On the energy–momentum tensor

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Abstract
We clarify the relation among canonical, metric and Belinfante's energy–momentum tensors for general tensor field theories. For any tensor field T, we define a new tensor field $\tilde{T}$ in terms of which metric and Belinfante's energy–momentum tensors are readily computed. We show that the latter is the one that arises naturally from Noether's theorem for an arbitrary spacetime and it coincides on-shell with the metric one.

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> *Symmetry as wide or as narrow as you may define*
> *its meaning, is one idea which man through the*
> *ages has tried to comprehend and create order,*
> *beauty, and perfection.*
>
> Hermann Weyl [1]

1. Introduction

For many decades a suitable definition for the energy–momentum tensor has been under investigation. This is more than merely a technical point, not only because T^{ab} should provide meaningful physical conserved quantities but also because it is the source of Einstein's gravitational field equations.

In flat spacetime the canonical energy–momentum tensor arises from Noether's theorem by considering the conserved currents associated with translation invariance. However, only for scalar fields does the energy–momentum tensor constructed in this way turn out to be symmetric. Moreover, for Maxwell's theory, it breaks gauge symmetry. Of course, it is possible to correct it through Belinfante's symmetrization procedure [2], although this is usually presented as an *ad hoc* prescription (see for example [3, 4]).

On the other hand, a completely different approach, based on the diffeomorphism invariance of the theory, leads to the metric energy–momentum tensor (see for example [5]) which is, by definition, symmetric and gauge invariant.

The aim of this paper is to clarify the relation among these tensors.

In section 2, we define the tensor $\widetilde{T}$, which turns out to be a very useful tool for the rest of our work. In section 3, we analyse the relation among the different energy–momentum tensors for general tensor field theories on an arbitrary spacetime of any dimension.

2. Lie derivatives and the tensor $\widetilde{T}$

Let ξ be a vector field on a (pseudo) Riemannian manifold of dimension n and ϕ_t a local one-parameter group of diffeomorphism generated by ξ. This diffeomorphism maps each tensor field T at p of the type (r, s) into $\phi_{t*}T|_{\phi(p)}$, the pullback of T.

As is well-known (see for instance [5, 6]), the Lie derivative $\mathcal{L}_\xi T$ of a tensor field T with respect to ξ is defined to be minus the derivative with respect to t of this family of tensor fields, evaluated at $t = 0$, i.e.

$$\mathcal{L}_\xi T = \lim_{t \to 0} \frac{1}{t}(T|_p - \phi_{t*}T|_p). \tag{1}$$

Thus, it measures how much the tensor field $T(x^a)$ deviates from being formally invariant under the infinitesimal transformation $x'^a = x^a - t\xi^a$, with $t \ll 1$.

The coordinate components are

$$\mathcal{L}_\xi T^{b_1...b_p}{}_{c_1...c_q} = \partial_a T^{b_1...b_p}{}_{c_1...c_q}\xi^a - T^{ab_2...b_p}{}_{c_1...c_q}\partial_a\xi^{b_1} - T^{b_1a...b_p}{}_{c_1...c_q}\partial_a\xi^{b_2} - \cdots$$
$$+ T^{b_1...b_p}{}_{ac_2...c_q}\partial_{c_1}\xi^a + T^{b_1...b_p}{}_{c_1a...c_q}\partial_{c_2}\xi^a + \cdots. \tag{2}$$

Of course, for any torsion free connection, the partial derivatives can be replaced by covariant ones.

The following definition will prove useful. For each tensor field $T^{c_1...c_p}{}_{d_1...d_q}$ of type (p, q) we define a tensor field $\widetilde{T}^{c_1...c_p}{}_{d_1...d_q}{}^a{}_b$ of type $(p + 1, q + 1)$, such that

$$\widetilde{T}^{c_1...c_p}{}_{d_1...d_q}{}^a{}_b := T^{ac_2...c_p}{}_{d_1...d_q}\delta^{c_1}_b + T^{c_1a...c_p}{}_{d_1...d_q}\delta^{c_2}_b + \cdots - T^{c_1...c_p}{}_{bd_2...d_q}\delta^a_{d_1}$$
$$- T^{c_1...c_p}{}_{d_1b...d_q}\delta^a_{d_2} - \cdots. \tag{3}$$

For a scalar field φ we define $\widetilde{\varphi} = 0$, since there is no index to be replaced. So, in terms of $\widetilde{T}$, (2) can be written as

$$\mathcal{L}_\xi T^{c_1...c_p}{}_{d_1...d_q} = \partial_a T^{c_1...c_p}{}_{d_1...d_q}\xi^a - \widetilde{T}^{c_1...c_p}{}_{d_1...d_q}{}^a{}_b\partial_a\xi^b$$
$$= \nabla_a T^{b_1...b_p}{}_{c_1...c_q}\xi^a - \widetilde{T}^{c_1...c_p}{}_{d_1...d_q}{}^a{}_b\nabla_a\xi^b, \tag{4}$$

where ∇_a denotes the covariant derivative associated with the Levi-Civita connection. In index free notation, our definition (3) reads

$$\widetilde{T}(\nabla\xi) := \nabla_\xi T - \mathcal{L}_\xi T. \tag{5}$$

Some simple examples are in order. For instance, for the tensor δ^a_b we have

$$\widetilde{\delta}^{ac}_b{}_d = \delta^c_b\delta^a_d - \delta^a_d\delta^c_b = 0, \tag{6}$$

which expresses the fact that $\mathcal{L}_\xi\delta^a_b = \nabla_\xi\delta^a_b = 0$. Moreover, for the metric tensor we have

$$\widetilde{g}_{ab}{}^c{}_d = -g_{db}\delta^c_a - g_{ad}\delta^c_b, \tag{7}$$

and so

$$\mathcal{L}_\xi g_{ab} = \nabla_c g_{ab}\xi^c - \widetilde{g}_{ab}{}^c{}_d\nabla_c\xi^d = g_{db}\nabla_a\xi^d + g_{ad}\nabla_b\xi^d = \nabla_a\xi_b + \nabla_b\xi_a. \tag{8}$$

On the other hand, the well-known expression for the derivative of the volume element

$$\mathcal{L}_\xi\sqrt{|g|} = \tfrac{1}{2}\sqrt{|g|}g^{ab}\mathcal{L}_\xi g_{ab} = \sqrt{|g|}\nabla_a\xi^a, \tag{9}$$

can also be obtained from

$$\tilde{\varepsilon}_{a_1 a_2 \ldots a_n}{}^b{}_c = -\varepsilon_{c a_2 \ldots a_n}\delta^b_{a_1} - \varepsilon_{a_1 c \ldots a_n}\delta^b_{a_2} - \cdots = -\varepsilon_{a_1 a_2 \ldots a_n}\delta^b_c, \tag{10}$$

where $\varepsilon_{a_1 a_2 \ldots a_n}$ is the Levi-Civita alternating symbol.

Consequently, with this notation, other classical formulae of Ricci calculus simplify:

$$\nabla_a T^{c_1 \ldots c_p}{}_{d_1 \ldots d_q} = \partial_a T^{c_1 \ldots c_p}{}_{d_1 \ldots d_q} + \Gamma^b{}_{ca}\tilde{T}^{c_1 \ldots c_p}{}_{d_1 \ldots d_q}{}^c{}_b, \tag{11}$$

or

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)T^{c_1 \ldots c_p}{}_{d_1 \ldots d_q} = R^d{}_{cab}\tilde{T}^{c_1 \ldots c_p}{}_{d_1 \ldots d_q}{}^c{}_d, \tag{12}$$

where $R^d{}_{cab}$ is the Riemann curvature tensor.

Note that

$$\widetilde{\mathbf{T} \otimes \mathbf{S}} = \tilde{\mathbf{T}} \otimes \mathbf{S} + \mathbf{T} \otimes \tilde{\mathbf{S}}, \tag{13}$$

and

$$\widetilde{\nabla_e T}^{c_1 \ldots c_p}{}_{d_1 \ldots d_q}{}^a{}_b = \nabla_e\left(\tilde{T}^{c_1 \ldots c_p}{}_{d_1 \ldots d_q}{}^a{}_b\right) - \delta^a_e \nabla_b T^{c_1 \ldots c_p}{}_{d_1 \ldots d_q}, \tag{14}$$

because there is an additional covariant index to be replaced in the left-hand side.

When there is no danger of confusion, we shall suppress the unnecessary indices and write, for instance, (14) as

$$\widetilde{\nabla_e T}^{\ldots}{}_{\ldots}{}^a{}_b = \nabla_e \tilde{T}^{\ldots}{}_{\ldots}{}^a{}_b - \delta^a_e \nabla_b T^{\ldots}{}_{\ldots}, \tag{15}$$

or, as in next section, even as

$$\widetilde{\nabla_e T}^a{}_b = \nabla_e \tilde{T}^a{}_b - \delta^a_e \nabla_b T. \tag{16}$$

We shall now consider the commutator $[\mathcal{L}_\xi, \nabla]$ between the Lie derivative $\mathcal{L}_\xi$ and the covariant one ∇ associated with the Levi-Civita connection. It is a map which takes each smooth tensor field of type (p, q) to a smooth $(p, q + 1)$ tensor field. In the index notation, we denote the tensor field resulting from the action of $[\mathcal{L}_\xi, \nabla]$ on $T^{b_1 \ldots b_p}{}_{c_1 \ldots c_q}$ by $\mathcal{D}_{\xi a}T^{b_1 \ldots b_p}{}_{c_1 \ldots c_q}$. On scalar fields it vanishes, for

$$[\mathcal{L}_\xi, \nabla]f = \mathcal{L}_\xi \, \mathrm{d}f - \mathrm{d}\mathcal{L}_\xi f = 0 \qquad \text{so} \quad \mathcal{D}_{\xi a}f = 0. \tag{17}$$

Moreover, its action on the metric tensor is very simple

$$\mathcal{D}_{\xi a}g_{bc} = -\nabla_a \mathcal{L}_\xi g_{bc} = -\nabla_a(\nabla_b \xi_c + \nabla_c \xi_b). \tag{18}$$

Since, for any two tensors $\mathbf{T}$ and $\mathbf{S}$

$$[\mathcal{L}_\xi, \nabla](\mathbf{T} \otimes \mathbf{S}) = [\mathcal{L}_\xi, \nabla]\mathbf{T} \otimes \mathbf{S} + \mathbf{T} \otimes [\mathcal{L}_\xi, \nabla]\mathbf{S}, \tag{19}$$

we have

$$\mathcal{D}_{\xi a}\left(T^{b_1 \ldots}{}_{c_1 \ldots}S^{d_1 \ldots}{}_{e_1 \ldots}\right) = \mathcal{D}_{\xi a}\left(T^{b_1 \ldots}{}_{c_1 \ldots}\right)S^{d_1 \ldots}{}_{e_1 \ldots} + T^{b_1 \ldots}{}_{c_1 \ldots}\mathcal{D}_{\xi a}\left(S^{d_1 \ldots}{}_{e_1 \ldots}\right). \tag{20}$$

We can derive the general formula for the action of $[\mathcal{L}_\xi, \nabla]$ on an arbitrary tensor field from the Leibniz rule (20) if we know its action on scalars and one-forms (or vectors). However, we can achieve it in an easier way by using the symbol $\tilde{T}^{\ldots}{}_{\ldots}$, for

$$\begin{aligned}
\mathcal{D}_{\xi a}T^{\ldots}{}_{\ldots} &= \mathcal{L}_\xi \nabla_a T^{\ldots}{}_{\ldots} - \nabla_a \mathcal{L}_\xi T^{\ldots}{}_{\ldots} \\
&= \xi^b \nabla_b \nabla_a T^{\ldots}{}_{\ldots} - \widetilde{\nabla_a T}^{\ldots}{}_{\ldots}{}^b{}_c \nabla_b \xi^c - \nabla_a\left(\xi^b \nabla_b T^{\ldots}{}_{\ldots} - \tilde{T}^{\ldots}{}_{\ldots}{}^b{}_c \nabla_b \xi^c\right) \\
&= \xi^b(\nabla_a \nabla_b - \nabla_b \nabla_a)T^{\ldots}{}_{\ldots} + \tilde{T}^{\ldots}{}_{\ldots}{}^b{}_c \nabla_a \nabla_b \xi^c \\
&= \left(R^c{}_{bda}\xi^d + \nabla_a \nabla_b \xi^c\right)\tilde{T}^{\ldots}{}_{\ldots}{}^b{}_c, \tag{21}
\end{aligned}$$

where we have used (15) and (12). By defining

$$\mathscr{C}_{\xi}{}^{c}{}_{ba} := R^{c}{}_{bda}\xi^{d} + \nabla_{a}\nabla_{b}\xi^{c}, \tag{22}$$

for any tensor field $T^{\cdots}{}_{\cdots}$, we can write

$$\mathscr{D}_{\xi a}T^{\cdots}{}_{\cdots} = \mathscr{C}_{\xi}{}^{c}{}_{ba}\widetilde{T}^{\cdots}{}_{\cdots}{}^{b}{}_{c}. \tag{23}$$

Note that $\mathcal{C}_{\xi}{}^{c}{}_{ba}$ is symmetric in the lower indices, for

$$\begin{aligned}
\mathscr{C}_{\xi}{}^{c}{}_{ba} &= R^{c}{}_{bda}\xi^{d} + \nabla_{b}\nabla_{a}\xi^{c} + (\nabla_{a}\nabla_{b} - \nabla_{b}\nabla_{a})\xi^{c} \\
&= \left(R^{c}{}_{bda} + R^{c}{}_{dab}\right)\xi^{d} + \nabla_{b}\nabla_{a}\xi^{c} \\
&= R^{c}{}_{adb}\xi^{d} + \nabla_{b}\nabla_{a}\xi^{c} = \mathscr{C}_{\xi}{}^{c}{}_{ab},
\end{aligned} \tag{24}$$

where we have used the symmetry properties of the Riemann tensor.

Now, for the metric tensor (23) reads

$$\mathscr{D}_{\xi a}g_{bc} = -\nabla_{a}\mathcal{L}_{\xi}g_{bc} = -\mathscr{C}_{\xi cba} - \mathscr{C}_{\xi bca}. \tag{25}$$

By index substitution, we also have

$$-\nabla_{b}\mathcal{L}_{\xi}g_{ca} = -\mathscr{C}_{\xi acb} - \mathscr{C}_{\xi cab}, \tag{26}$$

$$-\nabla_{c}\mathcal{L}_{\xi}g_{ab} = -\mathscr{C}_{\xi bac} - \mathscr{C}_{\xi abc}. \tag{27}$$

We add equations (25) and (26) and then subtract equation (27). Using the symmetry property of $\mathcal{C}_{\xi}{}^{c}{}_{ba}$ we find

$$\begin{aligned}
\mathscr{C}_{\xi}{}^{d}{}_{ab} &= \tfrac{1}{2}g^{dc}(\nabla_{a}\mathcal{L}_{\xi}g_{bc} + \nabla_{b}\mathcal{L}_{\xi}g_{ac} - \nabla_{c}\mathcal{L}_{\xi}g_{ab}) \\
&= \tfrac{1}{2}g^{dc}(\nabla_{a}(\nabla_{b}\xi_{c} + \nabla_{c}\xi_{b}) + \nabla_{b}(\nabla_{a}\xi_{c} + \nabla_{c}\xi_{a}) - \nabla_{c}(\nabla_{a}\xi_{b} + \nabla_{b}\xi_{a})).
\end{aligned} \tag{28}$$

Of course, (28) can be readily obtained by adding to the definition (22) the null term $3\nabla_{[a}\nabla_{b}\xi_{c]}$.

Therefore, we see that $\mathscr{C}_{\xi}{}^{c}{}_{ba}$ are linear combinations of the covariant derivatives of the Lie derivative of the metric tensor. Thus we can write (23) as

$$\mathscr{D}_{\xi a}T^{\cdots}{}_{\cdots} = \tfrac{1}{2}\widetilde{T}^{\cdots}{}_{\cdots}{}^{bc}(\nabla_{a}\mathcal{L}_{\xi}g_{bc} + \nabla_{b}\mathcal{L}_{\xi}g_{ac} - \nabla_{c}\mathcal{L}_{\xi}g_{ab}). \tag{29}$$

Note that the action of $[\mathcal{L}_{\xi}, \nabla]$ on any tensor field vanishes when $\nabla_{a}\mathcal{L}_{\xi}g_{bc} = \nabla_{a}(\nabla_{b}\xi_{c} + \nabla_{c}\xi_{b}) = 0$, and so Lie derivative and the covariant one commute in this case. In particular, it occurs when ξ^{b} is a Killing vector, so

$$\nabla_{a}\mathcal{L}_{\xi}T^{\cdots}{}_{\cdots} = \mathcal{L}_{\xi}\nabla_{a}T^{\cdots}{}_{\cdots}, \tag{30}$$

for any tensor field $T^{\cdots}{}_{\cdots}$ when ξ^{b} is a Killing vector field.

3. Energy–momentum tensor and the tensor $\widetilde{T}$

3.1. Diffeomophism invariance of the action

Let us consider a field theory where the Lagrangian $\mathscr{L}$ is a local function of a collection of tensor fields $\psi_{(\ell)}{}^{b_{1}...b_{p}}{}_{c_{1}...c_{q}}$ defined on a (pseudo)-Riemannian manifold, their first covariant derivatives $\nabla_{a}\psi_{(\ell)}{}^{b_{1}...b_{p}}{}_{c_{1}...c_{q}}$, and the metric tensor g_{ab}. Often we shall suppress all tensor indices and denote the fields by $\psi_{(\ell)}$.

As usual, we obtain the equations of motion by requiring that the action

$$S = \int_{\Omega}\mathscr{L}(\nabla_{a}\psi_{(\ell)}, \psi_{(\ell)}, g_{ab})\sqrt{|g|}\,\mathrm{d}^{n}x, \tag{31}$$
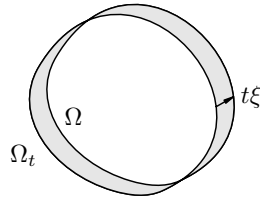
Figure 1. The boundary contribution to equation (34).

be stationary under arbitrary variations of the fields $\delta\psi_{(\ell)}$ in the interior of any compact region Ω. Thus, one obtains

$$\nabla_a\left(\frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\right) = \frac{\partial\mathscr{L}}{\partial\psi_{(\ell)}}. \tag{32}$$

The action (31) must be independent of the coordinates we choose. Needless to say, even in flat spacetime we are allowed to use curvilinear coordinates, so it must be invariant under general coordinate transformations. By making a change of coordinates generated by the vector field ξ^a, $x^a \to x^a - t\xi^a$, the action can be written as

$$S = \int_{\Omega_t} \mathscr{L}_t\sqrt{|g_t|}\,\mathrm{d}x, \tag{33}$$

where $\mathscr{L}_t = \mathscr{L}(\nabla_a\phi_{t*}(\psi_{(\ell)}), \phi_{t*}(\psi_{(\ell)}), \phi_{t*}(g_{ab}))$, that is the same function $\mathscr{L}$ evaluated on the Lie-dragged tensors fields, and $|g_t| = \det(\phi_{t*}(g_{ab}))$. Now, taking the derivative of (33) with respect to t and evaluating it at $t = 0$ we get three terms:

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(\mathscr{L}_t)_{t=0}\sqrt{|g|}\,\mathrm{d}^nx + \int_\Omega \mathscr{L}\frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{|g_t|})_{t=0}\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t}\mathscr{L}\sqrt{|g|}\,\mathrm{d}^nx\right)_{t=0} = 0. \tag{34}$$

The first one, by definition, contains the Lie derivative of $\mathscr{L}$; the second one, the derivative of the volume element (9), while the last one (see figure 1) is a boundary term which by using the Gauss theorem can be rewritten as a volume integral, so we get

$$\begin{aligned}
0 &= \int_\Omega [\mathcal{L}_\xi\mathscr{L} + \mathscr{L}\nabla_a\xi^a - \nabla_a(\mathscr{L}\xi^a)]\sqrt{|g|}\,\mathrm{d}^nx \\
&= \int_\Omega [\mathcal{L}_\xi\mathscr{L} - \nabla_a(\mathscr{L})\xi^a]\sqrt{|g|}\,\mathrm{d}^nx.
\end{aligned} \tag{35}$$

Therefore, taking into account that the vector field ξ^a is completely arbitrary, we have

$$\mathcal{L}_\xi\mathscr{L} - \nabla_a(\mathscr{L})\xi^a = 0, \tag{36}$$

which just reflects that the Lagrangian $\mathscr{L}$ must be a scalar function. Thus, the result is very simple, the invariance of the action under general coordinate transformations requires $\mathscr{L}$ to be an scalar function, and (36) must hold for any vector field ξ^a.

Now, taking into account that the Lagrangian $\mathscr{L}$ depends on the coordinates only through the tensor fields $\nabla_a\psi_{(\ell)}$, $\psi_{(\ell)}$ and g_{ab}, we can write (36) as

$$\frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\mathcal{L}_\xi\nabla_a\psi_{(\ell)} + \frac{\partial\mathscr{L}}{\partial\psi_{(\ell)}}\mathcal{L}_\xi\psi_{(\ell)} + \frac{\partial\mathscr{L}}{\partial g_{ab}}\mathcal{L}_\xi g_{ab} - \nabla_a(\mathscr{L})\xi^a = 0, \tag{37}$$

which is a linear combination of the vector field ξ_b and its first derivatives $\nabla_b\xi_c$.

Now, from (4) (see also (3) for our definition of $\widetilde{\psi_{(\ell)}}{}^c{}_d$) we can write

$$\mathcal{L}_\xi\psi_{(\ell)} = \nabla^b\psi_{(\ell)}\xi_b - \widetilde{\psi_{(\ell)}}{}^{bc}\nabla_b\xi_c, \tag{38}$$

and, consequently,

$$
\begin{aligned}
\mathcal{L}_\xi \nabla_a \psi_{(\ell)} &= \nabla^b \nabla_a \psi_{(\ell)} \xi_b - \widetilde{\nabla_a \psi_{(\ell)}}^{bc} \nabla_b \xi_c \\
&= \nabla^b \nabla_a \psi_{(\ell)} \xi_b - \nabla_a \widetilde{\psi_{(\ell)}}^{bc} \nabla_b \xi_c + \delta_a^b \nabla^c \psi_{(\ell)} \nabla_b \xi_c \\
&= \nabla_a \nabla^b \psi_{(\ell)} \xi_b - \nabla_a \widetilde{\psi_{(\ell)}}^{bc} \nabla_b \xi_c + \nabla^b \psi_{(\ell)} \nabla_a \xi_b + (\nabla^b \nabla_a - \nabla_a \nabla^b) \psi_{(\ell)} \xi_b,
\end{aligned}
\tag{39}
$$

where in the second line we have used (15). Using the field equations (32), (38) and (39) we can write the first two terms in (37) as

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} \mathcal{L}_\xi \nabla_a \psi_{(\ell)} + \frac{\partial \mathcal{L}}{\partial \psi_{(\ell)}} \mathcal{L}_\xi \psi_{(\ell)} &= \nabla_a \left(\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)} \xi_b \right) \\
&\quad - \nabla_c \left(\frac{\partial \mathcal{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} \right) \nabla_a \xi_b + \frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} (\nabla^b \nabla_a - \nabla_a \nabla^b) \psi_{(\ell)} \xi_b.
\end{aligned}
\tag{40}
$$

Now, we shall rewrite the last term. From (12) we can write

$$
\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} (\nabla^b \nabla_a - \nabla_a \nabla^b) \psi_{(\ell)} = \frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} R^{d\ b}_{c\ a} \widetilde{\psi_{(\ell)}}^c_d = R^b_{adc} \frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[cd]},
\tag{41}
$$

where, as usual $\widetilde{\psi_{(\ell)}}^{[cd]} = \frac{1}{2}\left(\widetilde{\psi_{(\ell)}}^{cd} - \widetilde{\psi_{(\ell)}}^{dc}\right)$.

Now, defining

$$
\widehat{T}^{abc} := \underbrace{\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[cb]} + \overbrace{\frac{\partial \mathcal{L}}{\partial \nabla_b \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[ac]} + \frac{\partial \mathcal{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[ab]}}^{\text{symmetric under } b \leftrightarrows c}}_{\text{antisymmetric under } a \leftrightarrows b},
\tag{42}
$$

we can rewrite (41) as

$$
\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} (\nabla^b \nabla_a - \nabla_a \nabla^b) \psi_{(\ell)} = -R^b_{adc} \widehat{T}^{acd},
\tag{43}
$$

since there is no contribution from the last two terms in $\widehat{T}^{acd}$, due to the antisymmetry of the Riemann tensor in the last two indices. But, using now the symmetry properties of R^b_{adc} and $\widehat{T}^{abc}$, we get

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} (\nabla^b \nabla_a - \nabla_a \nabla^b) \psi_{(\ell)} &= -\frac{1}{2}\left(R^b_{adc} \widehat{T}^{acd} + R^b_{cda} \widehat{T}^{cad} \right) \\
&= -\frac{1}{2}\left(R^b_{acd} + R^b_{cda} \right) \widehat{T}^{cad} = \frac{1}{2} R^b_{dac} \widehat{T}^{cad} \\
&= \frac{1}{2} (\nabla_a \nabla_c - \nabla_c \nabla_a) \widehat{T}^{cab} = \nabla_a \nabla_c \widehat{T}^{cab}.
\end{aligned}
\tag{44}
$$

So, we can write the last term in (40) as

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} (\nabla^b \nabla_a - \nabla_a \nabla^b) \psi_{(\ell)} \xi_b &= \nabla_a \nabla_c \widehat{T}^{cab} \xi_b \\
&= \nabla_a (\nabla_c \widehat{T}^{cab} \xi_b) - \nabla_c \widehat{T}^{cab} \nabla_a \xi_b.
\end{aligned}
\tag{45}
$$

Hence, the first two terms in (37) can be written as

$$
\nabla_a \left(\frac{\partial \mathcal{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)} \xi_b + \nabla_c \widehat{T}^{cab} \xi_b \right) - \nabla_c \left(\frac{\partial \mathcal{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} + \widehat{T}^{cab} \right) \nabla_a \xi_b.
\tag{46}
$$

Therefore, the requirement that $\mathscr{L}$ be scalar leads, for any ξ_b, to

$$\nabla_a \left(\frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)} \xi_b + \nabla_c \widehat{T}^{cab} \xi_b - \mathscr{L} \xi^a \right)$$
$$+ \left[2 \frac{\partial \mathscr{L}}{\partial g_{ab}} - \nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} + \widehat{T}^{cab} \right) + g^{ab} \mathscr{L} \right] \nabla_a \xi_b = 0, \tag{47}$$

where we have used (8) and the obvious symmetry of the tensor field $\frac{\partial \mathscr{L}}{\partial g_{ab}}$.

3.2. Definitions of the energy–momentum tensors $T_{\mathscr{C}}^{ab}$, $T_{\mathscr{B}}^{ab}$ and $T_{\mathscr{M}}^{ab}$

Now, we define the canonical energy–momentum tensor as

$$T_{\mathscr{C}}^{ab} := - \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)} + g^{ab} \mathscr{L} \tag{48}$$

and the metric one as

$$T_{\mathscr{M}}^{ab} := 2 \frac{\partial \mathscr{L}}{\partial g_{ab}} - \nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} + \widehat{T}^{cab} \right) + g^{ab} \mathscr{L}$$
$$= 2 \frac{\partial \mathscr{L}}{\partial g_{ab}} - \nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{(ab)} + \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[cb]} + \frac{\partial \mathscr{L}}{\partial \nabla_b \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[ca]} \right) + g^{ab} \mathscr{L}. \tag{49}$$

By definition, $T_{\mathscr{M}}^{ab}$ is symmetric.

By using these definitions, for any vector field ξ^a, we can write (47) as

$$\nabla_a \left(-T_{\mathscr{C}}^{ab} \xi_b + \nabla_c \widehat{T}^{cab} \xi_b \right) + T_{\mathscr{M}}^{ab} \nabla_a \xi_b = 0. \tag{50}$$

Therefore, defining the Belinfante energy–momentum tensor

$$T_{\mathscr{B}}^{ab} := T_{\mathscr{C}}^{ab} - \nabla_c \widehat{T}^{cab} = - \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)}$$
$$- \nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[ba]} + \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[cb]} + \frac{\partial \mathscr{L}}{\partial \nabla_b \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{[ca]} \right) + g^{ab} \mathscr{L}, \tag{51}$$

we finally get

$$\nabla_a \left(T_{\mathscr{B}}^{ab} \xi_b \right) - T_{\mathscr{M}}^{ab} \nabla_a \xi_b = 0, \tag{52}$$

or, alternatively,

$$\nabla_a \left(\left(T_{\mathscr{B}}^{ab} - T_{\mathscr{M}}^{ab} \right) \xi_b \right) + \nabla_a \left(T_{\mathscr{M}}^{ab} \right) \xi_b = 0. \tag{53}$$

Moreover, taking into account the symmetry of $T_{\mathscr{M}}^{ab}$, we can also write (52) as

$$\nabla_a \left(T_{\mathscr{B}}^{ab} \xi_b \right) - \tfrac{1}{2} T_{\mathscr{M}}^{ab} \mathcal{L}_\xi g_{ab} = 0. \tag{54}$$

Equation (52), a rewritten form of (36), which holds on shell for any vector field ξ^a, has several important consequences. In fact, we shall use it in five different ways.

3.3. Noether currents

Let us restrict attention to the case where ξ^a is a Killing vector field, i.e. a generator of an infinitesimal isometry, so $\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0$. From (54), we directly obtain the

Noether conserved current $\mathcal{J}_\xi^a$ associated with this symmetry

$$\nabla_a \mathcal{J}_\xi^a = \nabla_a \left(T_\mathcal{B}^{ab}\xi_b\right) = 0 \tag{55}$$

for, in this case, the last term in (54) clearly vanishes. So, we can think of $T_\mathcal{B}^{ab}$ as a linear function from covector fields to vector fields such that

$$\boldsymbol{T}_\mathcal{B}(\text{Killing covector}) = \text{conserved current.} \tag{56}$$

3.4. $T_\mathcal{B}^{ab}$ conservation

At any point of the manifold we can choose Riemannian normal coordinates x^α (i.e., a local inertial coordinate system). Moreover, we can choose for ξ_b any set of n linear independent covectors with constant components in this coordinate system. For instance, the dual basis covectors dx_b^α. So, in this local coordinate system, (54) reads

$$\partial_\alpha \left(T_\mathcal{B}^{\alpha\beta}\right)\xi_\beta + T_\mathcal{B}^{\alpha\beta}\partial_\alpha\xi_\beta - T_\mathcal{M}^{\alpha\beta}\partial_\alpha\xi_\beta = \partial_\alpha \left(T_\mathcal{B}^{\alpha\beta}\right)\xi_\beta = 0, \tag{57}$$

because of the vanishing of Christoffel symbols and partial derivatives of ξ_β. Hence, we get $\nabla_\alpha T_\mathcal{B}^{\alpha\beta} = \partial_\alpha T_\mathcal{B}^{\alpha\beta} = 0$. But this is a tensor relation, then

$$\nabla_a T_\mathcal{B}^{ab} = 0. \tag{58}$$

3.5. $T_\mathcal{M}^{ab}$ conservation

Now, we integrate (53) over any compact region Ω, taking arbitrary vector fields ξ^a vanishing everywhere except in its interior. The first contribution may be transformed into an integral over the boundary which vanishes as ξ^a is zero there. Since the second term must therefore be zero for arbitrary ξ^a, it follows that

$$\nabla_a T_\mathcal{M}^{ab} = 0. \tag{59}$$

3.6. On shell equality of the tensors $T_\mathcal{B}^{ab}$ and $T_\mathcal{M}^{ab}$

Now, coming back to (53), we see that the diffeomorphism invariance of the action yields not only $\nabla_a T_\mathcal{B}^{ab} = \nabla_a T_\mathcal{M}^{ab} = 0$, but also

$$\nabla_a \left(\left(T_\mathcal{B}^{ab} - T_\mathcal{M}^{ab}\right)\xi_b\right) = \left(T_\mathcal{B}^{ab} - T_\mathcal{M}^{ab}\right)\nabla_a\xi_b = 0, \tag{60}$$

for any covector field ξ_b. Therefore, since $\nabla_a\xi_b$ is arbitrary, we conclude that both tensors coincide

$$T_\mathcal{B}^{ab} = T_\mathcal{M}^{ab}. \tag{61}$$

Therefore, we have shown that

$$\nabla_a T_\mathcal{B}^{ab} = 0, \qquad \nabla_a T_\mathcal{M}^{ab} = 0, \qquad \text{and} \qquad T_\mathcal{B}^{ab} = T_\mathcal{M}^{ab}, \tag{62}$$

follow as a consequence of the diffeomophism invariance of the action.

3.7. On the canonical energy–momentum tensor $T_\mathcal{C}^{ab}$

For any covector ξ_b, due to the asymmetry of $\widehat{T}^{abc}$, it holds

$$\begin{aligned}
\nabla_a (\nabla_c \widehat{T}^{cab}\xi_b + \widehat{T}^{cab}\nabla_c\xi_b) &= \nabla_a \nabla_c (\widehat{T}^{cab}\xi_b) \\
&= \tfrac{1}{2}(\nabla_a\nabla_c - \nabla_c\nabla_a)(\widehat{T}^{cab}\xi_b) \\
&= \tfrac{1}{2}\left(R^c_{\ dac}\widehat{T}^{dab}\xi_b + R^a_{\ dac}\widehat{T}^{cdb}\xi_b\right) = R_{ac}\widehat{T}^{cab}\xi_b = 0,
\end{aligned} \tag{63}$$

because of the symmetry of the Ricci tensor R_{ab}. Thus, we can also write (50) as

$$\nabla_a\left(T_{\mathscr{C}}^{ab}\xi_b + \widehat{T}^{cab}\nabla_c\xi_b\right) - T_{\mathscr{M}}^{ab}\nabla_a\xi_b = 0. \tag{64}$$

The last term in (64) vanishes for any Killing vector field owing to the symmetry of $T_{\mathscr{M}}^{ab}$. So, besides $T_{\mathscr{B}}^{ab}\xi_b$, we get another conserved current

$$T_{\mathscr{C}}^{ab}\xi_b + \widehat{T}^{cab}\nabla_c\xi_b, \tag{65}$$

which is, in general, linear in the Killing vector field ξ^a and its first covariant derivatives. Of course, this current differs from $T_{\mathscr{B}}^{ab}\xi_b$ by the divergentless vector $\nabla_c(\widehat{T}^{cab}\xi_b)$.

For scalar fields $\psi_{(\ell)}$ the last term in this current is absent, since $\dot{\psi}_{(\ell)}$ vanishes in this case, and both currents coincide.

On the other hand, for general tensor fields, this vanishing also occurs if there exists a parallel Killing vector, i.e. $\nabla_a\xi_b = 0$. So,

$$\nabla_a\left(T_{\mathscr{C}}^{ab}\xi_b\right) = \nabla_a T_{\mathscr{C}}^{ab}\xi_b = 0, \qquad \text{for any parallel } \xi^b, \tag{66}$$

thus, the vector $\nabla_a T_{\mathscr{C}}^{ab}$ is orthogonal to ξ^b. Of course, this occurs in flat spacetime, where we can always find n linear independent parallel vectors, for example the Cartesian coordinates vectors. Then, in that case, we have $\nabla_a T_{\mathscr{C}}^{ab} = 0$. But, as we are going to see, this is an exception. $\nabla_a T_{\mathscr{C}}^{ab} \neq 0$ for curved spacetime.

Note that the conservation of the current (65) means that

$$\nabla_a\left(\frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\mathcal{L}_\xi\psi_{(\ell)} - \mathscr{L}\xi^a\right) = 0, \tag{67}$$

which holds for any Killing vector ξ^a and fields satisfying the field equations (32). This result can also be readily obtained from (36) using the fact, shown in the preceding section, that the Lie derivative with respect to a Killing vector field and the covariant one commute.

Some comments are in order. We want to point out that $T_{\mathscr{B}}^{ab}$ does not depend on Killing vectors. $T_{\mathscr{B}}^{ab}$ depends on the fields, their derivatives and the metric, and $\nabla_a T_{\mathscr{B}}^{ab} = 0$ is always true, even when the metric has no isometry at all. But, of course, a tensor by itself does not give rise to any conserved quantity[1] so, in order to construct conserved quantities, it is necessary to have a Killing vector at hand to construct the current $\mathcal{J}_\xi^a = T_{\mathscr{C}}^{ab}\xi_b$.

The $T_{\mathscr{B}}^{ab}$ as defined in (51) is the one that arises naturally from Noether's theorem, since (55) shows that if spacetime admits a Killing vector we obtain from $T_{\mathscr{B}}^{ab}$ a conserved current $\mathcal{J}_\xi^a$. Thus, for instance, the $n(n+1)/2$ currents in Minkowski spacetime are obtained from $T_{\mathscr{B}}^{ab}$ by contracting it with the corresponding Killing vectors.

The canonical energy–momentum tensor $T_{\mathscr{C}}^{ab}$ is not symmetric except for scalar fields. It is not even gauge invariant for gauge theories. Of course, in flat spacetime, it holds $\nabla_a T_{\mathscr{C}}^{ab} = 0$. But, as we mentioned above, it is worthwhile noticing that this is not even true for curved spacetime. Since, taking into account that the Lagrangian $\mathscr{L}$ depends on the coordinates only through the tensor fields $\nabla_a\psi_{(\ell)}$, $\psi_{(\ell)}$ and g_{ab}, we can compute

$$\begin{aligned}
\nabla_b(\mathscr{L}) &= \frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\nabla_b\nabla_a\psi_{(\ell)} + \frac{\partial\mathscr{L}}{\partial\psi_{(\ell)}}\nabla_b\psi_{(\ell)} \\
&= \frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\nabla_a\nabla_b\psi_{(\ell)} + \nabla_a\left(\frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\right)\nabla_b\psi_{(\ell)} - \frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}(\nabla_a\nabla_b - \nabla_b\nabla_a)\psi_{(\ell)} \\
&= \nabla_a\left(\frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\nabla_b\psi_{(\ell)}\right) - \frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}R^d{}_{cab}\widetilde{\psi_{(\ell)}}{}^c{}_d, \tag{68}
\end{aligned}$$

---

[1] For $\nabla_a T^{ab} = \frac{\partial_a(\sqrt{-g}T^{ab})}{\sqrt{-g}} + T^{ac}\Gamma_{ca}^b$.

where we have used the field equations (32), and (12). So, we get

$$\nabla_a T_{\mathscr{C}}^{ab} = \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} R^b{}_{adc} \widetilde{\psi_{(\ell)}}^{cd}. \tag{69}$$

Thus, except for scalar fields, $T_{\mathscr{C}}^{ab}$ is not 'conserved' when spacetime is curved.

Moreover, even in flat spacetime, for a Killing field ξ_b it holds $\nabla_a \left(T_{\mathscr{C}}^{ab} \xi_b \right) = T_{\mathscr{C}}^{[ab]} \nabla_a \xi_b$, so it vanishes only for parallel ξ_b for general tensor fields, since $T_{\mathscr{C}}^{ab}$ is not symmetric. Then we get from $T_{\mathscr{C}}^{ab}$ only n conserved currents associated with the parallel Killing vectors (translations). A similar result holds for curved spacetime, even though $\nabla_a T_{\mathscr{C}}^{ab} \neq 0$. In fact, if there exists a parallel Killing vector ($\nabla_a \xi^b = 0$), (66) shows that $\nabla_a \left(T_{\mathscr{C}}^{ab} \xi_b \right) = 0$.

Therefore, the canonical energy–momentum tensor $T_{\mathscr{C}}^{ab}$ is rather an exception that occurs only when spacetime admits parallel Killing vectors. Our computations clearly show that, in general, it is $T_{\mathscr{B}}^{ab}$ and not $T_{\mathscr{C}}^{ab}$ that arises naturally from Noether's theorem, so there is no reason to expect much from $T_{\mathscr{C}}^{ab}$. So, we find no reason to start from $T_{\mathscr{C}}^{ab}$ and then symmetrize it in order to get the right tensor $T_{\mathscr{B}}^{ab}$ (see for example [3, 4]). After all, we can always find a nonsense correction to a wrong result to get the right one.

3.8. Off-shell relation between the tensors $T_{\mathscr{B}}^{ab}$ and $T_{\mathscr{M}}^{ab}$

Notice that the on-shell equality $T_{\mathscr{B}}^{ab} = T_{\mathscr{M}}^{ab}$ means that for any scalar Lagrangian depending on the tensor fields $\nabla_a \psi_{(\ell)}$, $\psi_{(\ell)}$ and g_{ab}, for fields satisfying the field equations, it must hold

$$2 \frac{\partial \mathscr{L}}{\partial g_{ab}} = \nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} \right) - \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)}. \tag{70}$$

It is worthwhile noticing that (70) is a consequence of $\widetilde{\mathscr{L}} = 0$, since for any scalar $\mathscr{L}$ we have

$$\widetilde{\mathscr{L}}^{ab} = 0 = \frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\nabla_c \psi_{(\ell)}}^{ab} + \frac{\partial \mathscr{L}}{\partial \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} + \frac{\partial \mathscr{L}}{\partial g_{cd}} \widetilde{g_{cd}}^{ab}, \tag{71}$$

taking into account that $\widetilde{g_{cd}}^{ab} = -\delta_c^a \delta_d^b - \delta_d^a \delta_c^b$, we get

$$2 \frac{\partial \mathscr{L}}{\partial g_{ab}} = \frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\nabla_c \psi_{(\ell)}}^{ab} + \frac{\partial \mathscr{L}}{\partial \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab}. \tag{72}$$

But, from (15), we have

$$\widetilde{\nabla_c \psi_{(\ell)}}^{ab} = \nabla_c \widetilde{\psi_{(\ell)}}^{ab} - \delta_c^a \nabla^b \psi_{(\ell)}. \tag{73}$$

Thus

$$2 \frac{\partial \mathscr{L}}{\partial g_{ab}} = \frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \nabla_c \widetilde{\psi_{(\ell)}}^{ab} + \frac{\partial \mathscr{L}}{\partial \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} - \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)}. \tag{74}$$

Now, for fields satisfying the field equations (32), we get (70).

For instance, for a scalar field ϕ the first term in (70) vanishes, since $\widetilde{\phi} = 0$, and so we get

$$2 \frac{\partial \mathscr{L}}{\partial g_{ab}} = -\frac{\partial \mathscr{L}}{\partial \partial_a \phi} \partial^b \phi. \tag{75}$$

Moreover, as the right-hand side of (70) is a symmetric tensor field, so is the left-hand side. Hence, for fields satisfying the field equation, we have the nontrivial on-shell relations

$$\nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ba} \right) - \frac{\partial \mathscr{L}}{\partial \nabla_b \psi_{(\ell)}} \nabla^a \psi_{(\ell)} = \nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}^{ab} \right) - \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \nabla^b \psi_{(\ell)}. \tag{76}$$

In order to obtain the off-shell relation between the tensors $T_{\mathcal{B}}^{ab}$ and $T_{\mathcal{M}}^{ab}$, we subtract their definitions (49) and (51)

$$
\begin{aligned}
T_{\mathcal{M}}^{ab} - T_{\mathcal{B}}^{ab} &= 2\frac{\partial\mathscr{L}}{\partial g_{ab}} + \frac{\partial\mathscr{L}}{\partial\nabla_a\psi_{(\ell)}}\nabla^b\psi_{(\ell)} - \nabla_c\left(\frac{\partial\mathscr{L}}{\partial\nabla_c\psi_{(\ell)}}\widetilde{\psi_{(\ell)}}^{ab}\right) \\
&= -\left(\nabla_c\left(\frac{\partial\mathscr{L}}{\partial\nabla_c\psi_{(\ell)}}\right) - \frac{\partial\mathscr{L}}{\partial\psi_{(\ell)}}\right)\widetilde{\psi_{(\ell)}}^{ab},
\end{aligned}
\tag{77}
$$

where, in the second step, we have used equation (74), so the equality holds off-shell only for scalar fields.

On the other hand, while $T_{\mathcal{M}}^{ab}$ is symmetric, by definition, everywhere this is not so for $T_{\mathcal{B}}^{ab}$, for taking the antisymmetric part of (77) we get

$$
T_{\mathcal{B}}^{[ab]} = \left(\nabla_c\left(\frac{\partial\mathscr{L}}{\partial\nabla_c\psi_{(\ell)}}\right) - \frac{\partial\mathscr{L}}{\partial\psi_{(\ell)}}\right)\widetilde{\psi_{(\ell)}}^{[ab]},
\tag{78}
$$

and, again except for scalar fields, $T_{\mathcal{B}}^{ab}$ is not in general symmetric off-shell.

3.9. An example: the electromagnetic field

For the sake of clarity, we write down here the most important results of this section for electromagnetic fields. In this case, the Lagrangian is a local function $\mathscr{L} = \mathscr{L}(F_{ab}, g_{ab})$ of F_{ab} (the exterior derivative, $\partial_a A_b - \partial_b A_a$, of a one-form field A_b) and the metric tensor g_{ab}. The field equations are

$$
\nabla_b\left(\frac{\partial\mathscr{L}}{\partial\nabla_b A_a}\right) = 0,
\tag{79}
$$

since $\frac{\partial\mathscr{L}}{\partial A_a} = 0$.

Taking into account that $\widetilde{A_d}^{ab} = -\delta_d^a A^b$ and the obvious antisymmetry of the (2, 0) tensor field $\frac{\partial\mathscr{L}}{\partial\nabla_a A_d} = 2\frac{\partial\mathscr{L}}{\partial F_{ad}}$, equation (74) reads

$$
2\frac{\partial\mathscr{L}}{\partial g_{ab}} = -\frac{\partial\mathscr{L}}{\partial\nabla_c A_a}\nabla_c A^b - \frac{\partial\mathscr{L}}{\partial\nabla_a A_d}\nabla^b A_d = -\frac{\partial\mathscr{L}}{\partial\nabla_a A_c}F^b{}_c.
\tag{80}
$$

which is valid everywhere. As aforementioned, this last result is a consequence of

$$
\widetilde{\mathscr{L}}^{ab} = 0 = \frac{\partial\mathscr{L}}{\partial F_{cd}}\widetilde{F_{cd}}^{ab} + \frac{\partial\mathscr{L}}{\partial g_{cd}}\widetilde{g_{cd}}^{ab} = -2\frac{\partial\mathscr{L}}{\partial F_{ac}}F^b{}_c - 2\frac{\partial\mathscr{L}}{\partial g_{ab}}.
\tag{81}
$$

From (42) we get

$$
\widehat{T}^{cab} = \frac{\partial\mathscr{L}}{\partial\nabla_c A_d}\widetilde{A_d}^{[ba]} + \frac{\partial\mathscr{L}}{\partial\nabla_a A_d}\widetilde{A_d}^{[cb]} + \frac{\partial\mathscr{L}}{\partial\nabla_b A_d}\widetilde{A_d}^{[ca]} = \frac{\partial\mathscr{L}}{\partial\nabla_c A_a}A^b,
\tag{82}
$$

so, the second term in

$$
T_{\mathcal{M}}^{ab} = 2\frac{\partial\mathscr{L}}{\partial g_{ab}} - \nabla_c\left(\frac{\partial\mathscr{L}}{\partial\nabla_c A_d}\widetilde{A_d}^{ab} + \widehat{T}^{cab}\right) + g^{ab}\mathscr{L},
\tag{83}
$$

vanishes, and therefore we have

$$
T_{\mathcal{M}}^{ab} = 2\frac{\partial\mathscr{L}}{\partial g_{ab}} + g^{ab}\mathscr{L} = -\frac{\partial\mathscr{L}}{\partial\nabla_a A_c}F^b{}_c + g^{ab}\mathscr{L}.
\tag{84}
$$

On the other hand, from (51) and (82) we get

$$
\begin{aligned}
T_{\mathcal{B}}^{ab} &= -\frac{\partial\mathscr{L}}{\partial\nabla_a A_c}\nabla^b A_c - \nabla_c\left(\frac{\partial\mathscr{L}}{\partial\nabla_c A_a}A^b\right) + g^{ab}\mathscr{L} \\
&= -\frac{\partial\mathscr{L}}{\partial\nabla_a A_c}F^b{}_c + g^{ab}\mathscr{L} - \nabla_c\left(\frac{\partial\mathscr{L}}{\partial\nabla_c A_a}\right)A^b,
\end{aligned}
\tag{85}
$$

on shell, the last term vanishes, and so $T_{\mathscr{M}}^{ab}$ and $T_{\mathscr{B}}^{ab}$ coincide. Moreover it shows that the symmetry of $T_{\mathscr{B}}^{ab}$ only holds on shell.

3.10. The standard definition of $T_{\mathscr{M}}^{ab}$

Usually the metric energy–momentum tensor is defined through the variation of the action (31) (see for instance [5])

$$\delta S := \frac{1}{2} \int_{\Omega} T_{\mathscr{M}}^{ab} \delta g_{ab} \sqrt{|g|}\, \mathrm{d}^n x, \tag{86}$$

where δg_{ab} are arbitrary variations of the metric vanishing everywhere except in the interior of Ω. We can easily show that it coincides with the one defined in (49) for, under the change $g_{ab} \to g_{ab} + \delta g_{ab}$,

$$\delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \delta \nabla_a \psi_{(\ell)} + \frac{\partial \mathscr{L}}{\partial g_{ab}} \delta g_{ab}. \tag{87}$$

But, according to (11),

$$\delta \nabla_a \psi_{(\ell)} = \delta \left(\partial_a \psi_{(\ell)} + \Gamma_{ca}^b \widetilde{\psi_{(\ell)}}{}^c{}_b \right) = \delta \Gamma_{ca}^b \widetilde{\psi_{(\ell)}}{}^c{}_b. \tag{88}$$

Thus, by using the well-known relation

$$\delta \Gamma_{ca}^b = \tfrac{1}{2} g^{bd} (\nabla_a \delta g_{dc} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac}), \tag{89}$$

we can write the first term in (87) as

$$\frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \delta \nabla_a \psi_{(\ell)} = \frac{1}{2} \frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}{}^{cb} (\nabla_a \delta g_{bc} + \nabla_c \delta g_{ab} - \nabla_b \delta g_{ac})$$

$$= \frac{1}{2} \left(\frac{\partial \mathscr{L}}{\partial \nabla_a \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}{}^{bc} + \widehat{T}^{abc} \right) \nabla_a \delta g_{bc}. \tag{90}$$

Therefore, under the change $g_{ab} \to g_{ab} + \delta g_{ab}$

$$\delta (\mathscr{L} \sqrt{|g|}) = \frac{1}{2} \left(2 \frac{\partial \mathscr{L}}{\partial g_{ab}} - \nabla_c \left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}{}^{ab} + \widehat{T}^{cab} \right) + g^{ab} \mathscr{L} \right) \delta g_{ab} \sqrt{|g|}$$

$$+ \frac{1}{2} \nabla_c \left(\left(\frac{\partial \mathscr{L}}{\partial \nabla_c \psi_{(\ell)}} \widetilde{\psi_{(\ell)}}{}^{ab} + \widehat{T}^{cab} \right) \delta g_{ab} \right) \sqrt{|g|}, \tag{91}$$

where we have used the well-known result

$$\delta \sqrt{|g|} = \tfrac{1}{2} \sqrt{|g|}\, g^{ab} \delta g_{ab}. \tag{92}$$

Finally, by integrating (91) over any compact region Ω, taking arbitrary symmetric tensor fields δg_{ab} vanishing everywhere except in its interior, we show that definitions (49) and (86) coincide.

Equation (90) shows that the term between brackets in (49) arises from the Lagrangian dependence on the affine connection. In particular, it is absent for scalar or electromagnetic fields. Thus, in these cases, we have

$$T_{\mathscr{M}}^{ab} := 2 \frac{\partial \mathscr{L}}{\partial g_{ab}} + g^{ab} \mathscr{L}. \tag{93}$$

In these cases, the 'tilde calculus' also turns out to be unnecessary to write down $T_{\mathscr{M}}^{ab}$. In fact, there is a simpler definition for the energy–momentum tensor for Maxwell's theory [7]

$$T_{E.M.}^{ab} := -2 \frac{\partial \mathcal{L}}{\partial F_{ac}} F_{c}^{b} + g^{ab} \mathscr{L}, \tag{94}$$

which turns out to be symmetric and gauge-invariant for any field theory where the Lagrangian $\mathscr{L}$ is a local function of F_{ab}, the exterior derivative $\partial_a A_b - \partial_b A_a$ of a one-form field A_b.

4. Conclusions

Summarizing, we have shown that the Belinfante energy–momentum is the one that arises naturally from Noether's theorem when the metric has isometries, and all the currents are written as $\mathcal{J}_\xi^a = T_{\mathcal{B}}^{ab}\xi_b$. Moreover, it coincides on-shell with $T_{\mathcal{M}}^{ab}$ for general tensor field theories.

On the other hand, the utility of our definition of $\widetilde{T}$ is apparent if we take into account that most of the equations of this work contain at least one tilde.

Acknowledgments

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