subject of decomposition theory (see [59], chapter four). Given a C^*-algebra $\mathbf{A}$ with identity $\mathbf{1}$, the set of states forms a convex compact space of the dual space $\mathbf{A}^*$ (compact in the weak* topology). Given a state ρ such that $\rho \in K$, with K a closed convex set, one attempts to write an expression of the form

$$\rho = \int_K d\mu(\rho')\rho', \tag{25}$$

where μ is a measure supported by the set of extremal points of K (see Appendix). Equation (25) is referred to as the *barycentric decomposition of* ρ. The example of interest refers to the C^*-algebra $\mathcal{B}(\mathcal{H})$, the relevant set of states being $\mathcal{C}$, while $K = \mathcal{S}(\mathcal{H})$. In this specific instance, and according to the above definition, it is possible to show that [60]

$$\rho = \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} \rho_{\mathcal{H}_1} \otimes \rho_{\mathcal{H}_2} \nu(d\rho_{\mathcal{H}_1} d\rho_{\mathcal{H}_1}) \tag{26}$$

and also that

$$\rho = \int_{P(\mathcal{C}_1)} \int_{P(\mathcal{C}_2)} |\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi| \nu(d\psi d\varphi) \tag{27}$$

Before concentrating attention on more general convex subsets of $\mathcal{C}$, we will restrict ourselves in next section to a special mathematical construct, closely linked to $\mathcal{P}(\mathcal{H})$.

6 The Lattice $\mathcal{L}$

On $\mathcal{H}$ we find, among many, two particularly important constructs, namely, i) $\mathcal{A}$, the set of bounded *and* Hermitian operators together with ii) the set of ("merely") bounded operators (they map bounded sets to bounded sets), denoted by $\mathcal{B}(\mathcal{H})$. This set of bounded linear operators, together with the addition and composition operations, the norm and the adjoin operation, is a C*-algebra (see Appendix A). We *begin now to present our new results.*

Let us start by extending some developments and definitions of [31] to Hilbert spaces of arbitrary (possibly infinite) dimension. Let us define $G(\mathcal{A})$ as the lattice associated to the pair $(\mathcal{A}, \mathrm{tr})$

$$G(\mathcal{A}) := \{S \subset \mathcal{A} \,|\, S \text{ is a } \underline{\text{closed}}\ \mathcal{R}\text{-subspace}\}. \tag{28}$$

It is well-known that $G(\mathcal{A})$ is a modular, orthocomplemented, atomic and complete lattice (see Appendix B). It is not distributive and hence not a Boolean algebra. Let $\mathcal{L}$ be the associated, induced lattice in $\mathcal{C}$:

$$\mathcal{L} := \{S \cap \mathcal{C} \,|\, S \in G(\mathcal{A})\} \tag{29}$$

for which we will define now canonical lattice operations.

6.1 Characterizing $\mathcal{L}$

Note that there are many subspaces $S' \in G(\mathcal{A})$ such that $S \cap \mathcal{C} = S' \cap \mathcal{C}$. In order to deal with them and appropriately define the lattice induced in $\mathcal{C}$ by $G(\mathcal{A})$, we need a subspace that will represent the set $S \cap \mathcal{C}$. For each $L \in \mathcal{L}$ we will choose as this representative that subspace possessing the minimum dimension

$$S_L = \bigcap \{S \in G(\mathcal{A}) \,|\, S \cap \mathcal{C} = L\} \tag{30}$$

We need to characterize the subspace S_L. To this end notice that S_L satisfies

$$S_L \cap \mathcal{C} = L, \tag{31}$$

(notice that S_L may be infinite dimensional). Let consider the class $[S]$ of elements satisfying (31): $[S] = L$, with $S \in G(\mathcal{A})$ being an element of the class.

We introduce some math-notation now. First, closure of a set will be indicated by an overbar over the set's name. Secondly, given a set M we will denote by $< M >_E$ the set of linear combinations of $M-$elements with coefficients extracted from the set of scalars E. Then

$$S \cap \mathcal{C} \subseteq \overline{< S \cap \mathcal{C} >_{\mathcal{R}}} \subseteq S \Rightarrow S \cap \mathcal{C} \cap \mathcal{C} \subseteq \overline{< S \cap \mathcal{C} >_{\mathcal{R}}} \cap \mathcal{C} \subseteq S \cap \mathcal{C} \Rightarrow \tag{32}$$

$$\overline{< S \cap \mathcal{C} >} \cap \mathcal{C} = S \cap \mathcal{C} \tag{33}$$

Accordingly, $\overline{< S \cap \mathcal{C} >}$ and S are in the same class L. Note that $\overline{< S \cap \mathcal{C} >} \subseteq S$ and if S equals S_L, we will also have $S_L \subseteq \overline{< S_L \cap \mathcal{C} >}$ (because S_L is the intersection of all the elements in the class). Consequently,

$$\overline{< S_L \cap \mathcal{C} >} = S_L \tag{34}$$

The above equation also implies that S_L is the unique subspace with that property, because if we choose S' such that $S' \cap \mathcal{C} = S \cap \mathcal{C}$, then

$$S = \overline{< S \cap \mathcal{C} >} = \overline{< S' \cap \mathcal{C} >} = S' \tag{35}$$

Summing up, the representative of a class L is the unique $\mathcal{R}$-subspace $S \subseteq \mathcal{A}$ such that

$$S = \overline{< S \cap \mathcal{C} >_{\mathcal{R}}} \tag{36}$$

and we have proved that it is equal to S_L, that we call the *good representative*.

6.1.1 Math interlude

We need at this point to remind the reader of some lattice concepts (See appendix B). Let $\mathcal{P}$ be a poset, partially ordered by the order relation "less or equal". An element $a \in \mathcal{P}$ is called an atom if it covers some minimal element of $\mathcal{P}$. As a result, an atom is never minimal. A poset $\mathcal{P}$ is called atomic if for every element $p \in \mathcal{P}$ (that is not minimal) an atom a exist such that $a \leq p$. For instance,

1. Let A be a set and $\mathcal{P} = 2^A$ its power set. $\mathcal{P}$ is a poset ordered by the "inclusion" relation, with a unique minimal element. Thus, all singleton subsets a of A are atoms in $\mathcal{P}$ (a set is a singleton if and only if its cardinality is 1). Accordingly, in the set-theoretic construction of the natural numbers, the number 1 is defined as the singleton $\{0\}$.
2. The set of positive integeres is partially ordered if we define $a \leq b$ to mean that b/a is a positive integer. Then 1 is a minimal element and any prime number p is an atom.

Given a lattice $\mathcal{L}$ with underlying poset $\mathcal{P}$, an element $a \in \mathcal{L}$ is called an atom (of $\mathcal{L}$) if it is an atom in $\mathcal{P}$. A lattice is called an atomic lattice if its underlying poset is atomic. An atomistic lattice is an atomic lattice such that each element that is not minimal is a join of atoms.

Finally (see Appendix B), we call *modular* a lattice that satisfies the following self-dual condition (modular law) $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$, where $\leq$ is the partial order, and $\vee$ and $\wedge$ (join and meet, respectively) are the operations of the lattice.

6.2 Operations in $\mathcal{L}$

Let us now define "$\vee$", "$\wedge$" and "$\neg$" operations and a partial ordering relation "$\longrightarrow$" (or equivalently "$\leq$") in $\mathcal{L}$ as

1.
$$(S \cap \mathcal{C}) \wedge (T \cap \mathcal{C}) \Longleftrightarrow (\overline{< S \cap \mathcal{C} >} \cap \overline{< T \cap \mathcal{C} >}) \cap \mathcal{C} \tag{37}$$

2.
$$(S \cap \mathcal{C}) \vee (T \cap \mathcal{C}) \Longleftrightarrow (\overline{< S \cap \mathcal{C} >} + \overline{< T \cap \mathcal{C} >}) \cap \mathcal{C} \tag{38}$$

3.
$$(S \cap \mathcal{C}) \longrightarrow (T \cap \mathcal{C}) \Longleftrightarrow (S \cap \mathcal{C}) \subseteq (T \cap \mathcal{C}) \tag{39}$$

4.
$$\neg(S \cap \mathcal{C}) \Longleftrightarrow < S \cap \mathcal{C} >^{\perp} \cap \mathcal{C}. \tag{40}$$

Let us consider now some consequences of the above definitions. It is easy to see that (see Appendix B):

Proposition 6.1. *$\mathcal{L}$ is an atomic and complete lattice. If* $\dim(\mathcal{H}) < \infty$*, $\mathcal{L}$ is a modular lattice.*

$\mathcal{L}$ is not an orthocomplemented lattice, but it is easy to show that non-contradiction holds

$$L \wedge \neg L = \mathbf{0} \tag{41}$$

and also contraposition

$$L_1 \leq L_2 \Longrightarrow \neg L_2 \leq \neg L_1 \tag{42}$$

The following proposition is important because it *links atoms and quantum states*:

Proposition 6.2. *There is a one to one correspondence between the states of the system and the atoms of $\mathcal{L}$.*

Proof. Let $\rho \in \mathcal{C}$ be any state. Next, consider the subspace $S_\rho =< \rho >$. Then, $Q_\rho = S_\rho \cap \mathcal{C} = \rho$ is a singleton because ρ is the only trace one operator in S_ρ, and thus an atom since it covers the minimal element $\rho \in \mathcal{C}$. This completes the bijective correspondence. □

The reader is now adviced to peruse Appendix C and remember that a *face* of a convex set $\mathcal{C}$ is the intersection of $\mathcal{C}$ with a supporting hyperplane of $\mathcal{C}$. It is well known [50] that in the finite dimensional case there is a lattice isomorphism between the complemented and complete lattice of faces of the convex set $\mathcal{C}$ and $\mathcal{L}_{v\mathcal{N}}$ (see Appendix B for definitions). We also repeat that $\mathcal{L}$ is a set of intersections, Which ones? Those between $\mathcal{R}-$subspaces that belong to $G(\mathcal{A})$ and $\mathcal{C}$. Let us put forward the following proposition

Proposition 6.3. *Every face of $\mathcal{C}$ is an element of $\mathcal{L}$.*

Proof. If F is a face of $\mathcal{C}$, then there exists a closed hyperplane $H_F \subseteq \mathcal{A}$ such that $H_F \cap \mathcal{C} = F$ and $\mathcal{C}$ stands on one side of the half-spaces defined by H_F. If $\hat{H}_F$ is the continuous linear functional on $\mathcal{A}$ defined by the real sub-vector space H_F (that is, the map that assigns a scalar to each member of H_F), then there exists $\alpha \in \mathbb{R}$ such that

$$H_F = \{x \,|\, \hat{H}_F(x) = \alpha\}, \quad \mathcal{C} \subseteq \{x \,|\, \hat{H}_F(x) \leq \alpha\}.$$

Consider the continuous linear functional

$$\hat{L}(x) := \hat{H}_F(x) - \alpha \mathrm{tr}(x). \tag{43}$$

In the Banach space of trace class operators $\mathcal{B}_1$ we have the norm $\|T\| = \mathrm{tr}(|T|)$, then the linear functional $\mathrm{tr} : \mathcal{B}_1 \to \mathbb{R}$ is continuous. Given that $\hat{H}_F$ is continuous on $\mathcal{A}$ and $\mathcal{B}_1 \subseteq \mathcal{A}$ we have that $\hat{L}$ is continuous on $\mathcal{B}_1$. Its kernel, or null space (the set of all functions that the functional maps to zero) will be a closed subspace of $\mathcal{B}_1$ (we call it S). Now notice that $\mathcal{C} \subseteq \mathcal{B}_1$ and

$$S \cap \mathcal{C} = \{x \,|\, \hat{H}_F(x) = \alpha \mathrm{tr}(x)\} \cap \mathcal{C} = \{x \,|\, \hat{H}_F(x) = \alpha \mathrm{tr}(x) \,\mathrm{and}\, \mathrm{tr}(x) = 1\} \cap \mathcal{C} = \{x \,|\, \hat{H}_F(x) = \alpha\} \cap \mathcal{C} = H_F \cap \mathcal{C} = F. \tag{44}$$

We conclude that $F \in \mathcal{L}$.

□

Using this result and the isomorphism between faces of the convex set and subspaces, we conclude that (at least for the finite dimensional case)

Corollary 6.4. *The complete lattice of faces of the convex set $\mathcal{C}$ is essentially a subposet of $\mathcal{L}$.*

The previous Corollary shows that $\mathcal{L}$ and $\mathcal{L}_{v\mathcal{N}}$ are closely connected. We have stated that any convex subset of $\mathcal{C}$ can be considered as a probability space by itself, and certainly, the elements of $\mathcal{L}$ are convex sets, because they are built as the intersection of a closed subspace of $\mathcal{A}$ and a convex set ($\mathcal{C}$). Thus, we have found not only that closed subspaces of $\mathcal{H}$ may be considered as yes-no tests (via their one to one correspondence with projection operators), but also that they may be considered as probability spaces (endowed with "mixing principle" mentioned above), because of their one to one correspondence with the elements of $\mathcal{L}$.

The crucial question now is: what is the relationship between their respective operations? If F_1 and F_2 are faces we have

- ($\wedge$) $F_1, F_2 \in \mathcal{L}_{v\mathcal{N}}$, then $F_1 \wedge F_2$ in $\mathcal{L}_{v\mathcal{N}}$ is the same as in $\mathcal{L}$. Thus, the inclusion $\mathcal{L}_{v\mathcal{N}} \subseteq \mathcal{L}$ preserves the $\wedge$-operation.
- ($\vee$) $F_1 \vee_{\mathcal{L}} F_2 \leq F_1 \vee_{\mathcal{L}_{v\mathcal{N}}} F_2$ and $F_1 \leq F_2 \Rightarrow F_1 \vee_{\mathcal{L}} F_2 = F_1 \vee_{\mathcal{L}_{v\mathcal{N}}} F_2 = F_2$
- ($\neg$) $\neg_{\mathcal{L}} F \leq \neg_{\mathcal{L}_{v\mathcal{N}}} F$

Given two systems with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we can construct the lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. We can also built up $\mathcal{L}$, the lattice associated to the product space $\mathcal{H}_1 \otimes \mathcal{H}_2$. We define

$$\Psi : \mathcal{L}_1 \times \mathcal{L}_2 \longrightarrow \mathcal{L} \quad | \quad (S_1 \cap \mathcal{C}_1, S_2 \cap \mathcal{C}_2) \longrightarrow S \cap \mathcal{C} \tag{45}$$

where $S = (\overline{< S_1 \cap \mathcal{C}_1 >} \otimes \overline{< S_2 \cap \mathcal{C}_2 >})$. In terms of good representatives, $\Psi([S_1], [S_2]) = [S_1 \otimes S_2]$. An equivalent definition (in the finite dimensional case) states that Ψ is the induced morphism in the quotient lattices (See Appendix A) of the tensor map:

$$G(\mathcal{A}_1) \times G(\mathcal{A}_2) \to G(\mathcal{A}_1 \otimes_{\mathcal{R}} \mathcal{A}_2) \cong G(\mathcal{A}) \tag{46}$$

Given $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_2$, we can define the following convex tensor product:

$$\begin{tikzcd}
 & \mathcal{L} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] \arrow[d, shift left, "\tau"] & \\
\mathcal{L}_1 & \mathcal{L}_1 \times \mathcal{L}_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, shift left, "\psi"] & \mathcal{L}_2
\end{tikzcd}$$

Figure 1: The different maps between $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_1 \times \mathcal{L}_2$, and $\mathcal{L}$. π_1 and π_2 are the canonical projections.

Definition 6.5. $L_1 \widetilde{\otimes} L_2 := \{\sum \lambda_{ij} \rho_i^1 \otimes \rho_j^2 \,|\, \rho_i^1 \in L_1,\ \rho_j^2 \in L_2,\ \sum \lambda_{ij} = 1 \textit{ and } \lambda_{ij} \geq 0\}$

This product is formed by all possible convex combinations of tensor products of elements of L_1 and elements of L_2, and it is again a convex set. Let us compute $\mathcal{C}_1 \widetilde{\otimes} \mathcal{C}_2$. Remember that $\mathcal{C}_1 = [\mathcal{A}_1] \in \mathcal{L}_1$ and $\mathcal{C}_2 = [\mathcal{A}_2] \in \mathcal{L}_2$:

$$\mathcal{C}_1 \widetilde{\otimes} \mathcal{C}_2 = \{\sum \lambda_{ij} \rho_i^1 \otimes \rho_j^2 \,|\, \rho_i^1 \in \mathcal{C}_1,\ \rho_j^2 \in \mathcal{C}_2,\ \sum \lambda_{ij} = 1 \text{ and } \lambda_{ij} \geq 0\}. \tag{47}$$

Thus, if $\mathcal{S}(\mathcal{H})$ is the set of all separable states, we have by definition

$$\mathcal{S}(\mathcal{H}) = \overline{\mathcal{C}_1 \widetilde{\otimes} \mathcal{C}_2} \tag{48}$$

If the whole system is in a state ρ, using partial traces we can define states for the subsystems $\rho_1 = tr_1(\rho)$ and a similar definition for ρ_2. Then, we can consider the maps

$$\mathrm{tr}_i : \mathcal{C} \longrightarrow \mathcal{C}_j \quad | \quad \rho \longrightarrow \mathrm{tr}_i(\rho) \tag{49}$$

from which we can construct the induced projections:

$$\tau_i : \mathcal{L} \longrightarrow \mathcal{L}_i \quad | \quad S \cap \mathcal{C} \longrightarrow \mathrm{tr}_i(\overline{< S \cap \mathcal{C} >}) \cap \mathcal{C}_i = \mathrm{tr}_i(\overline{< S \cap \mathcal{C} >}) \cap \mathcal{C}_i \tag{50}$$

In terms of good representatives $\tau_i([S]) = [\mathrm{tr}_i(S)]$. As a consequence, we can define the product map

$$\tau : \mathcal{L} \longrightarrow \mathcal{L}_1 \times \mathcal{L}_2 \quad | \quad L \longrightarrow (\tau_1(L), \tau_2(L)) \tag{51}$$

The maps defined in this section are illustrated in Figure 6.2.

7 The Lattice of Convex Subsets

The elements of $\mathcal{L}$ are the intersections between closed (real) subspaces of $\mathcal{A}$ and $\mathcal{C}$. Now we drop the restriction and consider *all* possible probability subspaces of $\mathcal{C}$, i.e., all of their possible convex subsets. Because of linearity, partial trace operators preserve convexity and so they will map probability spaces of the system into probability spaces of the subsystems, as desired. Let us begin then by considering the set of all convex subsets of $\mathcal{C}$ (not only closed ones):

Definition 7.1. $\mathcal{L}_{\mathcal{C}} := \{C \subseteq \mathcal{C} \,|\, C \textit{ is a convex subset of } \mathcal{C}\}$

In order to give $\mathcal{L}_{\mathcal{C}}$ a lattice structure, we introduce the following operations (where $conv(A)$ stands for convex hull of a given set A):

Definition 7.2. *For all* $C, C_1, C_2 \in \mathcal{L}_{\mathcal{C}}$

$\wedge$ *:* $C_1 \wedge C_2 := C_1 \cap C_2$

$\vee$: $C_1 \vee C_2 := conv(C_1, C_2)$. *It is again a convex set, and it is included in* $\mathcal{C}$ *(using convexity).*

$\neg$: $\neg C := C^{\perp} \cap \mathcal{C}$

$\longrightarrow$: $C_1 \longrightarrow C_2 := C_1 \subseteq C_2$

With the operations of definition 7.2, it is apparent that $(\mathcal{L}_{\mathcal{C}}; \longrightarrow)$ is a poset. If we set $\emptyset = \mathbf{0}$ and $\mathcal{C} = \mathbf{1}$, then, $(\mathcal{L}_{\mathcal{C}}; \longrightarrow; \mathbf{0}; \emptyset = \mathbf{0})$ will be a bounded poset.
It is very easy to show that

Proposition 7.3. $(\mathcal{L}_{\mathcal{C}}; \longrightarrow; \wedge; \vee)$ *satisfies*

(*a*) $C_1 \wedge C_1 = C_1$

(*b*) $C_1 \wedge C_2 = C_2 \wedge C_1$

(*c*) $C_1 \vee C_2 = C_2 \vee C_1$

(*d*) $C_1 \wedge (C_2 \wedge C_3) = (C_1 \wedge C_2) \wedge C_3$

(*e*) $C_1 \vee (C_2 \vee C_3) = (C_1 \vee C_2) \vee C_3$

(*f*) $C_1 \wedge (C_1 \vee C_2) = C_1$

(*g*) $C_1 \vee (C_1 \wedge C_2) = C_1$

Regarding the "$\neg$" operation, if $C_1 \subseteq C_2$, then $C_2^{\perp} \subseteq C_1^{\perp}$. Accordingly, $C_2^{\perp} \cap \mathcal{C} \subseteq C_1^{\perp} \cap \mathcal{C}$, and hence

$$C_1 \longrightarrow C_2 \Longrightarrow \neg C_2 \longrightarrow \neg C_1 \tag{52}$$

Given that $C \cap (C^{\perp} \cap \mathcal{C}) = \emptyset$, we also have:

$$C \wedge (\neg C) = \mathbf{0} \tag{53}$$

Contraposition and non contradiction thus hold. But if we take the proposition $C = \{\frac{1}{N}1\}$, then an easy calculation yields $\neg C = \mathbf{0}$. Then, $\neg(\neg C) = \mathbf{1}$, and thus $\neg(\neg C) \neq C$ in general. Double negation does not hold. Consequently, $\mathcal{L}_{\mathcal{C}}$ is not an ortholattice. $\mathcal{L}_{\mathcal{C}}$ is a lattice which includes all convex subsets of the quantum space of states. It includes $\mathcal{L}$, and then all quantum states (including all improper mixtures), *as propositions* (understood as elements of the lattice). Compare with classical physics, where the lattice of propositions is formed by all measurable subsets of Gibbs' phase-space (the space of states).
As any convex subset of $\mathcal{C}$ is a probability space by itself, we can define on each of these subsets a lattice structure in analogous way to that used for $\mathcal{L}_{\mathcal{C}}$. In other words, we encounter an inheritance of the same structure.

7.1 Effects, mean values and maximum entropy principle

Let us discuss, as an example, the family of elements of $\mathcal{L}_{\mathcal{C}}$ that is associated to effects. Given an effect E, consider the set of states

$$C_{(E,\lambda)} := \{\rho \in \mathcal{C} \,|\, \mathrm{tr}(\rho E) = \lambda,\ \lambda \in [0,1]\}. \tag{54}$$

It is easy to verify that C_E is a convex set and, consequently, an element of $\mathcal{L}_{\mathcal{C}}$. $C_{(E,\lambda)}$ represents all the states for which the probability of having the effect E is equal to λ. Furthermore, there exists S, an $\mathcal{R}$-subspace of $\mathcal{A}$, such that

$$C_{(E,\lambda)} = S \cap \mathcal{C}, \tag{55}$$

and thus, C_E is also an element of $\mathcal{L}$. The proof of (55) is as follows. Consider the linear functional

$$F_{(E,\lambda)}(\rho) := \mathrm{tr}(E\rho) - \lambda \mathrm{tr}(\rho), \tag{56}$$

As in the proof of Proposition 6.3, we will have that the set $S = Ker(F_{(E,\lambda)})$ is a closed subspace of $\mathcal{B}_1$. Then, an element $\rho \in C_{(E,\lambda)}$ will be given by an element of $\mathcal{C}$ of trace one, and thus $F_{(E,\lambda)}(\rho) = \mathrm{tr}(E\rho) - \lambda$, plus the equality to λ requirement imply that ρ also belongs to S. We have thus demonstrated Eq. (55). More generally, if we have the equation for the mean value of an operator

$$\langle R \rangle = r, \tag{57}$$

the operator may be considered as represented by the set of density matrices which satisfy the above equality. The ensuing set is obtained as the intersection of the Kernel of the functional $F_R(\rho) := \mathrm{tr}(R\rho) - r\mathrm{tr}(\rho)$ with $\mathcal{C}$. Accordingly, each equation of the form (57) (understood as an equation to be solved) can be represented as an element $C \in \mathcal{L}$, and also as an element of $\mathcal{L}_{\mathcal{C}}$. Note that $\mathcal{C}$ is also a closed convex set because it is the intersection of the following closed half-spaces

$$H_x = \{\rho \in \mathcal{A} \,|\, x^\dagger \rho x = 0\}, \quad \mathcal{C} = \bigcap_{x \in \mathcal{H}} \{\rho \,|\, x^\dagger \rho x \geq 0\} \cap \{\rho \,|\, \mathrm{tr}(\rho) = 1\}.$$

With such materials at hand, we can now re-express the celebrated Jaynes' maximum entropy principle [33, 34] in lattice theoretical fashion. Jaynes assumes that for a quantum system one knows the mean values

$$\begin{aligned} \langle R_1 \rangle &= r_1 \\ \langle R_2 \rangle &= r_2 \\ &\vdots \\ \langle R_n \rangle &= r_n, \end{aligned} \tag{58}$$

and we want to determine the most unbiased density matrix compatible with conditions (58). Jaynes' MaxEnt principle asserts that the solution to this problem is given by the density matrix that maximizes entropy, given by

$$\rho_{max-ent} = \exp^{-\lambda_0 \mathbf{1} - \lambda_1 R_1 - \cdots - \lambda_n R_n}, \tag{59}$$

where the λ's are Lagrange multipliers satisfying

$$r_i = -\frac{\partial}{\partial \lambda_i} \ln Z, \tag{60}$$

while the partition function reads

$$Z(\lambda_1 \cdots \lambda_n) = \mathrm{tr}[\exp^{-\lambda_1 R_1 - \cdots - \lambda_n R_n}], \tag{61}$$

and the normalization condition is

$$\lambda_0 = \ln Z. \tag{62}$$

Our point here is that the set of conditions (58) can be expressed in an explicit lattice theoretical form as follows. Using a similar procedure as in (57) we easily conclude that each of the equations in (58) can be represented as a convex (and closed) sets C_{R_i}. In this way we can now express conditions (58) via the lattice theoretical expression

$$C_{max-ent} := \bigcap_i C_{R_i} = \bigwedge_i C_{R_i}. \tag{63}$$

Now, $C_{max-ent}$ is also an element of $\mathcal{L}_{\mathcal{C}}$ (but not necessarily of $\mathcal{L}$) and we must maximize entropy in it. We have thus encountered a MaxEnt-lattice theoretical expression: *given a set of conditions represented generally by convex subsets C_i, one should maximize the entropy in the set $C_{max-ent} = \bigwedge_i C_i$.*

7.2 The Relationship Between $\mathcal{L}_{v\mathcal{N}}$, $\mathcal{L}$ and $\mathcal{L}_{\mathcal{C}}$

Please, refer here to Appendix B.

Proposition 7.4. *For finite dimension $\mathcal{L}_{v\mathcal{N}} \subseteq \mathcal{L} \subseteq \mathcal{L}_{\mathcal{C}}$ as posets.*

Proof. We have already seen that $\mathcal{L}_{v\mathcal{N}} \subseteq \mathcal{L}$ as sets. Moreover it is easy to see that if $F_1 \leq F_2$ in $\mathcal{L}_{v\mathcal{N}}$ then $F_1 \leq F_2$ in $\mathcal{L}$. This is so because both orders are set theoretical inclusions. Similarly, if $L_1, L_2 \in \mathcal{L}$, because intersection of convex sets yields a convex set (and closed subspaces are convex sets also), $L_1, L_2 \in \mathcal{L}_{\mathcal{C}}$, then we obtain set theoretical inclusion. And, again, because of both orders are set theoretical inclusions, we obtain that they are included as posets. $\square$

Regarding the $\vee$ operation, let us compare $\vee_{\mathcal{L}_{v\mathcal{N}}}$, $\vee_{\mathcal{L}}$ and $\vee_{\mathcal{L}_{\mathcal{C}}}$. If $L_1, L_2 \in \mathcal{L}$, then they are convex sets and so, $L_1, L_2 \in \mathcal{L}_{\mathcal{C}}$. Thus we can compute

$$L_1 \vee_{\mathcal{L}_C} L_2 = conv(L_1, L_2). \tag{64}$$

On the other hand (if S_1 and S_2 are good representatives for L_1 and L_2), then:

$$L_1 \vee_{\mathcal{L}} L_2 = (\overline{< S_1 \cap \mathcal{C} >} + \overline{< S_2 \cap \mathcal{C} >}) \cap \mathcal{C}. \tag{65}$$

The direct sum of the subspaces $\overline{< S_1 \cap \mathcal{C} >}$ and $\overline{< S_2 \cap \mathcal{C} >}$ contains as a particular case all convex combinations of elements of L_1 and L_2. We then conclude (including infinite dimensional case)

$$L_1 \vee_{\mathcal{L}_C} L_2 \leq L_1 \vee_{\mathcal{L}} L_2. \tag{66}$$

The faces of $\mathcal{C}$ can be considered as elements of $\mathcal{L}_C$ because they are convex. If F_1 and F_2 are faces we can also state (finite dimension)

$$F_1 \vee_{\mathcal{L}_C} F_2 \leq F_1 \vee_{\mathcal{L}} F_2 \leq F_1 \vee_{\mathcal{L}_{v\mathcal{N}}} F_2. \tag{67}$$

Intersection of convex sets is the same as intersection of elements of $\mathcal{L}$ and so we have (infinite dimension)

$$L_1 \wedge_{\mathcal{L}_C} L_2 = L_1 \wedge_{\mathcal{L}} L_2, \tag{68}$$

and similarly (finite dimension)

$$F_1 \wedge_{\mathcal{L}_{v\mathcal{N}}} F_2 = F_1 \wedge_{\mathcal{L}_C} F_2 = F_1 \wedge_{\mathcal{L}} F_2. \tag{69}$$

What is the relationship between $\neg_{\mathcal{L}_{\mathcal{C}}}$ and $\neg_{\mathcal{L}}$? Suppose that $L_1 \in \mathcal{L}$, then they are convex sets as well and $L_1 \in \mathcal{L}_{\mathcal{C}}$. We can now compute $\neg_{\mathcal{L}_{\mathcal{C}}} L_1$ and obtain

$$\neg_{\mathcal{L}_{\mathcal{C}}} L_1 = L_1^{\perp} \cap \mathcal{C}. \tag{70}$$

On the other hand, if $L_1 = S \cap \mathcal{C}$, with S a good representative,

$$\neg_{\mathcal{L}} L_1 =< S \cap \mathcal{C} >^{\perp} \cap \mathcal{C}. \tag{71}$$

As $L_1 \subseteq < S \cap \mathcal{C} >$, then $< S \cap \mathcal{C} >^{\perp} \subseteq L_1^{\perp}$, and finally (infinite dimension)

$$\neg_{\mathcal{L}} L_1 \leq \neg_{\mathcal{L}_{\mathcal{C}}} L_1. \tag{72}$$

8 Identifying entanglement in the finite dimensional case

After introducing the new lattice theoretical frameworks, it is interesting to look back again to the finite dimensional case, condition that will we assume through this section. In doing so, we will find a separability criteria. We will include in this section proofs of various well established facts because in doing so, we illuminate the lattice structure behind those facts which allow us to deduce our novel separability criteria. In future works we will use these proofs to compute effectively *the volume of the space of separable states.* Let $\mathcal{S}_0(\mathcal{H})$ denote the space of product states

$$\mathcal{S}_0(\mathcal{H}) = \{a \otimes b \,|\, a \in \mathcal{C}_1, \, b \in \mathcal{C}_2\} \subseteq S(\mathcal{H}) \tag{73}$$

By definition, $S(\mathcal{H})$ is the convex hull of $\mathcal{S}_0(\mathcal{H})$. Using a well known fact about the theory of convex sets, we know that $S(\mathcal{H})$ is the intersection of all the closed half-hyperplanes containing $\mathcal{S}_0(\mathcal{H})$ [61, 11.5.1]. For any functional ℓ representing a half-plane and a real number m, we have

$$\mathcal{S}_0(\mathcal{H}) \subseteq \{\ell \geq m\} \iff \ell(\mathcal{S}_0(\mathcal{H})) \subseteq [m, +\infty) \iff \ell(\mathcal{S}_0(\mathcal{H})) \text{ has a minimum value } m. \tag{74}$$

Remark 8.1. *We work with i) $\mathcal{B}(\mathcal{H})$ and ii) its isomorphic space $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$. The composition for $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H})$ is $\rho_1\rho_2$, and for $a \otimes b, c \otimes d \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$, is $ac \otimes bd$. The induced trace in $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ is $tr(a \otimes b) = tr(a)tr(b)$. In $\mathcal{B}(\mathcal{H})$ we have a canonical inner product,*

$$\langle \rho_1, \rho_2 \rangle = tr(\rho_1 \rho_2^{\dagger})$$

In $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ we have the induced inner product,

$$\langle a \otimes b, c \otimes d \rangle = tr(ac^\dagger) tr(bd^\dagger)$$

In particular, in $\mathcal{A}(\mathcal{H}_1) \otimes \mathcal{A}(\mathcal{H}_2)$ one has

$$\langle a \otimes b, c \otimes d \rangle = tr(ac) tr(bd). \tag{75}$$

This inner product gives to $\mathcal{A}(\mathcal{H})$ the structure of an $\mathcal{R}$-vector space with an inner product. Hence, any linear functional ℓ is associated to a vector $\rho \in \mathcal{A}(\mathcal{H})$ such that $\ell = \langle \rho, - \rangle$.

It is important to notice here that any functional of the form $\ell = \langle \rho, - \rangle$ defines by varying $m \in \mathcal{R}$ a family of elements of $\mathcal{L}_C$ and $\mathcal{L}$ by considering the families of convex subsets $\{\ell \leq m\} \cap \mathcal{C}$ and $\{\ell = m\} \cap \mathcal{C}$ respectively. It is possible to show that

Proposition 8.2. *The space of product states $\mathcal{S}_0(\mathcal{H})$ is compact. In particular, every linear functional of the form ℓ_ρ on $\mathcal{S}_0(\mathcal{H})$ has a maximum M_ρ and a minimum m_ρ values.*

Proof. It is well known that for every matrix in $a \in \mathcal{C}_1$ there exist a unitary matrix $U \in \mathcal{U}(\mathcal{H}_1)$ such that $UaU^\dagger$ is diagonal. In particular, if we take the subgroup

$$\mathcal{U}(\mathcal{H}_1) \times \mathcal{U}(\mathcal{H}_2) \hookrightarrow \mathcal{U}(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) \,|\, UU^\dagger = I\}$$

we have that every product state is conjugate to a product of two diagonal matrixes

$$(U_1, U_2)(a_1 \otimes a_2)(U_1^\dagger, U_2^\dagger) = (U_1 a_1 U_1^\dagger) \otimes (U_2 a_2 U_2^\dagger) = d_1 \otimes d_2.$$

Note that the subgroup $\mathcal{U}(\mathcal{H}_1) \times \mathcal{U}(\mathcal{H}_2)$ preserves the space of product states and the space of separable states. Let us define the space of diagonal product states

$$\mathcal{D} = \{d_1 \otimes d_2 \in \mathcal{S} \,|\, d_1 = \mathrm{diag}(\alpha_1, \ldots, \alpha_n),\, d_2 = \mathrm{diag}(\beta_1, \ldots, \beta_m)\}.$$

Note that $\mathcal{D}$ is isomorphic, as a topological space, to the product of simplexes $\triangle^n \times \triangle^m$. Indeed, we have the following continuous surjective map from a compact space to $\mathcal{S}_0(\mathcal{H})$

$$\Gamma : \mathcal{U}(\mathcal{H}_1) \times \mathcal{U}(\mathcal{H}_2) \times \triangle^n \times \triangle^m \longrightarrow \mathcal{S}_0(\mathcal{H}), \tag{76}$$

$$\Gamma(U_1, U_2, (\alpha_1, \ldots, \alpha_n), (\beta_1, \ldots, \beta_m)) = U_1^\dagger \mathrm{diag}(\alpha_1, \ldots, \alpha_n) U_1 \otimes U_2^\dagger \mathrm{diag}(\beta_1, \ldots, \beta_m) U_2. \tag{77}$$

□

It is also known that

Corollary 8.3. *The space of separable states $S(\mathcal{H}) \subseteq A(\mathcal{H})$ is compact.*

Proof. We remind the reader that Carathéodory's theorem states that if a point x of $\mathcal{R}^d$ lies in the convex hull of a set P, there is a subset P' of P consisting of $d+1$ (or fewer) points such that x lies in the convex hull of P'. Equivalently, x lies in an $r-$simplex with vertices in P, where $r \leq d$. The result is named for Constantin Carathéodory, who proved the theorem in 1911 for the case when P is compact. In 1914, Ernst Steinitz expanded Carathéodory's theorem for any sets P in $\mathcal{R}^d$. Let $N = \dim_{\mathcal{R}} A(\mathcal{H})$. Then, it is seen by Carathéodory's theorem [61, 17.1] that the following continuous map is surjective,

$$\triangle^{N+1} \times (\mathcal{S}_0(\mathcal{H}))^{N+1} \xrightarrow{\Gamma} S(\mathcal{H}) \tag{78}$$

$$\Gamma((\lambda_0, \ldots, \lambda_N), (a_0, \ldots, a_N)) = \sum_{i=0}^{N} \lambda_i a_i. \tag{79}$$

Hence $S(\mathcal{H})$ is a compact convex space.

□

Entanglement witnesses are useful to characterize entanglement [50, 62] and have became a fundamental tool. They originate in geometry, being a consequence of the Hahn-Banach theorem [50]. This theorem is a central tool in functional analysis. It allows the extension of bounded linear functionals defined on a subspace of some vector space to the whole space, and it also guarantees that there are "enough" continuous linear functionals defined on every normed vector space to make the study of the dual space "interesting." Another version of the HB theorem, known also either i) as the HB-Banach separation theorem or ii) the separating hyperplane theorem, has numerous uses in convex geometry, and interest us here. If $\rho \notin \mathcal{S}(\mathcal{H})$, and because $\mathcal{S}(\mathcal{H})$ is closed and convex, then there exists a plane (and so a functional) which separates ρ and $\mathcal{S}(\mathcal{H})$. This means that ρ stands in one side of the plane and $\mathcal{S}(\mathcal{H})$ on the other. The existence of such a plane is closely linked to the existence of an observable W (and this is the entanglement witness) which satisfies that if $\rho \in \mathcal{E}(\mathcal{H})$, then $\operatorname{tr}(\rho W) < 0$ and $\operatorname{tr}(\sigma) \geq 0$ for any $\sigma \in \mathcal{S}(\mathcal{H})$. It is possible to show that for any entangled state there exists an entanglement witness. Also:

Proposition 8.4. *Let $\ell_\rho = \langle \rho, - \rangle$ an $\mathcal{R}$-linear functional with $\rho \in \mathcal{A}(\mathcal{H})$. Then*

$$\min_{S(\mathcal{H})} \ell_\rho = \min_{\mathcal{S}_0(\mathcal{H})} \ell_\rho = \min\{\langle \rho, vv^\dagger \otimes ww^\dagger \rangle \mid \|v\| = \|w\| = 1\} =: m_\rho,$$

$$\max_{S(\mathcal{H})} \ell_\rho = \max_{\mathcal{S}_0(\mathcal{H})} \ell_\rho = \max\{\langle \rho, vv^\dagger \otimes ww^\dagger \rangle \mid \|v\| = \|w\| = 1\} =: M_\rho.$$

Proof. Suppose that $\lambda_0 a_0 + \ldots + \lambda_s a_s \in S(\mathcal{H})$ is the maximum of ℓ_ρ and consider the following linear functional

$$(\alpha_0, \ldots, \alpha_s) \longrightarrow \sum_{i=0}^{s} \alpha_i \ell_\rho(a_i), \quad (\alpha_0, \ldots, \alpha_s) \in \triangle^{s+1}.$$

We know that its maximum is at $(\lambda_0, \ldots, \lambda_s)$ but let us appeal to some ideas from the theory of faces discussed in [61, p.162,163]. We learn there that a linear functional over a convex set $\triangle^{s+1}$, by definition, achieves its maximum over a face ($F = \triangle^{s+1} \cap H$) and given that every face contains 0-dimensional faces we have that every linear functional achieves its maximum at some 0-dimensional face. Recall that "a face of a face" is just a face of the convex set $\triangle^{s+1}$ and that the 0-dimensional faces of $\triangle^{s+1}$ are its vertexes. In conclusion, the linear functional considered above achieves its maximum at a vertex. Then such maximum refers to a product state (same for the minimum). For the second relationship invoked above we need still to show that every separable state is a convex combination of products of pure states, but this is a well established fact (given ρ separable, just express in diagonal forma the components of products which appear on its decomposition). □

What is interesting regarding the above proposition is the fact that we can effectively compute the numbers m_ρ and M_ρ, as demonstrated in [63]. Also, we can show that

Remark 8.5. *Over the convex set $\mathcal{C}$ it is easy to prove that the maximum and minimum of the lineal functional ℓ_ρ are the biggest and smallest eigenvalues of ρ. Given that $S(\mathcal{H}) \subseteq \mathcal{C}(\mathcal{H})$ we have that $\min_\mathcal{S} \ell_\rho \geq \lambda_{\min}$ and $\max_\mathcal{S} \ell_\rho \leq \lambda_{\max}$. In particular, for pure states we have $\max_\mathcal{S} \ell_{xx^\dagger} \leq 1$ and $\min_\mathcal{S} \ell_{xx^\dagger} \geq 0$.*

Let us define

Definition 8.6. *Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ be a canonical column vector and let c_{kl} be the matrix with 1 in place kl, i.e. $c_{kl} = e_k e_l^\dagger$. Let $x = \sum \lambda_i e_i$ be a vector such that $\|x\| = 1$ then*

$$x^\dagger \rho x = \sum_{kl} \overline{\lambda_k} \lambda_l e_k^\dagger \rho e_l = \sum_{kl} \overline{\lambda_k} \lambda_l tr(\rho c_{kl}) = tr(\rho \sum_{kl} \overline{\lambda_k} \lambda_l c_{kl}) = tr(\rho x x^\dagger).$$

Note that $xx^\dagger$ is a pure state (positive and rank one matrix) and $tr(xx^\dagger) = \|x\|^2 = 1$. Let us define the sphere in $\mathcal{H}$

$$S_1(\mathcal{H}) = \{x \in \mathcal{H} \mid \|x\| = 1\}.$$

Proposition 8.7. *The space of states $\mathcal{C}$ is compact.*

Proof. Let $n = \dim_\mathcal{R} \mathcal{H}$ and $N = \dim_\mathcal{R} \mathcal{A}(\mathcal{H})$. We can give here two proofs,

$$\triangle^{N+1} \times S_1(\mathcal{H})^{N+1} \xrightarrow{\Gamma_1} \mathcal{C}(\mathcal{H}), \quad \triangle^{N+1} \times \mathcal{U}(\mathcal{H})^n \times \triangle^n \xrightarrow{\Gamma_2} \mathcal{C}(\mathcal{H}).$$

$$\Gamma_1(a_0, \dots, a_N, x_0, \dots, x_N) = \sum_{i=0}^N a_i x_i x_i^\dagger.$$

$$\Gamma_2(a_0, \dots, a_N, U_1, \dots, U_n, d_1, \dots, d_n) = \sum_{i=0}^N a_i U_i^\dagger \mathrm{diag}(d_1, \dots, d_n) U_i.$$

Both are continuous surjective maps. □

Now we are in condition to characterize $\mathcal{S}(\mathcal{H})$ in terms of special convex sets

Theorem 8.8.

$$\mathcal{C} = \bigcap_{x \in S_1(\mathcal{H})} \{\ell_{xx^\dagger} \geq 0\}, \quad S(\mathcal{H}) = \bigcap_{\rho \in \mathcal{A}(\mathcal{H})} \{m_\rho \leq \ell_\rho \leq M_\rho\}.$$

In particular, if $s \in S(\mathcal{H})$ then $\|s\|^2 \leq M_s$.

Proof. We only have to prove the second equality. If $x \in \mathcal{A}(\mathcal{H})$ and $m_\rho \leq \ell_\rho(x) \leq M_\rho$ for all $\rho \in \mathcal{A}(\mathcal{H})$, then $x \in S(\mathcal{H})$. Suppose that $x \notin S(\mathcal{H})$. Then, there exists ℓ separating x and $S(\mathcal{H})$. Assume that there exist also $d \in \mathcal{R}$ such that

$$\ell(x) > d,\ \ell(s) \leq d \quad \forall s \in \mathcal{S}(\mathcal{H}).$$

Let M be the maximum of ℓ over $\mathcal{S}(\mathcal{H})$. Then $M \leq d$. Note that ℓ is one of the $\{\ell_\rho\}$ used in the intersection operation above, so that we would have $\ell(x) \leq M \leq d$, which is a contradiction! Accordingly,

$$\ell(x) < d,\ \ell(s) \geq d \quad \forall s \in \mathcal{S}(\mathcal{H}).$$

Now, let m be the minimum of ℓ over $\mathcal{S}$. Then one would have $d \leq m$, which is again a contradiction! □

Remark that any set of the form $\{m_\rho \le \ell_\rho \le M_\rho\} \cap \mathcal{C}$ is always convex (and thus an element of $\mathcal{L}_\mathcal{C}$). Consequently, the equality stated in the above theorem

$$S(\mathcal{H}) = \bigcap_{\rho \in \mathcal{A}(\mathcal{H})} \{m_\rho \le \ell_\rho \le M_\rho\} \tag{80}$$

may be recast in lattice theoretical terms as

$$S(\mathcal{H}) = \bigwedge_{\rho \in \mathcal{A}(\mathcal{H})} \{m_\rho \le \ell_\rho \le M_\rho\} \cap \mathcal{C} \tag{81}$$

The difference between equations (80) and (81) is both subtle and important: one of them is expressed in lattice form. In the previous theorem we aimed to characterize the separable states via a consideration of all the linear functionals. In practice we just want to know if a *given* state is separable. For such query we will use a theorem on projections over convex sets [64, V.2].

Proposition 8.9. *Let $\rho \in \mathcal{C}$ Then, there exists a linear functional ℓ and a real number M such that*

$$\rho \notin S(\mathcal{H}) \iff \ell(\rho) \ge M.$$

Proof. If $\rho \notin S(\mathcal{H})$ then the distance between ρ and $S(\mathcal{H})$ is positive. Let $s \in S(\mathcal{H})$ be its "projection" in the sense that the overlaps

$$\langle \rho - s, c\rangle \le \langle \rho - s, s\rangle, \quad \forall c \in S(\mathcal{H}).$$

Let $\ell := \langle \rho - s, \cdot\rangle$ and $M := \langle \rho - s, s\rangle = \ell(s)$,

$$\ell(\rho) = \ell(\rho) - M + M = \langle \rho - s, \rho - s\rangle + M = \|\rho - s\|^2 + M > M.$$

Then ℓ separates ρ and $S(\mathcal{H})$. If $\rho \in S(\mathcal{H})$ then $\ell(\rho) = \ell(s) = M$. □

The following definition and subsequent proposition will be useful for our separability criteria

Definition 8.10. *If we identify $x \in S_1(\mathcal{H})$ with a matrix $X \in \mathcal{C}^{n\times m}$, $X_{ij} = x_{m(i-1)+j}$, we have*

$$x^\dagger(v \otimes w) = v^\dagger X w.$$

Let $\{v_i\} \subseteq \mathcal{H}_1$, $\{w_i\} \subseteq \mathcal{H}_2$ and $\sigma_1 \ge \ldots \ge \sigma_r > 0$ be the singular value decomposition of X. It is known that the maximum of the bilinear form $v^\dagger X w$ over $S_1(\mathcal{H}_1) \times S_1(\mathcal{H}_2)$ is σ_1.

Proposition 8.11.

$$S(\mathcal{H}) \subseteq \bigcap_{x \in S_1(\mathcal{H})} \{\rho \in \mathcal{C}(\mathcal{H}) \,|\, \langle xx^\dagger, \rho\rangle \le \sigma_1(X)^2\} = \bigcap_{x \in S_1(\mathcal{H})} \{0 \le \ell_{xx^\dagger} \le \sigma_1(X)^2\}.$$

Proof. A general separable state is a convex combination of product states $vv^\dagger \otimes ww^\dagger$,

$$\langle xx^\dagger, vv^\dagger \otimes ww^\dagger\rangle = |\langle x, v \otimes w\rangle|^2 \le \sigma_1^2, \quad \langle xx^\dagger, v_1v_1^\dagger \otimes w_1w_1^\dagger\rangle = \sigma_1^2 \Longrightarrow \max_{S(\mathcal{H})} \ell_{xx^\dagger} = \sigma_1^2.$$

Note that $\sigma_1^2 = 1$ if and only if $xx^\dagger = v_1v_1^\dagger \otimes w_1w_1^\dagger \in S(\mathcal{H})$. Let $\rho \in \mathcal{C}(\mathcal{H})$ and assume that there exist a state $xx^\dagger$ such that $\langle xx^\dagger, \rho\rangle > \sigma_1(x)^2$. Then, $\rho \notin S(\mathcal{H})$. □

The first inclusion of Proposition 8.11 can be written in the language of $\mathcal{L}_C$. This is so because, for a fixed $x \in S_1$, the set

$$C_x = \{\rho \in \mathcal{C}(\mathcal{H}) \,|\, \langle xx^\dagger, \rho \rangle \leq \sigma_1(X)^2\}, \tag{82}$$

is convex. In order to better appreciate this fact, suppose that $\rho_1 \in C_x$ and $\rho_2 \in C_x$. Then, we have $\langle xx^\dagger, \rho_1 \rangle \leq \sigma_1(x)^2$ and $\langle xx^\dagger, \rho_2 \rangle \leq \sigma_1(x)^2$. Multiplying the first inequality by $\lambda \in (0,1)$ and the second one by $(1-\lambda)$, we easily find that $\langle xx^\dagger, (\lambda\rho_1 + (1-\lambda)\rho_2) \rangle \leq \sigma_1(x)^2$. This proves that C_x is convex. Thus, for each $x \in S_1$, $C_x \in \mathcal{L}_\mathcal{C}$. Accordingly, we can state (using a lattice theoretical language) that

Proposition 8.12. $\mathcal{S}(\mathcal{H}) \leq \bigwedge_{x \in S_1(\mathcal{H})} C_x$.

The above proposition shows that the set of separable states is included in the conjunction of a collection of special elements of $\mathcal{L}(\mathcal{C})$. This leads to a new (partial) separability criteria, because from Proposition 8.12 it rapidly follows that

Given the state ρ, if there exists $x \in S_1(\mathcal{H})$ such that $\rho \notin C_x$, then $\rho \in \mathcal{E}(\mathcal{H})$.

The above discussion can be easily rephrased to prove the following theorem

Theorem 1. *Use the Cholesky and the singular value decompositions to write $\rho \in \mathcal{C}(\mathcal{H})$ as a sum $\rho = \sum_{i=1}^{s} \lambda_i x_i x_i^\dagger$, with $x_i^\dagger x_j = 0$,*

$$\rho = LL^\dagger = (U\Sigma V^\dagger)(U\Sigma V^\dagger)^\dagger = U\Sigma^2 U^\dagger \Longrightarrow$$

$$\langle x_j x_j^\dagger, \rho \rangle = \langle x_j x_j^\dagger, \sum_{i=1}^{s} \lambda_i x_i x_i^\dagger \rangle = \sum_{i=1}^{s} \lambda_i \langle x_j x_j^\dagger, x_i x_i^\dagger \rangle = \lambda_j.$$

If for some j, $\lambda_j > \sigma_1(x_j)^2$ then, $\rho \notin S(\mathcal{H})$.

9 The characterization of entanglement using informational invariants

In this section we study the relationship between the lattice $\mathcal{L}_\mathcal{C}$ of a system S composed of subsystems S_1 and S_2, and the lattices of its subsystems, $\mathcal{L}_{\mathcal{C}1}$ and $\mathcal{L}_{\mathcal{C}2}$ respectively. As in [31], we do this by concocting a physical interpretation of the maps which can be defined between them. Recall that we are working with spaces of arbitrary dimension.

9.1 Separable States (Going Up)

Let us define:

Definition 9.1. *Given $C_1 \subseteq \mathcal{C}_1$ and $C_2 \subseteq \mathcal{C}_2$*

$$C_1 \otimes C_2 := \{\rho_1 \otimes \rho_2 \,|\, \rho_1 \in C_1, \rho_2 \in C_2\} \tag{83}$$

Then, we define the map:

Definition 9.2.

$$\Lambda : \mathcal{L}_{\mathcal{C}1} \times \mathcal{L}_{\mathcal{C}2} \longrightarrow \mathcal{L}_{\mathcal{C}}$$
$$(C_1, C_2) \longrightarrow \overline{conv(C_1 \otimes C_2)}$$

where the bar denotes closure respect to norm.

In the rest of this work we will implicitly use the following proposition (see for example [65]):

Proposition 9.3. *Let S be a subset of a linear space $\mathcal{L}$. Then $x \in conv(S)$ iff x is contained in a finite dimensional simplex Δ whose vertices belong to S.*

From equation (48) and definition 6.5 it should be clear that

$$\Lambda(\mathcal{C}_1, \mathcal{C}_2) = \mathcal{S}(\mathcal{H}) \tag{84}$$

Definition 6.5 also implies that for all $C_1 \subseteq \mathcal{C}_1$ and $C_2 \subseteq \mathcal{C}_2$:

$$\Lambda(C_1, C_2) = \overline{C_1 \tilde{\otimes} C_2} \tag{85}$$

Proposition 9.4. *Let $\rho = \rho_1 \otimes \rho_2$, with $\rho_1 \in \mathcal{C}_1$ and $\rho_2 \in \mathcal{C}_2$. Then $\{\rho\} = \Lambda(\{\rho_1\}, \{\rho_2\})$ with $\{\rho_1\} \in \mathcal{L}_{C1}$, $\{\rho_2\} \in \mathcal{L}_{C2}$ and $\{\rho\} \in \mathcal{C}$.*

Proof. We already know that atoms are special elements of lattices. Thus,

$$\Lambda(\{\rho_1\}, \{\rho_2\}) = conv(\{\rho_1 \otimes \rho_2\}) = \{\rho_1 \otimes \rho_2\} = \{\rho\} \tag{86}$$

□

Proposition 9.5. *Let $\rho \in \mathcal{S}(\mathcal{H})$, the set of separable states. Then, there exists $C \in \mathcal{L}_{\mathcal{C}}$, $C_1 \in \mathcal{L}_{\mathcal{C}_1}$ and $C_2 \in \mathcal{L}_{\mathcal{C}_2}$ such that $\rho \in C = \Lambda(C_1, C_2)$.*

Proof. Let $\{\rho_n\}_{n=1}^{\infty} \subseteq \mathcal{S}(\mathcal{H})$ be a sequence in the interior of $\mathcal{S}(\mathcal{H})$ such that $\rho_n \to \rho$, then $\rho_n = \sum_i \lambda_i \phi_i^n \otimes \psi_i^n$, with $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0$. Consider the convex sets:

$$C_1 = conv(\{\phi_i^n\}_{i,n}) \in \mathcal{L}_{\mathcal{C}1}, \quad C_2 = conv(\{\psi_i^n\}_{i,n}) \in \mathcal{L}_{\mathcal{C}2}, \quad C = \Lambda(C_1, C_2) \in \mathcal{L}_{\mathcal{C}}.$$

Clearly, $\phi_i^n \otimes \psi_i^n \in C_1 \otimes C_2$, and then $\rho_n \in C$ for all $n \in \mathcal{N}$. Given that C is closed, we have $\rho \in C$. □

9.2 Projections Onto $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathcal{C}_2}$ (Going Down)

Let us now study the projections onto $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathcal{C}_2}$. In the next proposition we will see that they are well defined. Using the partial trace maps we can construct the induced projections:

$$\tau_i : \mathcal{L}_{\mathcal{C}} \longrightarrow \mathcal{L}_{\mathcal{C}_i} \tag{87a}$$

$$C \mapsto \mathrm{tr}_i(C) \tag{87b}$$

Then we can define the product map

$$\tau : \mathcal{L}_{\mathcal{C}} \longrightarrow \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} \tag{88a}$$

$$C \mapsto (\tau_1(C), \tau_2(C)) \tag{88b}$$

We use the same notation for τ and τ_i (though they have different domains) as in [31] and section 6, and this should not introduce any difficulty. We can prove the following about the image of τ_i.

Proposition 9.6. *The maps τ_i preserve the convex structure, i.e., they map convex sets into convex sets.*

Proof. Let $C \subseteq \mathcal{C}$ be a convex set. Let C_1 be the image of C under τ_1 (a similar argument holds for τ_2). Let us show that C_1 is convex. Let ρ_1 and ρ_1' be elements of C_1. Consider $\sigma_1 = \alpha\rho_1 + (1-\alpha)\rho_1'$, with $0 \leq \alpha \leq 1$. Then, there exists $\rho, \rho' \in \mathcal{C}$ such that:

$$\sigma_1 = \alpha \mathrm{tr}_1(\rho) + (1-\alpha)\mathrm{tr}_1(\rho') = \mathrm{tr}_1(\alpha\rho + (1-\alpha)\rho') \tag{89}$$

where we have used the linearity of trace. Because of convexity of C, $\sigma := \alpha\rho + (1-\alpha)\rho' \in C$, and so, $\sigma_1 = \mathrm{tr}_1(\sigma) \in C_1$. □

Proposition 9.7. *The functions τ_i are surjective and preserve the $\vee$-operation. They are not injective.*

Proof. Take the convex set $C_1 \in \mathcal{L}_{\mathcal{C}_1}$. Choose an arbitrary element of $\mathcal{C}_2$, say ρ_2. Now consider the following element of $\mathcal{L}_{\mathcal{C}}$

$$C = C_1 \otimes \rho_2 \tag{90}$$

C is convex, and so belongs to $\mathcal{L}_{\mathcal{C}}$, because if $\rho \otimes \rho_2, \sigma \otimes \rho_2 \in C$, then any convex combination $\alpha\rho \otimes \rho_2 + (1-\alpha)\sigma \otimes \rho_2 = (\alpha\rho + (1-\alpha)\sigma) \otimes \rho_2 \in C$ (where we have used the convexity of C_1). It is clear that $\tau_1(C) = C_1$, because if $\rho_1 \in C_1$, then $\mathrm{tr}_1(\rho_1 \otimes \rho_2) = \rho_1$. So, τ_1 is surjective. On the other hand, the arbitrariness of ρ_2 implies that it is not injective. An analogous argument follows for τ_2.
Let us see that τ_i preserves the $\vee$-operation. Let C and C' be convex subsets of $\mathcal{C}$. We must compute $\mathrm{tr}_2(C \vee C')) = \mathrm{tr}_2(conv(C, C'))$. We ought to show that this is the same as $conv(\mathrm{tr}_2(C), \mathrm{tr}_2(C'))$. Take $x \in conv(\mathrm{tr}_2(C), \mathrm{tr}_2(C'))$. Then $x = \alpha\mathrm{tr}_2(\rho) + (1-\alpha)\mathrm{tr}_2(\rho')$, with $\rho \in C$, $\rho' \in C'$ and $0 \leq \alpha \leq 1$. Using the linearity of trace, $x = \mathrm{tr}_2(\alpha\rho + (1-\alpha)\rho')$. $\alpha\rho + (1-\alpha)\rho' \in conv(C, C')$, and so, $x \in \mathrm{tr}_2(conv(C, C'))$. Hence we have

$$conv(\mathrm{tr}_2(C), \mathrm{tr}_2(C')) \subseteq \mathrm{tr}_2(conv(C, C')) \tag{91}$$

In order to prove the other inclusion, take $x \in \mathrm{tr}_2(conv(C, C'))$. Then,

$$x = \mathrm{tr}_2(\alpha\rho + (1-\alpha)\rho') = \alpha\mathrm{tr}_2(\rho) + (1-\alpha)\mathrm{tr}_2(\rho') \tag{92}$$

with $\rho \in C_1$ and $\rho' \in C'$. Note that $\mathrm{tr}_2(\rho) \in \mathrm{tr}_2(C)$ and $\mathrm{tr}_2(\rho') \in \mathrm{tr}_2(C')$. This proves that

$$\mathrm{tr}_2(conv(C, C')) \subseteq conv(\mathrm{tr}_2(C), \mathrm{tr}_2(C'))$$

□

Let us now consider the $\wedge$-operation. If $x \in \tau_i(C \wedge C') = \tau_i(C \cap C')$ then $x = \tau_i(\rho)$ with $\rho \in C \cap C'$. But, if $\rho \in C$, then $x = \tau_i(\rho) \in \mathrm{tr}_i(C)$. As $\rho \in C'$ as well, a similar argument shows that $x = \tau_i(\rho) \in \mathrm{tr}_i(C')$. Then, $x \in \tau_i(C) \cap \tau_i(C')$ and

$$\begin{tikzcd}
 & \mathcal{L}_{\mathcal{C}} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] \arrow[d, shift left, "\tau"] & \\
\mathcal{L}_{\mathcal{C}_1} & \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2} \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[u, shift left, "\Lambda"] & \mathcal{L}_{\mathcal{C}_2}
\end{tikzcd}$$

Figure 2: The different maps between $\mathcal{L}_{\mathcal{C}_1}$, $\mathcal{L}_{\mathcal{C}_2}$, $\mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2}$, and $\mathcal{L}_{\mathcal{C}}$

$$\tau_i(C \cap C') \subseteq \tau_i(C) \cap \tau_i(C'), \tag{93}$$

which is tantamount to

$$\tau_i(C \wedge C') \leq \tau_i(C) \wedge \tau_i(C'). \tag{94}$$

These sets are not, in general, equal. The following example illustrates the assertion. Take $\{\rho_1 \otimes \rho_2\} \in \mathcal{L}$ and $\{\rho_1 \otimes \rho'_2\} \in \mathcal{L}$, with $\rho' \neq \rho$. It is clear that $\{\rho_1 \otimes \rho_2\} \wedge \{\rho_1 \otimes \rho'_2\} = \mathbf{0}$ and so, $\tau_1(\{\rho_1 \otimes \rho_2\} \wedge \{\rho_1 \otimes \rho'_2\}) = \mathbf{0}$. On the other hand, $\tau_1(\{\rho_1 \otimes \rho_2\}) = \{\rho_1\} = \tau_1(\{\rho_1 \otimes \rho'_2\})$, and then $\tau_1(\{\rho_1 \otimes \rho_2\}) \wedge \tau_1(\{\rho_1 \otimes \rho'_2\}) = \{\rho_1\}$. A similar reasoning holds for the $\neg$-operation.

9.3 Geometrical Characterization of Entanglement

We have shown that it is possible to extend $\mathcal{L}_{v\mathcal{N}}$ in order to deal with statistical mixtures and that $\mathcal{L}$ and $\mathcal{L}_{\mathcal{C}}$ are possible extensions. It would be interesting to search for a characterization of entanglement within this framework. Let us see first what happens with the functions $\Lambda \circ \tau$ and $\tau \circ \Lambda$. We have:

Proposition 9.8. *$\tau \circ \Lambda(C_1, C_2) = (C_1, C_2)$ for every closed convex sets $C_1 \subseteq \mathcal{C}_1$ and $C_2 \subseteq \mathcal{C}_2$.*

Proof.

$$\tau_1(\Lambda(C_1, C_2)) = \tau_1(\overline{conv(C_1 \otimes C_2)}) = \mathrm{tr}_1(\overline{conv(C_1 \otimes C_2)}) = \overline{C_1} = C_1$$

$$\tau_2(\Lambda(C_1, C_2)) = \tau_2(\overline{conv(C_1 \otimes C_2)}) = \mathrm{tr}_2(\overline{conv(C_1 \otimes C_2)}) = \overline{C_2} = C_2$$

Then, $\tau(\Lambda(C_1, C_2)) = (C_1, C_2)$. □

Again, as in [31], if we take into account simple physical considerations, $\Lambda \circ \tau$ is not the identity function, because when we take partial traces we face the risk of losing information, that will not be recovered when we multiply states. Thus we reach the same conclusion as before [31]: *"going down and then going up is not the same as going up and then going down"*. We depict w the pertinent maps in Figure 9.2. How is this stuff related to entanglement? If we restrict $\Lambda \circ \tau$ to the set of product states, then it does reduce itself to the identity function. Indeed, if $\rho = \rho_1 \otimes \rho_2$, then:

$$\Lambda \circ \tau(\{\rho\}) = \{\rho\}. \tag{95}$$

On the other hand, it should be clear that if ρ is an entangled state

$$\Lambda \circ \tau(\{\rho\}) \neq \{\rho\}, \tag{96}$$

because $\Lambda \circ \tau(\{\rho\}) = \{\mathrm{tr}_2(\rho) \otimes \mathrm{tr}_1(\rho)\} \neq \{\rho\}$ for any entangled state. This property can be regarded as a signpost for entanglement. There are mixed states which are not product states. Thus, entangled states are not the only ones satisfying equation (96). What is the condition satisfied for a general mixed state? The following proposition summarizes the preceding considerations.

Proposition 9.9. *If ρ is a separable state, then there exists a convex set, $S_\rho \subseteq \mathcal{S}(\mathcal{H})$ such that $\rho \in S_\rho$ and $\Lambda \circ \tau(S_\rho) = S_\rho$. More generally, for a convex set $C \subseteq \mathcal{S}(\mathcal{H})$, there exists a convex set $S_C \subseteq \mathcal{S}(\mathcal{H})$ such that $\Lambda \circ \tau(S_C) = S_C$. For a product state, we can choose $S_\rho = \{\rho\}$. If ρ can be written as a finite convex sum of product states, then the convex set S_ρ can be taken as a polytope. On the other hand, for any $C \in \mathcal{L}_C$ which has at least one non-separable state, there is NO convex set S such that $C \subseteq S$ and $\Lambda \circ \tau(S) = S$.*

Proof. Product case. We have already seen above that if ρ is a product state, then $\Lambda \circ \tau(\{\rho\}) = \{\rho\}$, and so $S_\rho = \{\rho\}$.
Finite combination case. If ρ can be written as a finite convex combination of product states, then there exists $\rho_k^A \in \mathcal{C}_1$, $\rho_k^B \in \mathcal{C}_2$ and $\alpha_k^i \geq 0, \sum_{k=1}^N \alpha_k^i = 1$ such that

$$\rho = \sum_{k=1}^{N} \alpha_k \rho_k^A \otimes \rho_k^B \tag{97}$$

Define first

$$S_\rho = conv(\{\rho_k^A \otimes \rho_l^B\}) \tag{98}$$

S_ρ is the closed set of all convex combinations of products of the elements appearing in the decomposition of ρ. It should be clear that $\rho \in S_\rho$. Let us compute $\Lambda \circ \tau(S_\rho)$.

$$\mathrm{tr}_1(\sigma) = \sum_{k=1}^{N} (\sum_{l=1}^{N} \lambda_{kl}) \rho_k^A = \sum_{k=1}^{N} \mu_k \rho_k^A, \quad \mu_k := \sum_{l=1}^{N} \lambda_{kl}. \tag{99}$$

In an analogous way we may show that an element of $\tau_2(S_\rho)$ has the form $\sum_{l=1}^N \nu_l \rho_l^A$ with $\nu_l = \sum_{k=1}^N \lambda_{kl}$. Note that $\sum_{k=1}^N \mu_k = \sum_{l=1}^N \nu_l = 1$. In order to compute $\Lambda(\tau_1(S_\rho), \tau_2(S_\rho))$ we must construct the convex hull of the set

$$\tau_1(S_\rho) \otimes \tau_2(S_\rho) = \{\sigma_1 \otimes \sigma_2 | \sigma_1 \in \tau_1(S_\rho), \sigma_2 \in \tau_2(S_\rho)\} = \{\sum_{k,l=1}^{N} \mu_k \nu_l \rho_k^A \otimes \rho_l^B\}, \tag{100}$$

and we conclude that

$$\Lambda \circ \tau(S_\rho) = conv(\{\sum_{kl=1}^{N} \mu_k \nu_l \rho_k^A \otimes \rho_l^B\}) = conv(\{\rho_k^A \otimes \rho_l^B\}) = S_\rho. \tag{101}$$

It is apparent that S_ρ is a polytope.

Limit point case. There is still another possibility. Namely if ρ cannot be written as in (97), but there exists $\rho_{ik}^A \in \mathcal{C}_1$, $\rho_{ik}^B \in \mathcal{C}_2$ and $\alpha_{ik} \geq 0, \sum_{k=1}^{N_i} \alpha_{ik} = 1$ such that $\rho^i = \sum_{k=1}^{N_i} \alpha_{ik} \rho_{ik}^A \otimes \rho_{ik}^B$ converges to ρ as i goes to infinity. Consider the set of all possible products of states which appear in the decomposition of the ρ^i, namely

$$S_0 := \{\rho_{ik}^A \otimes \rho_{i'l}^B, \text{for all } i, i', k, l\} \tag{102}$$

and define the closure of its convex hull as

$$S_\rho := \overline{conv(S_0)} \tag{103}$$

Remember that convex hull means only taking finite sums. It is clear that $\rho \in S_\rho$ and that S_ρ is convex by construction (the closure of a convex set is also convex). Let us see what happens when we apply $\Lambda \circ \tau$ to S_ρ,

$$\Lambda(\tau(S_\rho)) = \overline{conv(\tau_1(S_\rho) \otimes \tau_2(S_\rho))} \tag{104}$$

Any of the ρ_{ik}^A belong to $\tau_1(S_\rho)$ (the same for $\rho_{i'k}^B$ and $\tau_2(S_\rho)$). Then, it is clear that $S_0 \subseteq conv(\tau_1(S_\rho) \otimes \tau_2(S_\rho))$. As $conv(\tau_1(S_\rho) \otimes \tau_2(S_\rho))$ is convex and the closure of sets preserves the inclusion, then we have $S_\rho \subseteq \Lambda \circ \tau(S_\rho))$ (look at Equation (103)). On the other hand, any element ρ_1 of $\tau_1(S_\rho)$ can be written as a finite sum $\rho_1 = \sum \alpha_{ik} \rho_{ik}^A$ ($\sum \alpha_{ik} = 1$, $\alpha_{ik} \geq 0$) or as a limit of such finite sums (we are using a property of partial traces tr_i: they are continuous linear maps). The same happens for an element $\rho_2 \in \tau_2(S_\rho)$ (taking the tensor product of density operators produces a continuous map). Then, any element of $\tau_1(S_\rho) \otimes \tau_2(S_\rho)$ may be written as a finite sum $\sum \alpha_{ik} \beta_{i'l} \rho_{ik}^A \otimes \rho_{i'l}^B$ or as a limit of such sums. This means that any element of $\tau_1(S_\rho) \otimes \tau_2(S_\rho)$ is also an element of S_ρ. As S_ρ is convex and closed by construction, we will have $\Lambda \circ \tau(S_\rho) = \overline{conv(\tau_1(S_\rho) \otimes \tau_2(S_\rho))} \subseteq S_\rho$, which proves that $\Lambda \circ \tau(S_\rho) = S_\rho$.

The space of separable states $\mathcal{S}(\mathcal{H})$ is a convex set. Let us see that it is invariant under $\Lambda \circ \tau$. First of all, we know that $\mathcal{S}(\mathcal{H})$ is formed by the closure of all possible convex combinations of products of the form $\rho_1 \otimes \rho_2$, with $\rho_1 \in \mathcal{C}_1$ and $\rho_2 \in \mathcal{C}_2$. But each one of these tensor products, $\Lambda \circ \tau(\{\rho_1 \otimes \rho_2\}) = \{\rho_1 \otimes \rho_2\}$, belongs to $\Lambda \circ \tau(\mathcal{S}(\mathcal{H}))$. Given that $\Lambda \circ \tau(\mathcal{S}(\mathcal{H}))$ is a closed convex set, we have $\Lambda \circ \tau(\mathcal{S}(\mathcal{H})) \supseteq \mathcal{S}(\mathcal{H})$. On the other hand we know that the image of $\Lambda \circ \tau$ is always separable, so we can conclude that

$$\Lambda \circ \tau(\mathcal{S}(\mathcal{H})) = \mathcal{S}(\mathcal{H}) \tag{105}$$

Now, consider $C \in \mathcal{L}_{\mathcal{C}}$ such that there exists $\rho \in C$, with ρ nonseparable. Given that $\Lambda \circ \tau(S) \subseteq \mathcal{S}(\mathcal{H})$ for all $S \in \mathcal{L}_{\mathcal{C}}$, it could never happen that there exists $S \in \mathcal{L}_{\mathcal{C}}$ such that $C \subseteq S$ and $\Lambda \circ \tau(S) = S$.

□

From the last proposition, we conclude that there exists an interesting property which the convex subsets of separable states satisfy, while convex subsets which include non-separable states do not. This "existence theorem" motivates the following definition for the proposition $C \in \mathcal{L}_{\mathcal{C}}$:

Definition 9.10. *$C \in \mathcal{L}_{\mathcal{C}}$ is a separable proposition if there exists $S_C \in \mathcal{L}_{\mathcal{C}}$ such that $\Lambda \circ \tau(S_C) = S_C$ and $C \subseteq S_C$. Otherwise, it is a non-separable or entangled proposition. The definition is equivalent to the statement $C \subseteq \mathcal{S}(\mathcal{H})$.*

Another conclusion of proposition 9.9 is that a density matrix ρ is separable iff there exists a convex set S_ρ such that $\rho \in S_\rho$ and $\Lambda \circ \tau(S_\rho) = S_\rho$. Thus, proposition 9.8 also provides an entanglement criterium **which includes the infinite dimensional case** (see also [36]):

$$\boxed{\rho \in \mathcal{S}(\mathcal{H}) \Longleftrightarrow \text{ there exists a convex set } S_\rho \text{ with } \rho \in S_\rho \text{ such that } \Lambda \circ \tau(S_\rho) = S_\rho. \quad (106)}$$

9.4 An unifying generalization for the entanglement of mixed states

In the last section we have introduced a new separability criterium which is also valid for the infinite dimensional case. Now we proceed to an important issue regarding pure states (see also the discussions in [36]). It is a well known fact that pure states are separable, if and only if they are product states. This means that $|\psi\rangle\langle\psi|$ will be separable if and only if there exist $|\varphi_1\rangle$ and $|\varphi_2\rangle$ such that $|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle$. This implies that the state $|\psi\rangle\langle\psi|$ is invariant under the map

$$\begin{aligned} \Omega : \mathcal{C} &\longrightarrow \mathcal{C} \\ \rho &\mapsto \rho^A \otimes \rho^B, \end{aligned} \tag{107}$$

and this in turn means that

$$|\psi\rangle\langle\psi| \in \mathcal{S}(\mathcal{H}) \Longleftrightarrow \Omega(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|. \tag{108}$$

Such simple separability criterium for the pure case is unfortunately invalid for the mixed case. In what follows we show that our separability criteria allows for an interesting unifying generalization.

First of all, notice that only product states exhibit the property of being invariant under Ω. Is there any generalization of Ω and of the notion of product states? Let us look in more detail to the invariance under Ω-property. In mathematical terms, suppose that a state ρ satisfies

$$\Omega(\rho) = \rho. \tag{109}$$

This is equivalent to stating that ρ can be fully recovered from its reduced states by using local operations. It is easy to show that the function $\Lambda \circ \tau$ satisfies

$$\Lambda \circ \tau(\Lambda \circ \tau(C)) = \Lambda \circ \tau(C), \tag{110}$$

and this is equivalent to

$$(\Lambda \circ \tau)^2 = \Lambda \circ \tau, \tag{111}$$

a property that Ω also satisfies, i.e.

$$\Omega^2 = \Omega, \tag{112}$$

as may be easily checked out. It is trivially shown that, when restricted to "one point" convex subsets of the form $\{\rho\}$ (an arbitrary state), $\Lambda \circ \tau$ coincides with Ω, that is

$$\Lambda \circ \tau(\{\rho\}) = \{\rho^A \otimes \rho^B\} = \{\Omega(\rho)\}. \tag{113}$$

Equations (111), (112) and (113) clearly suggest that $\Lambda \circ \tau$ is a suitable generalization of Ω to arbitrary convex subsets (a single state being a particular case of one point convex sets). The separability criterium presented in section 9 provides the clue for generalizing product states to convex subsets, i.e., the convex set generalization of a product state C will satisfy

$$\Lambda \circ \tau(C) = C, \tag{114}$$

and this reduces to the the separability properties defined in section 9. The special subsets of $\mathcal{C}$ that we are concerned with exhibit the following property: they can be fully recovered via all possible tensor products and mixtures of its sets of reduced states. More specifically, given a convex set C satisfying (114), it can be recovered from the sets of its reduced states, namely $\tau_1(C)$ and $\tau_2(C)$ via all possible tensor products and all possible convex mixtures. In physical terms this means that they can be recovered using classical and local operations (just adding systems via all possible tensor products and then considering all possible mixtures of the resulting states). The content of this discussion is compactly encapsulated into equation (114). Convex subsets with this property where termed *Convex Invariant Subsets* (CSS) in [36]. Now it should be clear that CSS are proper generalizations of product states to arbitrary convex subsets.

We can now generalize equation (108) to arbitrary states as follows. Separability criterium 106 implies that a state ρ is separable iff it belongs to a CSS C such that $\Lambda \circ \tau(C) = C$. The analogy with the pure-states case is clear if we effect the identification $\rho \longrightarrow \{\rho\}$ (i.e., the state considered as an element to the state considered as a particular case of convex subset).

We have thus shown that the map $\Lambda \circ \tau$ is a suitable generalization of Ω. The sets invariant under Ω are product states and the sets invariant under $\Lambda \circ \tau$ are CSS, a suitable generalization of product states. We may now generalize equation (108) to any state as follows:

$$\rho \in \mathcal{S}(\mathcal{H}) \Longleftrightarrow \exists C \text{ (a CSS)}, \, \Lambda \circ \tau(C) = C, \tag{115}$$

a neat extension of (108). For the finite dimensional case the analogy is stronger still: criterium (108) can be rephrased using von Neumann's entropy

$$S(\rho) = -\mathrm{tr}(\rho \ln(\rho)), \tag{116}$$

as follows:

$$\rho \in \mathcal{S}(\mathcal{H}) \Longleftrightarrow S(\rho^A) = 0 = S(\rho^B), \tag{117}$$

where ρ^A and ρ^B are the reduced states of ρ. As products of pure states generate (in the convex sense) all separable states, it is possible to show that the CSS-criterium may be, in particular, chosen to be generated by products of pure states. von Neumann's entropy reaches its minimum value in such an instance. Summing up:

$\rho \in \mathcal{S}(\mathcal{H}) \Longleftrightarrow$ there exist C such that $\rho \in C$ and $\Lambda \circ \tau(C) = C \Longleftrightarrow$ (finite dimension) there exist C such that $\rho \in C$, $\Lambda \circ \tau(C) = C$ and $\inf\{S(\sigma) \,|\, \sigma \in C\} = 0$.

The fact that the above structure may be found for arbitrary states is a clear conceptual simplification for the characterization of entanglement, providing a unifying framework which generalizes (108) to arbitrary states. In the following section we outline how this geometrical structure extends to arbitrary COMs, and thus to any statistical theory.

10 Entanglement and separability in arbitrary convexity models

In Section 3 we reviewed how to construct a general setting for convex operational models out of which the quantum case was a particular example. In this section we study how to extend our geometrical formulation of entanglement to arbitrary statistical models.

Given two convex operational models $(\mathbf{A}, \mathbf{A}^\sharp, u_{\mathbf{A}})$ and $(\mathbf{B}, \mathbf{B}^\sharp, u_{\mathbf{B}})$, a morphism between them will be given by a positive linear map $\phi : \mathbf{A} \to \mathbf{B}$ such that the linear adjoint map $\phi^* : \mathbf{B}^* \to \mathbf{A}^*$ is positive with respect to the cones $\mathbf{A}^\sharp_+$ and $\mathbf{B}^\sharp_+$.

A link between (or process from)) $\mathbf{A}$ - $\mathbf{B}$ will be represented by a morphism $\phi : \mathbf{A} \to \mathbf{B}$ such that, for every state $\alpha \in \Omega_{\mathbf{A}}$, $u_{\mathbf{B}}(\phi(\alpha)) \leq 1$ (this is a normalization condition). $u_{\mathbf{B}}(\phi(\alpha))$ will

represent the probability that the process represented by ϕ take place. For the special case of quantum mechanics, we will show that the above processes preserve the convex structure of the cone of positive self adjoint operators. Also, we demonstrate that when the processes preserve trace (i.e., when they map density operators into density operators and thus represent quantum evolutions), they will also preserve the lattice structure of $\mathcal{L}_{\mathcal{C}}$.
In the preceding Section we saw how to characterize entanglement and separability using maps between elements of $\mathcal{L}_{\mathcal{C}}$ and $\mathcal{L}_{\mathcal{C}_i}$. The interesting point here is that the most salient feature of our lattices is their convex structure, and this will allow us to extend the notions of entanglement and separability to any COM. This is done as follows. In [5] extensions of COM's are studied (we review here their definition of extension slightly modifying the reference's notation). A COM $(\mathbf{C}, \mathbf{C}^{\sharp}, u_{\mathbf{C}})$ will be said to be an extension of $(\mathbf{A}, \mathbf{A}^{\sharp}, u_{\mathbf{A}})$ if there exists a morphism $\phi : \mathbf{C} \to \mathbf{A}$ which is surjective.
In order to look for a generalization of entanglement which captures the results of previous Sections we must look at triads of COM's $(\mathbf{C}, \mathbf{C}^{\sharp}, u_{\mathbf{C}})$, $(\mathbf{C}_1, \mathbf{C}_1^{\sharp}, u_{\mathbf{C}_1})$, and $(\mathbf{C}_2, \mathbf{C}_2^{\sharp}, u_{\mathbf{C}_2})$, such that there exist two morphisms ϕ_1 and ϕ_2 with $(\mathbf{C}, \mathbf{C}^{\sharp}, u_{\mathbf{C}})$ an extension of $(\mathbf{C_1}, \mathbf{C_1}^{\sharp}, u_{\mathbf{C_1}})$ and $(\mathbf{C_2}, \mathbf{C_2}^{\sharp}, u_{\mathbf{C_2}})$. It is clear that $\phi = (\phi_1, \phi_2)$ may be considered as the best candidate for a generalization of τ. Now, if we want an analogue of Λ, we must demand additional requirements. We are looking for a map Ψ with the following property. Ψ maps any pair of non-empty convex subsets (C_1, C_2) of $\mathbf{C}_1 \times \mathbf{C}_2$ into a non-empty convex subset C of $\mathbf{C}$ with this particular property: for any $c \in C$, we must have $\phi_1(c) \in C_1$ and $\phi_2(c) \in C_2$. Such property guarantees that for any pair of states a_1 and a_2 there will always exist at least one state $c \in \mathbf{C}$ such that $\phi_1(c) = a_1$ and $\phi_2(c) = b_1$. Why? Because if $C_1 = \{c_1\}$ and $C_2 = \{c_2\}$, then we must have $\phi_1(c) = a_1$ and $\phi_2(c) = b_2$, which guarantees that for any states c_1 and c_2 there will always exist a state c for which c_1 and c_2, respectively, are the reduced states relative to the maps ϕ_i. As the maps ϕ_i are morphisms, using them it is possible to define canonically induced functions on convex subsets, and them to map convex subsets of $\mathbf{C}$ into convex subsets of $\mathbf{C_i}$ (there is an analogy with the earlier language involving τ_i's and partial traces). With some abuse of notation we will keep calling them $\phi_i' s$, without undue harm.

Summing up:

Definition 10.1. *A triad* $(\mathbf{C}, \mathbf{C}^{\sharp}, u_{\mathbf{C}})$, $(\mathbf{C}_1, \mathbf{C}_1^{\sharp}, u_{\mathbf{C}_1})$, *and* $(\mathbf{C}_2, \mathbf{C}_2^{\sharp}, u_{\mathbf{C}_2})$ *will be called a* triple compound system *if*

1. *There exist morphisms* ϕ_1 *and* ϕ_2 *such that* $(\mathbf{C}, \mathbf{C}^{\sharp}, u_{\mathbf{C}})$ *is an extension of* $(\mathbf{C}_1, \mathbf{C}_1^{\sharp}, u_{\mathbf{C}_1})$ *and* $(\mathbf{C}_2, \mathbf{C}_2^{\sharp}, u_{\mathbf{C}_2})$.
2. *There exists a map* $\Psi : \mathcal{P}(\mathbf{C}_1) \times \mathcal{P}(\mathbf{C}_2) \to \mathcal{P}(\mathbf{C})$ *which maps pair of non-empty convex subsets* $(C_1, C_2) \in \mathcal{P}(\mathbf{C}_1) \times \mathcal{P}(\mathbf{C}_2)$ *into a nonempty convex subset* $C \in \mathcal{P}(\mathbf{C})$, *such that for every* $c \in C$, $\phi(c) = (\phi_1(c), \phi_2(c)) \in C_1 \times C_2$.

If the map Ψ *of the triple compound system satisfies that for any* c_1 *and* c_2, $\Psi(\{c_1\}, \{c_2\}) = \{c\}$ *for some* $c \in \mathbf{C}$, *we will say that it is a* strictly two-components triple compound system.

With this constructions at hand, let us restrict for the sake of simplicity to *strictly two-components triple compound systems* and look for a generalization of entanglement and separability. It is clear now that the analogues of the maps Λ and τ are Ψ and ϕ, respectively. Thus, it is natural now state

Definition 10.2. *Given a* strictly two-components triple compound system $(\mathbf{C}, \mathbf{C}^{\sharp}, u_{\mathbf{C}})$, $(\mathbf{C}_1, \mathbf{C}_1^{\sharp}, u_{\mathbf{C}_1})$, *and* $(\mathbf{C}_2, \mathbf{C}_2^{\sharp}, u_{\mathbf{C}_2})$, *with an up-map* Ψ *and a down-map* ϕ, *then*

1. *A state* $c \in \mathbf{C}$ *will be called* non-product state *if* $\Psi \circ \phi(\{c\}) \neq \{c\}$*. Otherwise, it will be called a* product state*.*

2. *For an* invariant convex subset C *one has* $C \in \mathcal{P}(\mathbf{C})$*, such that* $\Psi \circ \phi(C) = C$

3. *If there exist a largest (in the sense of the lattice order) invariant subset, we will denote it by* $\mathcal{S}(\mathbf{C})$*.*

4. *A* strictly two-components triple compound system *for which there exists* $\mathcal{S}(\mathbf{C})$ *and is strictly included in* $\mathbf{C}$*, will be said to be an* entanglement operational model*.*

5. *In an* entanglement operational model *a state* c *which satisfies* $c \notin \mathcal{S}(\mathbf{C})$ *will be said to be entangled.*

It is clear that using these constructions we can export the quantum entanglement structure to a much wider class of COM's, and for that reason, to many new statistical physical systems. It should be clear also that quantum mechanics is the best example for entanglement, and that all states in classical mechanics are separable. Remark that the properties of a *strictly two-components triple compound systems* will depend, in a strong sense, on the choice of the functions Ψ and ϕ. These should be selected as the canonical ones, i.e., the ones which are somehow natural for the physics of the problem under study. Notice that nothing prevents us from make more general choices for practical purposes. The physical criterium for the construction of ψ should be that the simple addition of the systems involved should not generate new correlations. We can also "postulate" a generalized separability criterium:

Definition 10.3.
A state $c \in \mathbf{C}$ *in an* entanglement operational model *is said to be separable iff there exists* $C \subseteq \mathcal{S}(\mathbf{C})$ *containing* c *such that* $\Psi \circ \phi(C) = C$*.*

These constructions may be useful to develop and search for generalizations/corrections of quantum mechanics and for the study of quantum entanglement in theories more general than quantum mechanics. Our constructions are a valid alternative to others that one can find in the literature. An interesting open problem would be that of finding the way in which we can express the violation of Bell's inequalities using this approach.
In this section we restricted ourselves to *strictly two-components compound triples*. An important example of a *two-components compound triple* which is not strict is to look at a quantal three-components systems out of which we only consider two subsystems. In that case, to any product state of the first two subsystems we can add any other state of the third one, and the map Ψ will yield a convex subset of more than one element.

11 Conclusions

In this work we studied different mathematical structures of the convex subsets of the quantum set of states. We showed that these sets are endowed with a canonical lattice structure and extended previous results to the infinite dimensional case. This lattice structure reveals interesting algebraic and geometrical properties of the quantum set of states.
We showed in Section 8 that the lattice structure is strongly linked to functionals and entanglement witness. Thus, many of previous results might be translated into our language. At the end of this Section we also provided a new (partial) entanglement criteria easily expressible in lattice theoretical language. We also showed how this algebraic and geometrical convex set-viewpoint can be used to reformulate the Max-Ent principle in a form extensible to any statistical theory,

via the COM approach. In particular, it may be useful to include fussy measurements (POVM's) into the Max-Ent formalism.
We also extended a previous abstract separability criterium, strongly linked to the lattice structure of convex subsets, to the infinite dimensional case. Furthermore, we showed that this geometrical setting can be exported to any arbitrary statistical model via the COM approach, which may be useful to analyze the classicality of theories which generalize quantum mechanics, and also for the study of semiclassical models.

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A Basic mathematical concepts used in the text

1. A function is surjective (onto) if every possible image is mapped to by at least one argument. In other words, every element in the codomain has non-empty preimage. Equivalently, a function is surjective if its image is equal to its codomain. A surjective function is a surjection.

2. A linear functional (also called a one-form or covector) is a linear map from a vector space to its field of scalars K. In general, if V is a vector space over a field K, then a linear functional f is a function from V to K, which is linear. Linear functionals are particularly important in quantum mechanics. Quantum mechanical systems are represented by Hilbert spaces, which are anti-isomorphic to their own dual spaces. A state of a quantum mechanical system can be identified with a linear functional.

3. Suppose that K is a field (for example, the real numbers) and V is a vector space over K. If $v_1, \ldots, v_n$ are vectors and $a_1, \ldots, a_n$ are scalars, then the linear combination of those vectors with those scalars as coefficients is, of course, $\sum_{i=1}^{n} a_i\, v_i$. By restricting the coefficients used in linear combinations, one can define the related concepts of affine combination, conical combination, and convex combination, together with the associated notions of sets closed under these operations. If $\sum_{i=1}^{n} a_i = 1$, we have an affine combination, its span being an affine subspace while the model space is an hyperplane. If all $a_i \geq 0$, we have instead a conical combination, a convex cone and a quadrant, respectively. Finally, if all $a_i \geq 0$ plus $\sum_{i=1}^{n} a_i = 1$, we have now a convex combination, a convex set and a simplex, respectively.

4. By a $\sigma-$algebra one means a collection of sets that satisfy certain properties, used in the definition of measures: it is the collection of sets over which a measure is defined. The concept is important in probability theory, being there interpreted as the collection of events which can be assigned probabilities. Such an algebra, over a set X, is a nonempty collection S of subsets of X (including X itself) that is closed under complementation and countable unions of its members. It is an algebra of sets, completed to include countably infinite operations. The pair (X, S) is also a field of sets, called a measurable space.

5. A quotient space (also called an identification space) is, intuitively speaking, the result of identifying certain points of a given space. The points to be identified are specified by an equivalence relation. This is commonly done in order to construct new spaces from given ones. Let (X, τ_X) be a topological space, and let R be an equivalence relation on X. The quotient space $Y = X/R$ is defined to be the set of equivalence classes of elements of X:
$$Y = \{[x] : x \in X\} = \{\{v \in X : vRx\} : x \in X\},$$

equipped with the topology where the open sets are defined to be those sets of equivalence classes whose unions are open sets in X. Equivalently, we can define them to be those sets with an open pre-image under the quotient map which sends a point in X to the equivalence class containing it.

6. Banach spaces are vector spaces V with a norm $||.||$ such that every Cauchy sequence (with respect to the metric $d(x, y) = ||x - y||$ in V) has a limit in V (with respect to the topology induced by that metric). As for general vector spaces, a Banach space over the real numbers is called a real Banach space, and a Banach space over the complex numbers is called a complex Banach space.

7. Algebras: general vector spaces do not possess a multiplication between vectors. A vector space equipped with an additional bilinear operator defining the multiplication of two vectors is an algebra over a field. Many algebras stem from functions on some geometrical object: since functions with values in a field can be multiplied, these entities form algebras.

8. In functional analysis, a Banach algebra is an associative algebra A over the real or complex numbers which at the same time is also a Banach space The algebra multiplication and the Banach space norm are required to be related by the following inequality: $\forall x, y \in A$: $\|x\, y\| \ \leq \|x\|\, \|y\|$ (i.e., the norm of the product is less than or equal to the product of the norms). This ensures that the multiplication operation is continuous. This property is found in the real and complex numbers; for instance.

9. A C^*-algebra is a Banach algebra with an antiautomorphic involution $*$ which satisfies $(x^*)^* = x$ (1); $x^* y^* = (yx)^*$ (2); $x^* + y^* = (x + y)^*$ (3); and $(cx)^* = c^*\, x^*$ (4), where c^* is the complex conjugate of c, and whose norm satisfies $||xx^*|| = ||x||^2$.

10. C^*-algebras are an important area of research in functional analysis. An outstanding example is the complex algebra of linear operators on a complex Hilbert space with two additional properties:
it is a topologically closed set in the norm topology of operators and
is closed under the operation of taking adjoints of operators.
It is generally believed that these algebras were first considered primarily for their use in quantum mechanics to model algebras of physical observables, beginning with Werner Heisenberg's matrix mechanics and developed further by Pascual Jordan circa 1933. Afterwards, John von Neumann established a general framework for them which culminated in papers on rings of operators, considered as a special class of C*-algebras known as von Neumann algebras.

11. It is now generally accepted that the description of quantum mechanics in which all self-adjoint operators represent observables is untenable. For this reason, observables are identified to elements of an abstract C*-algebra A (that is one without a distinguished representation as an algebra of operators) and states are positive linear functionals on A. However, by using the GNS construction, we can recover Hilbert spaces which realize A as a subalgebra of operators. Geometrically, a pure state on a C*-algebra A is a state which is an extreme point of the set of all states on A. By properties of the GNS construction these states correspond to irreducible representations of A. The states of the C*-algebra of compact operators $K(\mathcal{H})$ correspond exactly to the density operators and therefore the pure states of $K(\mathcal{H})$ are exactly the pure states in the sense of quantum mechanics. The C*-algebraic formulation can be seen to include both classical and quantum systems. When the system is classical, the algebra of observables become an abelian C*-algebra. In that case the states become probability measures.

12. In functional analysis, given a C*-algebra A, the Gelfand-Naimark-Segal (GNS) construction establishes a correspondence between cyclic *-representations of A and certain linear functionals on A (called states). The correspondence is shown by an explicit construction of the *-representation from the state.

13. A *-representation of a C*-algebra A on a Hilbert space $\mathcal{H}$ is a mapping π from A into the algebra of bounded operators on $\mathcal{H}$.

14. Point-wise convergence is one of various senses in which a sequence of functions can converge to a particular function. Suppose $\{f_n\}$ is a sequence of functions sharing the same domain and codomain. The sequence $\{f_n\}$ converges pointwise to f, often written as $\lim_{n\to\infty} f_n = f$ point wise iff for every x in the domain one has $\lim_{n\to\infty} f_n(x) = f(x)$.

15. Every subset Q of a vector space is contained within a smallest convex set (called the convex hull of Q), namely the intersection of all convex sets containing Q,

16. A set with a binary relation R on its elements that is reflexive (for all a in the set, aRa), antisymmetric (if aRb and bRa, then $a = b$) and transitive (if aRb and bRc, then aRc) is described as a partially ordered set or poset,

17. Let X be a space. Its dual space X^* consists of all linear functions from X into the base field K which are continuous with respect to the prevailing topology.

18. The *weak* topology on X is the coarsest topology (the topology with the fewest open sets) such that each element of X^* is a continuous function.

19. The predual of a space D is a space D' whose dual space is D. For example, the predual of the space of bounded operators $\mathcal{B}(\mathcal{H})$ is the space of trace class operators,

20. The ultraweak topology, also called the weak-* topology, on the set $\mathcal{B}(\mathcal{H})$ is the weak-topology obtained from the trace class operators on $\mathcal{H}$. In other words it is the weakest topology such that all elements of the predual are continuous (when considered as functions on $\mathcal{H}$),

21. A partially-ordered group is a group $(G, +)$ equipped with a partial order "$\vdash$" that is translation-invariant. That is, "$\vdash$" has the property that, for all a, b, and g in G, if $a \vdash b$ then $a + g \vdash b + g$ and $g + a \vdash g + b$,

22. An element x of G is called positive element if $0 \vdash x$. The set of elements $0 \vdash x$ is often denoted with $G+$, and it is called the *positive cone* of G. So we have $a \vdash b$ if and only if $-a + b \in G+$.

23. For the general group G, the existence of a positive cone specifies an order on G. A group G is a partially-ordered group if and only if there exists a subset J (which is $G+$) of G such that: $0 \in J$; if $a \in J$ and $b \in J$ then $a + b \in J$; if $a \in J$ then $-x + a + x \in J$ for each x of G; if $a \in J$ and $-a \in J$ then $a \vdash 0$.

24. In linear algebra, a matrix decomposition is a factorization of a matrix into some canonical form. There are many different matrix decompositions; each finds use among a particular class of problems. *The Cholesky decomposition* is applicable to any square, symmetric, positive definite matrix A in the form $A = U^T U$, where U is upper triangular with positive diagonal entries. The Cholesky decomposition is a special case of the symmetric LU decomposition, with $L = U^T$. The Cholesky decomposition is unique and also applicable

for complex hermitian positive definite matrices. *The singular value decomposition* is applicable to m times n matrix A in the fashion $A = UDV^\dagger$, where D is a nonnegative diagonal matrix while U, V are unitary matrices, and $V^\dagger$ denotes the conjugate transpose of V (or simply the transpose, if V contains real numbers only). The diagonal elements of D are called the singular values of A.

25. The orthogonal complement $W^\perp$ of a subspace W of an inner product space V is the set of all vectors in V that are orthogonal to every vector in W, i.e.,

$$W^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in W\}.$$

26. A topological space is called *separable* if it contains a countable dense subset. In other words, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

27. A cover of a set X is a collection of sets whose union contains X as a subset.

28. A topological space X is called compact if each of its open covers has a finite subcover. Otherwise it is called non-compact.

29. A relatively compact subspace (or relatively compact subset) Y of a topological space X is a subset whose closure is compact.

30. T is a compact operator on Hilbert's space if the image of each bounded set under T is relatively compact.
Compact operators on Hilbert spaces are a direct extensions of matrices. In such spaces they are the closure of finite-rank operators. As such, results from matrix theory can sometimes be extended to compact operators using similar arguments. In contrast, the study of general operators on infinite dimensional spaces often requires a genuinely different approach. For example, the spectral theory of compact operators on Banach spaces takes a form that is very similar to the Jordan canonical form of matrices. In the context of Hilbert spaces, a square matrix is unitarily diagonalizable if and only if it is normal. A corresponding result holds for normal compact operators on Hilbert spaces.

31. A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimension. Specifically, an $n-$simplex is an $n-$dimensional polytope which is the convex hull of its $n + 1$ vertices. A 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron. A single point may be considered a 0-simplex, and a line segment may be considered a 1-simplex. *A simplex may be defined as the smallest convex set containing the given vertices.*

B Lattices

A lattice $\mathcal{L}$ (also called a poset) is a partially ordered set (also called a poset) in which any two elements a and b have a unique supremum (the elements' least upper bound "$a \vee b$"; called their join) and an infimum (greatest lower bound "$a \wedge b$"; called their meet). Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order ($>$, $<$) theory and universal algebra. Semilattices include lattices, which in turn include Heyting and Boolean algebras. These "lattice-like" structures all admit order-theoretic as well as algebraic descriptions.

A *bounded* lattice has a greatest (or maximum) and least (or minimum) element, denoted 1 and 0 by convention (also called top and bottom, respectively). Any lattice can be converted into a bounded lattice by adding a greatest and least element, and every non-empty finite lattice is bounded. For any set A, the collection of all subsets of A (called the power set of A) can be ordered via subset inclusion to obtain a lattice bounded by A itself and the null set. Set intersection and union represent the operations meet and join, respectively.

A poset is called a complete lattice if all its subsets have both a join and a meet. In particular, every complete lattice is a bounded lattice. While bounded lattice homomorphisms in general preserve only finite joins and meets, complete lattice homomorphisms are required to preserve arbitrary joins and meets.

Any quantum system represented by an $N-$dimensional Hilbert space $\mathcal{H}$ has associated a lattice formed by all its convex subspaces $\mathcal{L}_{v\mathcal{N}}(\mathcal{H}) =< \mathcal{P}(\mathcal{H}),\ \cap,\ \oplus,\ \neg,\ 0,\ 1 >$, where 0 is the empty set $\emptyset$, 1 is the total space $\mathcal{H}$, $\oplus$ the closure of the sum, and $\neg(S)$ is the orthogonal complement of a subspace S [41]. This lattice was called "Quantum Logic" by Birkhoff and von Neumann. One refers to this lattice as the von Neumann-lattice $\mathcal{L}_{v\mathcal{N}}(\mathcal{H}))$ [41].

Let $\mathcal{L}$ be a bounded lattice with greatest element 1 and least element 0. Two elements x and y of the lattice are complements of each other if and only if: $x \bigvee y = 1$ and $x \bigwedge y = 0$. In the case the complement is unique, we write $\neg x = y$ and equivalently, $\neg y = x$. A bounded lattice for which every element has a complement is called a *complemented* lattice. The corresponding unitary operation over the lattice, called complementation, introduces an analogue of logical negation into lattice theory. The complement is not necessarily unique, nor does it have a special status among all possible unitary operations over $\mathcal{L}$.

Distributive lattices are lattices for which the operations of join and meet distribute over each other. The prototypical examples of such structures are collections of sets for which the lattice operations can be given by set union and intersection. Indeed, these lattices of sets describe the scenerio completely. A complemented lattice that is also distributive is a Boolean algebra. For a distributive lattice, the complement of x, when it exists, is unique.

The concept of lattice's atom is of great physical importance. If $\mathcal{L}$ has a null element 0, then an element x of $\mathcal{L}$ is an *atom* if $0 < x$ and there exists no element y of $\mathcal{L}$ such that $0 < y < x$. One says that $\mathcal{L}$ is:
i) *Atomic*, if for every nonzero element x of $\mathcal{L}$, there exists an atom a of $\mathcal{L}$ such that $a = x$
ii) Atomistic, if every element of $\mathcal{L}$ is a supremum of atoms.

A *modular* lattice is one that satisfies the following self-dual condition (modular law) $x \leq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$, where $\leq$ is the partial order, and $\vee$ and $\wedge$ (join and meet, respectively) are the operations of the lattice.

Modular lattices arise naturally in algebra and in many other areas of mathematics. For example, the subspaces of a vector space (and more generally the submodules of a module over a ring) form a modular lattice. Every distributive lattice is modular. In a not necessarily modular lattice, there may still be elements b for which the modular law holds in connection with arbitrary elements a and x ($\leq b$). Such an element is called a modular element. Even more generally, the modular law may hold for a fixed pair (a, b). Such a pair is called a modular pair, and there are various generalizations of modularity related to this notion and to semi-modularity.

For a, $b \in \mathcal{L}$, to assert that a is orthogonal to b ($a \perp b$) implies $a \wedge b = 0$. Equivalently, in "order" terms, one says that $a \leq b^{\perp}$. Now, $\mathcal{L}$ is an orthocomplemented lattice if whenever $a \perp b$ then

$b \leq a^{\perp}$. $\perp$ is a symmetric relation.

For any $a \in \mathcal{L}$, define $M(a) := \{c \in \mathcal{L} | c \perp a, \text{and } 1 = c \vee a\}$. An element of $M(a)$ is called an orthogonal complement of a. We have $a^{\perp} \in M(a)$, and any orthogonal complement of a is a complement of a. If we replace the unity in $M(a)$ by an arbitrary element $b \geq a$, then we have the set $M(a, b) := \{c \in \mathcal{L} | c \vee a \,\text{and}\, b = c \vee a\}$. An element of $M(a, b)$ is called an orthogonal complement of a relative to b . Clearly, $M(a) = M(a, 1)$. Also, for $a \leq cb$, $c \in M(a, b)$, iff $a \in M(c, b)$. As a result, we can define still another symmetric binary operator $\oplus$ on $[0, b]$, given by $b = a \oplus c$ iff $c \in M(a, b)$. Note that $b = b \oplus 0$. A final operation is the "difference" $b - a = b \wedge a$. Some properties: (1) $a - a = 0$, $a - 0 = a$, $0 - a = 0$, $a - 1 = 0$, and $1 - a = a^{\perp}$; (2) $b - a = a - b$; (3) if $a \leq b$, then $a \wedge (b - a)$ and $a \oplus (b - a) \leq b$.

Definition: A lattice $\mathcal{L}$ is called an orthomodular lattice if i) $\mathcal{L}$ is orthocomplemented, and (orthomodular law) ii) if $x \leq y$, then $y = x \oplus (y - x)$. The orthomodular law can be recasted as follows: if $x \leq y$, then $y = x \vee (y \wedge x^{\perp})$. Equivalently, $x \leq y$ implies $y = (y \wedge x) \vee (y \wedge x^{\perp})$. Such relation is automatically true in an arbitrary distributive lattice, even without the assumption that $x \leq y$. For example, the lattice $\mathcal{L}(H)$ of closed subspaces of a Hilbert space H is orthomodular. $\mathcal{L}(H)$ is modular iff H is finite dimensional. In addition, if we give the set $\mathcal{P}_p(H)$ of (bounded) projection operators on H an ordering structure by defining $P \leq Q$ iff $\mathcal{P}(H) \leq \mathcal{Q}(H)$, then $\mathcal{P}_p(H)$is lattice isomorphic to $\mathcal{L}(H)$, and hence orthomodular [11].

C Faces of a convex set

We define here a convex set's *face* in a real vector space of finite dimension. Let $\mathcal{C}$ be a convex subset of $\mathbb{R}^n$ and let us introduce the auxiliary notions of oriented hyperplanes and supporting hyperplanes. Given $\mathbf{n}, \mathbf{p} \in \mathbb{R}^n$ let us define the hyperplane $H(\mathbf{n}, \mathbf{p})$ via

$$H(\mathbf{n}, \mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0\}.$$

If $\mathbf{n} = 0$ it is equal to $\mathbb{R}^n$ and we call it degenerate. As long as $H(\mathbf{n}, \mathbf{p})$ is nondegenerate, its removal disconnects $\mathbb{R}^n$. The upper halfspace of $\mathbb{R}^n$ determined by $H(\mathbf{n}, \mathbf{p})$ is $H(\mathbf{n}, \mathbf{p})^+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) \geq 0\}$. A hyperplane $H(\mathbf{n}, \mathbf{p})$ <u>is a supporting hyperplane</u> for $\mathcal{C}$ if its upper halfspace contains $\mathcal{C}$, that is, if $\mathcal{C} \subset H(\mathbf{n}, \mathbf{p})^+$.

Using this terminology, we can define a **face** of a convex set $\mathcal{C}$ to be the intersection of $\mathcal{C}$ with a supporting hyperplane of $\mathcal{C}$. Notice that we still get both the empty set and $\mathcal{C}$ itself as improper faces of $\mathcal{C}$. For the definition of a face in the infinite dimensional case we extend the definition of a supporting hyperplane to a real Hilbert space $\mathcal{H}$. Given $\mathbf{n}, \mathbf{p} \in \mathcal{H}$, we say that $H(\mathbf{n}, \mathbf{p})$,

$$H(\mathbf{n}, \mathbf{p}) = \{\mathbf{x} \in \mathcal{H} : \langle \mathbf{n}, \mathbf{x} - \mathbf{p} \rangle = 0\},$$

is a supporting hyperplane if $\mathcal{C} \subset H(\mathbf{n}, \mathbf{p})^+$. Note that $H(\mathbf{n}, \mathbf{p})$ is closed and using Riesz representation theorem, for every closed hyperplane H there exists $\mathbf{n}, \mathbf{p} \in \mathcal{H}$ such that $H = H(\mathbf{n}, \mathbf{p})$.

In the general case (in a Banach space) we say that <u>F is a face of $\mathcal{C}$</u> if there exist a <u>closed</u> hyperplane H such that $F = \mathcal{C} \cap H$. A closed hyperplane is given by a continuos lineal functional.

Remarks: Let $\mathcal{C}$ be a convex set. Then:

- If $F_1 = \mathcal{C} \cap H(\mathbf{n_1}, \mathbf{p_1})$ and $F_2 = \mathcal{C} \cap H(\mathbf{n_2}, \mathbf{p_2})$ are faces of $\mathcal{C}$ intersecting at a point p then $H(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{p})$ is a supporting hyperplane of $\mathcal{C}$ and $F1 \cap F2 = C \cap H(\mathbf{n}_1 + \mathbf{n}_2, \mathbf{p})$. This shows that the faces of $\mathcal{C}$ form a meet-semilattice.

- Since each proper face lies on the base of the upper halfspace of some supporting hyperplane, each such face must lie on the relative boundary of $\mathcal{C}$.

An extreme point of a convex set $\mathcal{C}$ in a real vector space is a point in $\mathcal{C}$ which does not lie in any open line segment joining two points of $\mathcal{C}$. Intuitively, an extreme point is a "corner" of $\mathcal{C}$. The Krein-Milman theorem states that if $\mathcal{C}$ is convex and compact in a locally convex space, then $\mathcal{C}$ is the closed convex hull of its extreme points. In particular, such a set has extreme points.

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