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# Dimensional Transmutation and Dimensional Regularization in Quantum Mechanics

## II. Rotational Invariance

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A thorough analysis is presented of the class of central fields of force that exhibit: (i) dimensional transmutation and (ii) rotational invariance. Using dimensional regularization, the two-dimensional delta-function potential and the $D$-dimensional inverse square potential are studied. In particular, the following features are analyzed: the existence of a critical coupling, the boundary condition at the origin, the relationship between the bound-state and scattering sectors, and the similarities displayed by both potentials. It is found that, for rotationally symmetric scale-invariant potentials, there is a strong-coupling regime, for which quantum-mechanical breaking of symmetry takes place, with the appearance of a unique bound state as well as of a logarithmic energy dependence of the scattering with respect to the energy.


## I. INTRODUCTION

Dimensional transmutation [1] has become a standard concept for the analysis of quantum field theories devoid of intrinsic dimensional parameters. In the first paper in this series [2], we have shown that dimensional transmutation is a more general phenomenon related to dimensional analysis and renormalization, and we provided a general theory for its description in nonrelativistic quantum mechanics. The main purpose of this paper is to apply the general theory developed in Ref. [2] to an exhaustive analysis of examples that exhibit rotational invariance. Parenthetically, as shown in Ref. [2], scale-invariant potentials—those that posses scale symmetry at the classical level and do not exhibit any explicit dimensional scale—are



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exactly described by homogeneous functions of degree $-2$; this class includes contact potentials as well as ordinary potentials of the form $1/r^2$, with a possible angular dependence. Thus, within the subset of such potentials with rotational invariance, the following two problems stand out: the two-dimensional delta-function potential [3–6] and the inverse square potential [7–16]. In addition to solving these potentials, we will expand the general framework presented in [2] in order to encompass further understanding of the scattering sector—including a partial-wave analysis—as well as a reexamination of the peculiar boundary conditions satisfied by these potentials at the origin.

The plan of this paper is as follows. In Section II we discuss the two-dimensional delta-function potential and compare the results with those earlier derived in Ref. [2]. In Section III we summarize the known properties of the inverse square potential (but generalizing them to $D$ dimensions) and examine the issue of the boundary condition at the origin; the conclusion of that section is that renormalization is needed in the strong-coupling regime. In Section IV we provide the required dimensional renormalization of the inverse square potential and find a complete solution to the problem by means of a duality transformation. Section V summarizes our main conclusions and outlines a strategy for answering additional questions. The appendices explicitly use the concept of rotational invariance and deal with the $D$-dimensional central-force problem, the $D$-dimensional partial-wave expansion, and the duality transformation.

We conclude the introduction with some comments about notation and background, with which we assume that the reader has some familiarity from Ref. [2]. In particular, in this paper, we will solve the Schrödinger equation associated with

$$V(\mathbf{r}) = -\lambda W(\mathbf{r}), \tag{1.1}$$

for the two-dimensional delta-function potential

$$W(\mathbf{r}) = \delta^{(2)}(\mathbf{r}) \tag{1.2}$$

(Section II), and for the inverse square potential

$$W(\mathbf{r}) = \frac{1}{r^2} \tag{1.3}$$

(Sections III and IV). The regularized solutions will be obtained with dimensional continuation from $D_0$ to $D$ dimensions and will depend on the parameter $\varepsilon = D_0 - D$; from these solutions, we will identify the energy generating function $\Xi(\varepsilon)$, by comparison with the master eigenvalue equation

$$\lambda \mu^{\varepsilon} \Xi(\varepsilon) \, |E(\varepsilon)|^{-\varepsilon/2} = 1, \tag{1.4}$$

where $\lambda$ is the dimensionless coupling constant, $E$ the energy, and $\mu$ the renormalization scale that arises from the bare coupling $\lambda_B = \mu^{\varepsilon}\lambda$. The ensuing analysis of existence of bound states and related concepts will often refer to properties of the energy generating function $\Xi(\varepsilon)$ and of the critical couplings

$$\lambda_n^{(*)} = [\lim_{\varepsilon \to 0} \Xi_n(\varepsilon)]^{-1}, \tag{1.5}$$

as discussed in Ref. [2], even though the derived formulas can mostly be analyzed on their own right and, for the most part, this paper is essentially self-contained.

## II. TWO-DIMENSIONAL DELTA-FUNCTION POTENTIAL

The delta-function potential belongs to the class of pseudopotentials that arises in the low-energy limit of effective quantum field theory [17]. As in Ref. [2], in this section we will use dimensional regularization and will focus only on those features that are associated with the dimensional transmutation of the two-dimensional representative of this class, Eqs. (1.1) and (1.2).

The two-dimensional delta-function potential has been extensively studied in the literature using a variety of regularization techniques: (i) momentum-cutoff regularization [3, 4, 18], (ii) real-space short-distance regularization (using a circular-well potential) [19–22], (iii) Pauli–Villars regularization [23], (iv) dimensional regularization [2, 23], and (v) method of self-adjoint extensions [4, 5]. As expected, the final renormalized results are independent of the regularization scheme, despite the dissimilar appearance of the regularized expressions. Even though our treatment will be solely based on dimensional regularization, the establishment of a peculiar boundary condition at the origin shows a natural connection with method (v), where the singular nature of the potential at the origin is replaced by a boundary condition in order to preserve the self-adjoint character of the Hamiltonian.

The dimensionally regularized Schrödinger equation for the potential (1.2) can be solved in a number of different ways. In Ref. [2] we solved it in momentum space—a procedure that works due to the zero-range nature of this potential [3]. The same feature permits an effective solution in hyperspherical coordinates as if it were a central potential (according to the framework developed in Appendix A). In fact, the similarities between the two-dimensional delta-function and inverse square potentials will become more transparent, by the use of hyperspherical coordinates for both.

According to the dimensional-continuation prescription of Ref. [2], we Fourier transform Eq. (1.2) in $D_0 = 2$ dimensions, then perform a dimensional jump to $D$ dimensions in Fourier space, and finally return to position space in $D$ dimensions, with the obvious result

$$[-\nabla^2_{\mathbf{r},D} - \lambda\mu^{\varepsilon}\delta^{(D)}(\mathbf{r})]\,\Psi(\mathbf{r}) = E\Psi(\mathbf{r}). \tag{2.1}$$

From the generalized angular-momentum analysis for central potentials of Appendix A, it follows that the radial part of Eq. (2.1) in hyperspherical coordinates reduces to the homogeneous form

$$\left[ \frac{d^2}{dr^2} + E - \frac{(l+\nu)^2 - 1/4}{r^2} \right] u_l(r) = 0, \tag{2.2}$$

for any $r \neq 0$. In Eq. (2.2), as well as in our subsequent analysis, both for the two-dimensional delta-function and inverse square potentials, the number $D$ of dimensions will usually appear in terms of the variable

$$\nu = D/2 - 1, \tag{2.3}$$

which will thereby simplify the form of most formulas.

Equation (2.2) is indistinguishable from a free particle except for the stringent boundary condition enforced by the delta-function singularity at the origin. We now move on to analyze this boundary condition.

### A. *Boundary Condition at the Origin for the Two-Dimensional Delta-Function Potential*

Simple inspection of Eqs. (2.1) and (2.2) shows that the delta-function singularity represents the only difference between them. Even though the correct equation is (2.1), one can still safely use (2.2), provided that the singularity be replaced by an appropriate boundary condition at the origin. This can be established by means of the small-argument behavior of the wave function (neglecting the energy terms as $r \to 0$)

$$-\nabla^2_{\mathbf{r},D} \Psi(\mathbf{r}) \overset{(r \to 0)}{\sim} \lambda \mu^{\varepsilon} \Psi(\mathbf{0})\, \delta^{(D)}(\mathbf{r}), \tag{2.4}$$

which can be evaluated by comparison with the identity

$$-\nabla^2_{\mathbf{r},D} \left[ \frac{r^{-(D-2)}}{(D-2)\, \Omega_D} \right] = \delta^{(D)}(\mathbf{r}). \tag{2.5}$$

Equations (2.4) and (2.5) imply that the general form of $\Psi(\mathbf{r})$ near the origin is

$$\Psi(\mathbf{r}) \overset{(r \to 0)}{\sim} \Psi(\mathbf{0}) \frac{\lambda \mu^{\varepsilon}}{(D-2)\, \Omega_D} r^{-(D-2)} + \Psi_h(\mathbf{r}), \tag{2.6}$$

where $\Psi_h(\mathbf{r})$ is the general solution of the homogeneous equation. Moreover, Eq. (2.6) can be made more explicit by means of

$$\Psi_h(\mathbf{r}) = \sum_{l=0}^{\infty} \left[ C_l^{(+)} r^l + C_l^{(-)} r^{-(l+2\nu)} \right] Y_L(\Omega^{(D)}) \tag{2.7}$$

and

$$-\nabla^2_{\mathbf{r},D}[r^{-(l+2\nu)}Y_L(\Omega^{(D)})] \propto \nabla^l \delta^{(D)}(\mathbf{r}), \tag{2.8}$$

with $\nabla^l$ being a certain linear combination of $l$th order derivatives; then, it follows that $C_l^{(-)}=0$ for all $l$, for otherwise $C_l^{(-)}$ would modify the inhomogeneous part of Eq. (2.6). Then, resolving the wave function into individual angular momentum components,

$$\Psi(\mathbf{r}) = \sum_{l=0}^{\infty} R_l(r)\, Y_L(\Omega^{(D)}), \tag{2.9}$$

we have the following set of boundary conditions. First, for $l=0$, the value of the constant $C_0^{(+)}$ corresponds to $\Psi(\mathbf{0})$, as required by self-consistency of the value of the wave function at the origin; then,

$$R_0(r) \overset{(r\to 0)}{\sim} R_0(0)\left[\frac{\lambda\mu^{-2\nu}\Gamma(\nu)}{4\pi^{\nu+1}}\, r^{-2\nu}+1\right]. \tag{2.10}$$

In addition, for $l\neq 0$,

$$R_l(r) \overset{(r\to 0)}{\sim} C_l^{(+)} r^l. \tag{2.11}$$

Equations (2.10) and (2.11) are the required boundary conditions at the origin. In particular, Eq. (2.10) implies that $D<2$ is the condition for regularity, while the case $D=2$ is "critical."

B. *Bound-State Sector for a Two-Dimensional Delta-Function Potential*

The coupling $\lambda$ in Eq. (1.2) defines an "attractive" zero-range potential for $\lambda>0$ and a "repulsive" one for $\lambda<0$. By the form of the potential, this physical argument implies that all states with $E>0$ are of the scattering type, while states with $E<0$ can only be bound and are impossible for $\lambda<0$. In other words, there exists a critical coupling

$$\lambda^{(*)}=0, \tag{2.12}$$

which separates the theory into two regimes. Therefore, for a delta-function potential, the strong-coupling regime ($\lambda>\lambda^{(*)}=0$) coincides with the set of attractive potentials, while the weak-coupling regime ($\lambda<\lambda^{(*)}=0$) amounts to repulsive potentials.

Let us now consider the bound-state sector, with energy $E=-\kappa^2<0$, for an attractive two-dimensional delta-function potential. The corresponding solution of Eq. (2.2) is

$$\frac{u_l(r)}{\sqrt{r}} = \{I_{l+\nu}(\kappa r), K_{l+\nu}(\kappa r)\}, \tag{2.13}$$

where the symbol { , } stands for linear combination, and $I_p(z)$ and $K_p(z)$ are the modified Bessel functions of the first and second kinds, respectively [24].

Of course, the boundary conditions restrict the selection in Eq. (2.13). First, the boundary condition at infinity leads to the rejection of the modified Bessel function of the first kind, so that (from Eq. (A14))

$$R_l(r) = A_l r^{-\nu} K_{l+\nu}(\kappa r). \tag{2.14}$$

Next, the boundary condition at the origin, Eqs. (2.10) and (2.11), can be enforced at the level of Eq. (2.14) by considering the small-argument behavior of the modified Bessel function of the second kind [24, 25],

$$K_p(z) \overset{(z\to 0)}{\sim} \frac{1}{2}\left[\Gamma(p)\left(\frac{z}{2}\right)^{-p} + \Gamma(-p)\left(\frac{z}{2}\right)^{p}\right][1+O(z^2)]. \tag{2.15}$$

For $l \neq 0$, Eq. (2.14) gives a singular term proportional to $r^{-p}$, with $p = l + \nu$, according to Eq. (2.15). Therefore, the boundary condition can *only* be satisfied for

$$l = 0, \tag{2.16}$$

so that the delta-function potential, being of zero range, can only sustain bound states in the absence of a centrifugal barrier ($s$ states). Then, for the radial wave function with $l = 0$,

$$R_0(r) = A_0 r^{-\nu} K_\nu(\kappa r) \tag{2.17}$$

$$\overset{(r\to 0)}{\sim} \frac{A_0}{2}\left(\frac{\kappa}{2}\right)^{\nu}\left[\Gamma(-\nu) + \Gamma(\nu)\left(\frac{\kappa r}{2}\right)^{-2\nu}\right][1+O(r^2)]. \tag{2.18}$$

In conclusion, compatibility of Eqs. (2.10) and (2.18) requires that the following two conditions be simultaneously met: (i) that

$$A_0 = 2R_0(0)\left(\frac{2}{\kappa}\right)^{-\nu}\frac{1}{\Gamma(-\nu)}, \tag{2.19}$$

which just enforces the condition of finiteness at the origin and requires $D < 2$ for regularity; and (ii) that the bound-state energies $E$ satisfy the eigenvalue condition

$$\frac{\lambda\mu^{-2\nu}}{4\pi}\left(\frac{|E|}{4\pi}\right)^{\nu}\Gamma(-\nu) = 1. \tag{2.20}$$

Equations (2.18), (2.19), and (2.20) are in complete agreement with the results obtained in Ref. [2], to which we refer the reader for additional comments.

As there is no other quantum number in this problem, and $l = 0$ is required, we see that Eq. (2.17) represents the ground state of the regularized system. Parenthetically, the proportionality constant $A_0$ can be found by normalization by means of the identity [26]

$$\int_0^{\infty} x[K_\nu(x)]^2\, dx = \tfrac{1}{2}\Gamma(1+\nu)\,\Gamma(1-\nu), \tag{2.21}$$

so that

$$\Psi(\mathbf{r}) = \frac{\kappa}{\pi^{(\nu+1)/2}[\Gamma(1-\nu)]^{1/2}} \frac{K_\nu(\kappa r)}{r^\nu}. \tag{2.22}$$

Equations (2.20) and (2.22) indeed reduce to the known expressions for $D = 1$ [27]. Here, no attempt is made to draw any conclusions about the cases $D > 2$, as they require a separate regularization procedure.

As discussed in Ref. [2], Eq. (2.20) is equivalent to the master eigenvalue equation (1.4), with an energy generating function

$$\Xi(\varepsilon) = \frac{1}{4\pi}(4\pi)^{\varepsilon/2}\,\Gamma\left(\frac{\varepsilon}{2}\right), \tag{2.23}$$

which, not having any discrete labels, produces just a ground state

$$E_{(\mathrm{gs})} = -\mu^2 e^{g^{(0)}} \rightsquigarrow -\mu^2, \tag{2.24}$$

but no excited states. In Eq. (2.24), the symbol $\rightsquigarrow$ refers to the freedom to make the choice $g^{(0)} = 0$, which means no loss of generality, as both $g^{(0)}$ and $\mu$ are totally arbitrary [2]. Moreover, the critical coupling is $\lambda^{(*)} = 0$, as anticipated earlier by the argument leading to Eq. (2.12); more precisely, the coupling constant is asymptotically critical according to the scheme

$$\lambda(\varepsilon) = 2\pi\varepsilon\left\{1 + \frac{\varepsilon}{2}\,[g^{(0)} - (\ln 4\pi - \gamma)]\right\}, \tag{2.25}$$

where $\gamma$ is the Euler–Mascheroni constant.

Finally, the ground state wave function of the renormalized two-dimensional delta-function potential can be obtained in the limit $\nu \to 0$ of Eq. (2.22), and using Eq. (2.24), with the result

$$\Psi_{(\mathrm{gs})}(\mathbf{r}) = \frac{\mu}{\sqrt{\pi}} K_0(\mu r), \tag{2.26}$$

which was derived by using a renormalized path integral in Ref. [18].

C. *Scattering Sector for a Two-Dimensional Delta-Function Potential*

Scattering under the action of a two-dimensional delta-function potential will also be analyzed by an explicit partial-wave resolution in hyperspherical coordinates according to the theory of Appendix B. The first step is to solve Eq. (2.2), which for the scattering sector, $E = k^2 > 0$, admits the solution

$$u_l(r) = \sqrt{r}\,[A_l^{(+)} H_{l+\nu}^{(1)}(kr) + A_l^{(-)} H_{l+\nu}^{(2)}(kr)], \tag{2.27}$$

where the $H_p^{(1,2)}(z)$ are Hankel functions, whose behavior near the origin implies the asymptotic dependence

$$u_l(r) \overset{(r\to 0)}{\sim} \frac{\sqrt{r}}{\pi i}\left\{[A_l^{(+)} e^{-i\pi(l+\nu)} - A_l^{(-)} e^{i\pi(l+\nu)}]\,\Gamma(-\nu)\left(\frac{kr}{2}\right)^{l+\nu} + [A_l^{(+)} - A_l^{(-)}]\,\Gamma(\nu)\left(\frac{kr}{2}\right)^{-(l+\nu)}\right\}. \tag{2.28}$$

Equations (2.28) and (A14) yield

$$R_l(r) \overset{(r\to 0)}{\sim} \frac{A^{(-)}}{\pi i}[S_l^{(D)}(k)\,e^{-i\pi(l+\nu)} - e^{i\pi(l+\nu)}]\,\Gamma(-\nu)\left(\frac{kr}{2}\right)^{l} \times\left\{1 + \frac{[S_l^{(D)}(k) - 1]\,\Gamma(\nu)(2/k)^{2\nu}}{[S_l^{(D)}(k)\,e^{-i\pi(l+\nu)} - e^{i\pi(l+\nu)}]\,\Gamma(-\nu)}\,r^{-(2l+2\nu)}\right\}, \tag{2.29}$$

where the scattering matrix elements $S_l^{(D)}(k)$ have been explicitly introduced by means of Eq. (B12).

Equation (2.29) should be compared again with the boundary condition at the origin. According to Eq. (2.8), for $l \neq 0$, the boundary condition (2.11) cannot be satisfied, unless

$$S_l^{(D)}(k)|_{l\neq 0} = 1; \tag{2.30}$$

in particular, for the phase shifts,

$$\delta_l^{(D)}(k)|_{l\neq 0} = 0, \tag{2.31}$$

with the conclusion that there is no scattering for $l \neq 0$. On the other hand, for the $s$ wave, Eqs. (2.10) and (2.29) yield a scattering matrix element

$$S_0^{(D)}(k) = \frac{1 - \lambda e^{i\pi\nu}\,\Xi(-2\nu)(E/\mu^2)^{\nu}}{1 - \lambda e^{-i\pi\nu}\,\Xi(-2\nu)(E/\mu^2)^{\nu}}, \tag{2.32}$$

where

$$\Xi(-2\nu) = \frac{\Gamma(-\nu)}{(4\pi)^{\nu+1}} \tag{2.33}$$

is identical to the energy generating function, Eq. (2.23). Equation (2.32) provides the phase shift $\delta_0^{(D)}(k)$ through Eq. (B7), whence

$$\tan \delta_0^{(D)}(k) = \tan \pi\nu \left[ 1 - \frac{\sec \pi\nu}{\lambda (k/\mu)^{2\nu}\, \Xi(-2\nu)} \right]^{-1}. \tag{2.34}$$

We are now ready to derive the expressions for the two-dimensional delta-function potential. First, for $\lambda < 0$ (repulsive potential), $\lambda$ remains unrenormalized and the limit $\varepsilon = -2\nu \to 0$ yields

$$S_0^{(D)}(k)|_{\lambda<0} = 1, \tag{2.35}$$

so that there is no scattering whatsoever for repulsive two-dimensional delta-function potentials. Instead, for $\lambda > 0$ (attractive potential), in the limit $\varepsilon = -2\nu \to 0$ these expressions should be renormalized by means of Eq. (2.25), whence

$$S_0^{(2)}(k)|_{\lambda>0} = \frac{\ln(E/\mu^2) - g^{(0)} + i\pi}{\ln(E/\mu^2) - g^{(0)} - i\pi}. \tag{2.36}$$

Equation (2.36) can be further simplified with Eq. (2.24), leading to

$$S_0^{(2)}(k) = \frac{\ln(k^2/|E_{(gs)}|) + i\pi}{\ln(k^2/|E_{(gs)}|) - i\pi}, \tag{2.37}$$

and [20, 21, 28, 29]

$$\tan \delta_0^{(2)}(k) = \frac{\pi}{\ln(k^2/|E_{(gs)}|)}, \tag{2.38}$$

where, from this point on, it will be understood that scattering is nontrivial only for $\lambda > 0$. In particular, from Eqs. (2.30), (2.37), (B5), (B8), and (B9), the scattering amplitude becomes

$$\begin{aligned} f_k^{(2)}(\Omega^{(2)}) &= \sqrt{\frac{2}{\pi k}}\, \frac{S_0^{(2)}(k) - 1}{2i} \\ &= \sqrt{\frac{2\pi}{k}} \left[ \ln\left(\frac{k^2}{E_{(gs)}}\right) - i\pi \right]^{-1}. \end{aligned} \tag{2.39}$$

Equation (2.39) is identical to the corresponding expression derived in Ref. [2] and coincides with the corresponding expressions derived in Refs. [19, 20, 22, 23, 28, 29].

A number of consequences follow from Eqs. (2.37), (2.38), and (2.39):

1. The unique pole of the scattering matrix (2.37) corresponds to the unique bound state.

2. The phase shifts can be alternatively renormalized using a floating scale $\mu$, in the form

$$\frac{1}{\tan \delta_0^{(2)}(k)} = \frac{1}{\tan \delta_0^{(2)}(\mu)} + \frac{1}{\pi} \ln \left( \frac{k}{\mu} \right)^2. \tag{2.40}$$

3. Equation (2.39) yields a differential scattering cross section

$$\frac{d\sigma^{(2)}(k, \Omega^{(2)})}{d\Omega_2} = \frac{2\pi}{k} \left\{ \left[ \ln \left( \frac{k^2}{E_{(\mathrm{gs})}} \right) \right]^2 + \pi^2 \right\}^{-1}. \tag{2.41}$$

4. The total scattering cross section becomes

$$\sigma_2(k) = \frac{4\pi^2}{k} \{ [\ln(E/|E_{(\mathrm{gs})}|)]^2 + \pi^2 \}^{-1}. \tag{2.42}$$

5. All the relevant quantities are logarithmic with respect to the energy and agree with the predictions of generalized dimensional analysis [2].

6. Equations (2.37), (2.38), (2.39), (2.41), and (2.42) relate the bound-state and scattering sectors of the theory and confirm that the two-dimensional delta-function potential is renormalizable.

## III. INVERSE SQUARE POTENTIAL: INTRODUCTION

It is well known that the inverse square potential, $V(\mathbf{r}) \propto r^{-2}$, displays a number of unusual features in its quantum-mechanical version. In a sense, it represents the boundary between regular and singular power-law potentials (Appendix C). As we will see in this section, its marginally singular nature can be traced back to its interplay with the centrifugal barrier and produces two regimes separated by a critical coupling. These features have become part of the standard background on central potentials in quantum mechanics [7, 9, 10]. The difficulties encountered in the strong-coupling regime correspond to the classical picture of the "fall of the particle to the center" [10]. It should be pointed out that this problem is relevant in polymer physics [30], as well as in molecular physics, where it appears in a modified form as a dipole potential [31, 32].

In this section, our goal is to review the origin of this singular behavior and pave the way for the regularization and renormalization of the theory, which we will implement in Section IV.

A. *Exact Solution for the Unregularized Inverse Square Potential*

Stationary eigenstates of energy and orbital angular momentum of the unregularized inverse square potential are of the factorized form (A4), with an effective radial function (A14) that satisfies the $D_0$-dimensional radial Schrödinger equation (Eqs. (A11)–(A17)),

$$\left[\frac{d^2}{dr^2}+E-\frac{(l+\nu_0)^2-\lambda-1/4}{r^2}\right]u_l(r)=0, \tag{3.1}$$

where $\nu_0=D_0/2-1$ and $\lambda>0$ corresponds to an attractive potential. Then solutions to Eq. (3.1) are of the form

$$u_l(r)=\sqrt{r}\,Z_{s_l}(\sqrt{E}\,r), \tag{3.2}$$

where $Z_{s_l}(z)$ represents an appropriate linear combination of Bessel functions of order

$$s_l=\sqrt{\lambda_l^{(*)}-\lambda}, \tag{3.3}$$

with

$$\lambda_l^{(*)}=(l+\nu_0)^2. \tag{3.4}$$

It is immediately apparent that the inverse square potential in Eq. (3.1) has the same dependence on the radial variable as the centrifugal potential, so that its effect is solely to modify the strength of the centrifugal barrier; correspondingly, the solution (3.2) is essentially a free-particle wave function where the replacement

$$(l+\nu_0)\rightarrow s_l \tag{3.5}$$

has been made. The parameter $\lambda_l^{(*)}$ in Eq. (3.4) plays the role of a critical coupling, i.e., the nature of the solutions changes abruptly around the value $\lambda=\lambda_l^{(*)}$, for any state with angular momentum $l$. Thus, $\lambda_l^{(*)}$ acts as the boundary between two regimes: (i) weak coupling, characterized by $\lambda<\lambda_l^{(*)}$, for which the order $s_l$ is real; and (ii) strong coupling, characterized by $\lambda>\lambda_l^{(*)}$, for which the order $s_l=i\Theta_l$ is imaginary, with

$$\Theta_l=\sqrt{\lambda-\lambda_l^{(*)}}. \tag{3.6}$$

The character of the solutions also depends on the other relevant parameter in Eq. (3.1), namely, the energy $E$, in such a way that: (i) scattering states are only possible if $E=k^2>0$, with the argument of the Bessel functions in Eq. (3.2) being $kr$ (real); while (ii) bound states are only possible if $E=-\kappa^2<0$, with the argument of the Bessel functions in Eq. (3.2) being $kr=i\kappa r$ (imaginary). In conclusion, the solutions (3.2) fall into the following four families, according to the nature of the two relevant variables $s_l$ and $E$:

1. Bound-state sector of the weak-coupling regime, with

$$\frac{u_l(r)}{\sqrt{r}} = \{I_{s_l}(\kappa r), K_{s_l}(\kappa r)\}; \tag{3.7}$$

2. Scattering sector of the weak-coupling regime, with

$$\frac{u_l(r)}{\sqrt{r}} = \{H^{(1)}_{s_l}(kr), H^{(2)}_{s_l}(kr)\}; \tag{3.8}$$

3. Bound-state sector of the strong-coupling regime, with

$$\frac{u_l(r)}{\sqrt{r}} = \{I_{i\Theta_l}(\kappa r), K_{i\Theta_l}(\kappa r)\}; \tag{3.9}$$

4. Scattering sector of the strong-coupling regime, with

$$\frac{u_l(r)}{\sqrt{r}} = \{H^{(1)}_{i\Theta_l}(kr), H^{(2)}_{i\Theta_l}(kr)\}; \tag{3.10}$$

where the { , } stands again for linear combination. In Eqs. (3.7)–(3.10) we use the standard notation $H^{(1,2)}_{s_l}(z)$ for the Hankel functions, and $I_{s_l}(z)$ and $K_{s_l}(z)$ for the modified Bessel functions of the first and second kinds, respectively [24].

Let us first look at the weak-coupling regime. For bound states, the requirement that the solution be finite at infinity implies the rejection of the component $I_{s_l}(\kappa r)$ in Eq. (3.7), whereas the boundary condition at the origin, Eq. (A21), leads to the elimination of the component $K_{s_l}(\kappa r)$. In other words, there exist no bound states for a weak coupling. Physically, this state of affairs is reminiscent of the non-existence of bound states for a repulsive potential; in fact, the combination of the potential itself and the centrifugal potential is effectively "repulsive" when $\lambda < \lambda_l^{(*)}$. On the other hand, such a potential can produce nontrivial scattering, which, according to Eq. (3.8), is described by the solution

$$\begin{aligned} u_l(r) &= \sqrt{r}\,[\tilde{A}_l^{(+)} H^{(1)}_{sl}(kr) + \tilde{A}_l^{(-)} H^{(2)}_{sl}(kr)] \\ &\overset{(r\to\infty)}{\sim} \sqrt{r}\left[\tilde{A}_l^{(+)} H^{(1)}_{l+\nu_0}\left(kr + (l+\nu_0 - s_l)\frac{\pi}{2}\right)\right. \\ &\quad \left. + \tilde{A}_l^{(-)} H^{(2)}_{l+\nu_0}\left(kr + (l+\nu_0 - s_l)\frac{\pi}{2}\right)\right], \end{aligned} \tag{3.11}$$

where $\tilde{A}_l^{(+)} = \tilde{A}_l^{(-)}$ and the asymptotic behavior determines the energy-independent phase shifts

$$\delta^{(D_0)} = [(l+\nu_0) - \sqrt{(l+\nu_0)^2 - \lambda}]\,\frac{\pi}{2}. \tag{3.12}$$

Equation (3.12) manifestly displays the scale invariance of the theory, generalizing the well-known three-dimensional results [7, 33]. Notice that the scattering matrix $S_l^{(D_0)} = \exp[2i\delta_l^{(D_0)}]$ has no poles, which is in agreement with the absence of bound states. Parenthetically, our conclusions relied on the choice of the boundary condition (A21) at the origin; the validity of this procedure for the inverse square potential will be proved in the next section.

Let us now consider the strong-coupling regime. For bound states, the requirement that the solution be finite at infinity implies again the rejection of the component $I_{i\Theta_l}(\kappa r)$ in Eq. (3.9). However, the boundary condition at the origin, Eq. (A21), can neither eliminate the component $K_{s_l}(\kappa r)$ nor restrict the possible values of the energy. In effect, from Eq. (2.15) and elementary properties of the gamma function [24], the small-argument behavior of the modified Bessel function of the second kind and imaginary index is

$$K_{i\Theta_l}(\kappa r) \overset{(r\to 0)}{\sim} -\sqrt{\frac{\pi}{\Theta_l \sinh(\pi\Theta_l)}}\,\sin\left[\Theta_l \ln\left(\frac{\kappa r}{2}\right) - \delta_{\Theta_l}\right][1 + O(r^2)], \tag{3.13}$$

where

$$\delta_{\Theta_l} = \Im[\ln \Gamma(1 + i\Theta_l)] \tag{3.14}$$

is the phase of $\Gamma(1 + i\Theta_l)$. Then if Eq. (3.13) were a bound-state wave function, it would display an oscillatory behavior with an ever increasing frequency as $r \to 0$. In particular, starting with any finite value of $r$, the function defined by Eq. (3.13) has an infinite number of zeros, at the points $r_n = 2\exp[(\delta_{\Theta_l} - n\pi)/\Theta_l]/\kappa$ (with $n$ integer), whence it represents a state lying above infinitely many bound states. In other words, the boundary condition has become ineffective as a screening tool, a situation that we may describe as a "loss" of the boundary condition, and which permits the existence of a continuum of bound states extending from $E = -\infty$ to $E = 0$, in agreement with the conclusions of Ref. [2]. In particular, the potential has no ground state, so that the Hamiltonian is no longer bounded from below and has lost its self-adjoint character [34]. After due reflection, this situation makes sense: the potential has overcome the centrifugal barrier, a phenomenon that corresponds to the classical "fall of the particle to the center" [10]. With regard to the scattering in this strong-coupling regime, due to the loss of the boundary condition, it is impossible to determine the phase of the wave function or relative values of the coefficients in Eq. (3.10); in other words, the scattering parameters are ill-defined.

In summary, for $\lambda < \lambda_l^{(*)}$, the inverse square potential is incapable of producing bound states but it scatters in a manifestly scale-invariant way; while for $\lambda > \lambda_l^{(*)}$, it destroys the discrete character of the bound-state spectrum, the uniqueness of the scattering solutions, and the self-adjoint nature of the Hamiltonian. Here we recognize some of the familiar features of unregularized transmuting potentials.

The failure of the inverse square potential to provide a discriminating boundary condition at the origin for the strong-coupling regime is now seen as the source of the singular behavior required by dimensional transmutation. In some sense, the key to our regularization procedure will be the restoration of a sensible boundary condition for $r=0$.

## B. *Loss of the Boundary Condition at the Origin for the Inverse Square Potential*

Let us now reexamine the boundary condition at the origin, which seems to be the most important ingredient in the analysis of Subsection III.A. For the inverse square potential, the standard argument leading to Eq. (A21) should be modified with the replacement (3.5), so that

$$R_l(r) = \frac{u_l(r)}{r^{(D_0-1)/2}} \overset{(r\to 0)}{\sim} \{r^{\alpha_{+,l}}, r^{\alpha_{-,l}}\}, \tag{3.15}$$

where

$$\alpha_{\pm,l} = -\nu_0 \pm s_l \tag{3.16}$$

are the exponents arising from the associated indicial equation, with $s_l$ given by Eq. (3.3). Obviously, Eq. (3.16) is the small-argument limit of Eqs. (3.7)–(3.10). However, the power-law functions (3.15) are no longer solutions of Laplace's equation, so that the rejection of the second solution, in the analysis leading to Eq. (A21), is not justified. In order to clarify this issue it proves convenient to consider a truncated potential [10]

$$V_a(r) = \begin{cases} -\lambda/a^2 & \text{for} \quad r<a \\ -\lambda/r^2 & \text{for} \quad r \geqslant a, \end{cases} \tag{3.17}$$

which clearly satisfies $V(r) = \lim_{a\to 0} V_a(r)$. The truncated potential permits the use of regular boundary conditions for finite $a$, as a means of deducing the limiting behavior when $a \to 0$. The solution in the presence of the truncated potential (3.17), for $r$ sufficiently small, is

$$R_l(r) \overset{(r\to 0,\, a\to 0)}{\sim} \begin{cases} C_l^{(+)} r^{\alpha_{+,l}} + C_l^{(-)} r^{\alpha_{-,l}} & \text{for} \quad r>a \\ B_l r^l & \text{for} \quad r<a, \end{cases} \tag{3.18}$$

where the exponents $\alpha_{\pm,l}$ are given by Eq. (3.16). Continuity of the logarithmic derivative for $r=a$, in the limit $a \to 0$, provides the condition

$$\Omega_l(a) = \frac{C_l^{(+)}}{C_l^{(-)}} = -\frac{\sqrt{\lambda_l^{(*)}} + \sqrt{\lambda_l^{(*)} - \lambda}}{\sqrt{\lambda_l^{(*)}} - \sqrt{\lambda_l^{(*)} - \lambda}}\, a^{-2\sqrt{\lambda_l^{(*)} - \lambda}}. \tag{3.19}$$

For the weak-coupling regime, $\lambda < \lambda_l^{(*)}$, the ratio (3.19) goes to infinity as $a \to 0$, and the boundary condition becomes

$$R_l(r) \overset{(r \to 0)}{\sim} C_l^{(+)} r^{\alpha_{+,l}}, \tag{3.20}$$

so that

$$u_l(r) \overset{(r \to 0)}{\sim} C_l^{(+)} \sqrt{r}\, r^{s_l}. \tag{3.21}$$

In other words, the choice is made in favor of the least divergent solution; then Eq. (3.21) implies that the boundary condition for the function $u_l(r)$ is (A21), just as for regular potentials. This confirms, a posteriori, that (A21) is a valid condition to apply in the weak-coupling case, as was assumed in the analysis of Subsection III.A, which led to the proper selection of Bessel functions in Eqs. (3.7) and (3.8).

However, for the strong-coupling regime, $\lambda > \lambda_l^{(*)}$, the exponents $\alpha_{\pm}$ are complex conjugate values and both solutions have the same behavior. Then, no criterion can be given to discriminate the "good" from the "bad" solutions, so that the boundary condition at the origin is "lost." Correspondingly, the ratio of Eq. (3.19) has no definite limit for $a \to 0$, displaying the same oscillatory behavior found in Subsection III.A, i.e.,

$$\Omega_l(a) \propto \exp(-2i\Theta_l \ln a). \tag{3.22}$$

Then, as $a \to 0$, the general solution reduces to the oscillatory form (3.13). Notice that Eq. (A21) is still satisfied, but it is *not* a boundary condition, as it has lost its discriminating power: both solutions are now allowed (cf. Eq. (3.15)).

In short, focusing on the behavior near the origin exclusively, we have clarified the meaning of the loss of boundary condition and rediscovered the basic features that had already been predicted in Subsection III.A.

## IV. INVERSE SQUARE POTENTIAL: RENORMALIZATION

In the first detailed treatment of the inverse square potential, Ref. [8], it was proposed that an additional orthogonality constraint be imposed on the eigenfunctions (3.9) in the strong-coupling regime. Effectively, this procedure restores the discrete nature of the spectrum—a direct application of Eq. (3.13) gives the energies

$E_{n'l}/E_{nl} = e^{2\pi(n'-n)/\Theta_l}$. Nonetheless, the allowed bound-state levels still extend to $-\infty$, i.e., the Hamiltonian remains unbounded from below. Subsequently, a number of different regularization techniques were introduced [11–13] by properly adding regularization parameters of various kinds. It was soon realized that the strong-coupling Hamiltonian of the inverse square potential fails to be self-adjoint, despite it being a symmetric operator; as its deficiency indices [34] are (1, 1), its solutions can be regularized with a single parameter in the form of self-adjoint extensions [14]. Alternative attempts simply abandoned self-adjointness in favor of other requirements or interpretations; for example, in Ref. [15], the picture of the "fall of the particle to the center" is explicitly implemented by a non-Hermitian condition.

In our approach, we will follow the traditional path of enforcing the self-adjoint nature of the Hamiltonian—this time using field-theoretic methods. Recently, a renormalized solution was presented along these lines [16], which replaced the orthogonality criterion of Ref. [8] by a regular boundary condition at a cutoff point near the origin. However, this proposal was just limited to the one-dimensional case and only relied on real-space cutoff regularization. A second step along this path was the formulation of the $D$-dimensional generalization of the cutoff-regularization approach [35]. In this section, we extend the results of Refs. [16, 35] using dimensional regularization and emphasize those features associated with dimensional transmutation.

## A. *Dimensional Regularization of the Inverse Square Potential*

Dimensional regularization of the inverse square potential is done as follows. As discussed in Ref. [2], dimensional analysis implies that the $D$-dimensional regularization of the inverse square potential (Eqs. (1.1) and (1.3)) is of the homogeneous form

$$W^{(D)}(r) \propto r^{-(2-\varepsilon)}. \tag{4.1}$$

We would like no to confirm this prediction and find the proportionality coefficient $\mathcal{J}(\varepsilon)$. First, we should find the momentum-space expression by means of a $D_0$-dimensional Fourier transform,

$$\tilde{W}(\mathbf{k}) = \mathcal{F}_{(D_0)}\{W(\mathbf{r})\} = \int d^{D_0}r \, e^{-i\mathbf{k}\cdot\mathbf{r}} \frac{1}{r^2}, \tag{4.2}$$

which we can be computed by means of Bochner's theorem [36], i.e.,

$$\tilde{W}(\mathbf{k}) = \frac{(2\pi)^{D_0/2}}{k^{D_0/2-1}} \int_0^\infty dr \, r^{D_0/2-2} J_{D_0/2-1}(kr). \tag{4.3}$$

Applying the identity [26]

$$\int_0^\infty x^\alpha J_\beta(az)\,dz = 2^\alpha a^{-\alpha-1}\frac{\Gamma(1/2+\beta/2+\alpha/2)}{\Gamma(1/2+\beta/2-\alpha/2)}, \tag{4.4}$$

which is valid for $-\mathrm{Re}(\beta)-1<\mathrm{Re}(\alpha)<1/2$, the required transform is found to be

$$\tilde{W}(\mathbf{k}) = \frac{(4\pi)^{D_0/2}}{4k^{D_0-2}}\,\Gamma(D_0/2-1). \tag{4.5}$$

Equation (4.5) seems to be restricted to $2<\mathrm{Re}(D_0)<5$; however, the right-hand side is a meromorphic function with poles at $D_0=2,\ 0,\ -2, \ldots$ and is the desired analytic continuation of the integral. As we will see next, when the inverse Fourier transform is applied, the final analytically continued expression will have no such restriction. In effect, applying the dimensional-continuation prescription of Ref. [2] to $W^{(2)}(\mathbf{r})=1/r^2$, we get

$$\begin{aligned}
W^{(D)}(\mathbf{r}_D) &= \int \frac{d^Dk_D}{(2\pi)^D}\,e^{i\mathbf{k}_D\cdot\mathbf{r}_D}[\tilde{W}(\mathbf{k}_{D_0})]_{\mathbf{k}_{D_0}\to\mathbf{k}_D},\\
&= \frac{(4\pi)^{D_0/2}}{4}\,\Gamma(D_0/2-1)\int\frac{d^Dk_D}{(2\pi)^D}\,e^{i\mathbf{k}_D\cdot\mathbf{r}_D}[k_{D_0}^{-D_0+2}]_{\mathbf{k}_{D_0}\to\mathbf{k}_D}\\
&= \frac{(2\pi)^{\varepsilon/2}}{r^{D/2-1}}\,2^{D_0/2-2}\Gamma(D_0/2-1)\int_0^\infty dk\,k^{D/2-D_0+2}J_{D/2-1}(kr),
\end{aligned} \tag{4.6}$$

which, by means of Eq. (4.4), amounts to

$$W^{(D_0-\varepsilon)}(r) = \pi^{\varepsilon/2}\Gamma\left(1-\frac{\varepsilon}{2}\right)r^{-(2-\varepsilon)}. \tag{4.7}$$

Even though the last integration is restricted to $3-D_0<\mathrm{Re}(D_0-D)<2$, analytic continuation of the right-hand of Eq. (4.7) allows us to extend its validity for arbitrary values of $\varepsilon\neq-2, -4, \ldots$ (notice that the previous restriction has been lifted); this is perfectly fine anyway, because all that is needed is the limit $\varepsilon=0^+$.

Then, for the dimensionally regularized problem, using the notation

$$\mathscr{I}(\varepsilon) = \pi^{\varepsilon/2}\Gamma\left(1-\frac{\varepsilon}{2}\right), \tag{4.8}$$

we have the $D$-dimensional Schrödinger equation

$$\left[\nabla_D^2+E+\frac{\lambda\mu^\varepsilon\mathscr{I}(\varepsilon)}{r^{2-\varepsilon}}\right]\Psi(\mathbf{r})=0, \tag{4.9}$$

which, for the state of angular momentum $l$, reads

$$\left\{ \frac{d^2}{dr^2} + E + \frac{\lambda \mu^{\varepsilon} \mathscr{J}(\varepsilon)}{r^{2-\varepsilon}} - \frac{[l + D(\varepsilon)/2 - 1]^2 - 1/4}{r^2} \right\} u_l(r) = 0. \tag{4.10}$$

We are now ready to solve Eq. (4.10). As it stands, it does not look familiar when $\varepsilon$ is not an integer. However, it can be conveniently transformed into an asymptotically soluble problem by means of a duality transformation [37], whose properties we study in Appendix C. For the particular case at hand, applying the transformation (cf. (C22))

$$\begin{cases} |E|^{1/2}\, r = z^{2/\varepsilon} \\ |E|^{-D/4}\, u(r) = w(z)\, z^{1/\varepsilon - 1/2}, \end{cases} \tag{4.11}$$

the dual of Eq. (4.10) becomes Eq. (C26), which we write in the condensed form

$$\left\{ \frac{d^2}{dz^2} + \tilde{\eta} - \tilde{\mathscr{V}}_{\varepsilon}(z) - \frac{[p_l(\varepsilon)]^2 - 1/4}{z^2} \right\} w_{l,\varepsilon}(z) = 0, \tag{4.12}$$

with a dual dimensionless energy

$$\tilde{\eta} = \frac{4\lambda \mu^{\varepsilon} \mathscr{J}(\varepsilon)}{\varepsilon^2} |E|^{-\varepsilon/2}, \tag{4.13}$$

a dual potential energy term,

$$\tilde{\mathscr{V}}_{\varepsilon}(z) = -\sigma \frac{4}{\varepsilon^2} z^{4/\varepsilon - 2}, \tag{4.14}$$

and a dual angular momentum variable

$$p_l(\varepsilon) = \frac{2}{\varepsilon} \left[ l + \frac{D(\varepsilon)}{2} - 1 \right]. \tag{4.15}$$

In our subsequent analysis, we will often rewrite Eq. (4.15) in the form

$$p = p_l(\varepsilon) = \frac{2}{\varepsilon} (\lambda_l^{(*)})^{1/2} - 1 = \frac{2}{\varepsilon} (\lambda_l^{(*)})^{1/2} \left[ 1 - \frac{\varepsilon}{2} (\lambda_l^{(*)})^{-1/2} \right], \tag{4.16}$$

in terms of the critical coupling (3.4). Equation (4.16) provides an alternative expansion parameter $p$, such that, as $\varepsilon \to 0$, $p \to \infty$. It should be noticed in passing that Eq. (4.13) already provides the functional form (1.4) required for any possible eigenvalue equation of the inverse square potential, with

$$\Xi(\varepsilon) = \frac{4\mathscr{J}(\varepsilon)}{\varepsilon^2} \tilde{\eta}^{-1}. \tag{4.17}$$

As usual, as the problem is now regularized, Eq. (4.12) should be solved with the regular boundary condition at the origin, Eq. (A21), so that

$$w_{l,\varepsilon}(0)=0. \tag{4.18}$$

Even though the solutions to Eq. (4.12) cannot be expressed in terms of any standard special functions for arbitrary $\varepsilon>0$, it is still possible to write their asymptotic form with respect to $\varepsilon\to 0$, in terms of Bessel functions, as we show below. The key to this asymptotic analysis lies in the singular limiting form of the transformed potential (4.14), namely,

$$\lim_{\varepsilon\to 0}\tilde{\mathscr{V}}_\varepsilon(z)=\begin{cases}0 & \text{for}\quad z<1\\ -\sigma\infty & \text{for}\quad z>1;\end{cases} \tag{4.19}$$

in particular, for $E<0$ (i.e., $\sigma=-1$), $\tilde{\mathscr{V}}_\varepsilon(z)$ behaves as an infinite hyperspherical potential well. Because of the singular nature of the limit (4.19), an asymptotically exact hierarchy emerges, whereby Eq. (4.12) is split into two distinct differential equations: the first one, without the term $\tilde{\mathscr{V}}_\varepsilon(z)$, for the interior region $z<z_2\approx 1$,

$$\left\{\frac{d^2}{dz^2}+\tilde{\eta}-\frac{[p_l(\varepsilon)]^2-1/4}{z^2}\right\}w^{(<)}_{l,\varepsilon}(z)=0; \tag{4.20}$$

and the second one, without the terms proportional to $1/z^2$, for the exterior region $z>z_2\approx 1$,

$$\left[\frac{d^2}{dz^2}-\tilde{\mathscr{V}}_\varepsilon(z)\right]w^{(>)}_{l,\varepsilon}(z)=0. \tag{4.21}$$

Equation (4.20) is a Bessel differential equation of order $p=p_l(\varepsilon)$, whose regular solutions that satisfy the boundary condition (4.18) are of the form

$$w^{(<)}_{l,\varepsilon}(z)=B_l\sqrt{z}\,J_p(\tilde{\eta}^{1/2}z). \tag{4.22}$$

On the other hand, Eq. (4.21) can be taken to a standard Bessel form by another duality transformation; it is easy to verify that

$$w^{(>)}_{l,\varepsilon}(z)=\sqrt{z}\,\mathscr{C}_{\varepsilon/4}(\sqrt{\sigma}\,z^{2/\varepsilon}), \tag{4.23}$$

where $\mathscr{C}_{\varepsilon/4}$ stands for a linear combination of ordinary Bessel functions of order $\varepsilon/4$; in fact, more explicitly,

$$w^{(>)}_{l,\varepsilon}(z)=\begin{cases}A_l\sqrt{z}\,K_{\varepsilon/4}(z^{2/\varepsilon}) & \text{for}\quad \sigma=-1\\ \sqrt{z}\,[\tilde{A}^{(+)}_l H^{(1)}_{\varepsilon/4}(z^{2/\varepsilon})+\tilde{A}^{(-)}_l H^{(2)}_{\varepsilon/4}(z^{2/\varepsilon})] & \text{for}\quad \sigma=1,\end{cases} \tag{4.24}$$

where the first line includes a modified Bessel function of the second kind, whereas the second line includes a linear combination of Hankel functions. Their physical meaning becomes transparent when transformed back from the dual to the original space, in which

$$u_l(r) \overset{(r\to\infty,\,\varepsilon\to 0)}{\sim} \begin{cases} A_l\sqrt{r}\,K_0(\kappa r) & \text{for} \quad E=-\kappa^2<0 \\ \sqrt{r}\,[\tilde{A}_l^{(+)}H_0^{(1)}(kr)+\tilde{A}_l^{(-)}H_0^{(2)}(kr)] & \text{for} \quad E=k^2>0. \end{cases} \tag{4.25}$$

Equation (4.25) represents the asymptotic behavior of the wave function, when $r$ is large, for the bound-state ($E<0$) and scattering ($E>0$) sectors of the inverse square potential. Of course, the asymptotic behavior of Eq. (4.25) justifies a posteriori the choice of Bessel functions in Eq. (4.24).

The boundary point $z_2$ separating the two regimes described by Eqs. (4.20) and (4.21) is infinitesimally close to $z=1$, but further precision is required. In effect, the critical separation takes place at the point $z_2$ where

$$\left|\tilde{\eta}-\frac{[p_l(\varepsilon)]^2-1/4}{z_2^2}\right|=|\tilde{\mathscr{V}}_\varepsilon(z_2)|, \tag{4.26}$$

because, away from it, the term $\tilde{\mathscr{V}}_\varepsilon(z_2)$ will abruptly become negligible for $z<z_2$ and will overwhelmingly dominate over the other terms for $z>z_2$. From Eqs. (4.13)–(4.16), it follows that the condition (4.26) amounts to

$$p^2\left(\frac{\lambda}{\lambda_l^{(*)}}-1\right)\left[1+O\left(\frac{1}{p}\right)\right]=\frac{p^2}{\lambda_l^{(*)}}\,z_2^{4/\varepsilon}\left[1+O\left(\frac{1}{p}\right)\right], \tag{4.27}$$

so that

$$z_2=\Theta_l^{\varepsilon/2}, \tag{4.28}$$

where $\Theta_l$ is given by Eq. (3.6). With the value of $z_2$ so determined, the solutions to Eqs. (4.20) and (4.21) should be matched at $z=z_2$ through the logarithmic derivative

$$\mathscr{L}(\Theta_l)=z\,\frac{d\ln(w/\sqrt{z})}{dz}\bigg|_{z=z_2}. \tag{4.29}$$

For the interior solution (4.22), the logarithmic derivative takes the form

$$\mathscr{L}^{(<)}(\Theta_l)=y\,\frac{d\ln J_p(y)}{dy}\bigg|_{y=\tilde{\eta}^{1/2}z_2}, \tag{4.30}$$

whereas, for the exterior solution (4.22), it becomes

$$\mathscr{L}^{(>)}(\Theta_l)=\frac{2}{\varepsilon}\,\Theta_l\,\frac{d\ln\mathscr{C}_{\varepsilon/4}(\Theta_l)}{d\Theta_l}. \tag{4.31}$$

Having developed the general framework for the analysis of the inverse square potential in terms of its dual problem, we now turn to the distinct details of the bound-state and scattering sectors.

B. *Bound-State Sector for the Inverse Square Potential*

The bound states of the inverse square potential (if any) should be characterized by the energy condition $E<0$, which essentially converts the dual potential into an infinite hyperspherical well in the limit $\varepsilon \to 0^+$, as displayed by Eq. (4.19). This suggests that, as a first approximation, the boundary condition at $z_2$ may be replaced by

$$w^{(<)}_{l,\varepsilon}(1) \approx 0, \tag{4.32}$$

whence Eq. (4.22) immediately gives the spectrum through the condition

$$J_p(\tilde{\eta}^{1/2}) \approx 0, \tag{4.33}$$

from which the corresponding eigenvalues are

$$\tilde{\eta}_{n_r} \approx (j_{p,n_r})^2, \tag{4.34}$$

where $j_{p,n_r}$ is the $n_r$th zero of the Bessel function of order $p$. Therefore, from Eq. (4.13), the regularized eigenvalue condition becomes

$$\lambda \left[ \frac{4\mathcal{J}(\varepsilon)(j_{p,n_r})^{-2}}{\varepsilon^2} \right] \left( \frac{|E|}{\mu^2} \right)^{-\varepsilon/2} \approx 1, \tag{4.35}$$

where $n_r$ is now confirmed as the radial quantum number—the ordinal number for the stationary radial wave functions. Equation (4.35) is of the form of the master eigenvalue Eq. (1.4), with an energy generating function

$$\Xi_{n_r l}(\varepsilon) \approx \frac{4\mathcal{J}(\varepsilon)}{\varepsilon^2} (j_{p,n_r})^{-2}. \tag{4.36}$$

Even though the remarks of the previous paragraph are essentially correct, for a full comparison with the scattering sector of the theory, it is necessary to evaluate the logarithmic derivatives (4.30) and (4.31), which will give extra terms in the energy expressions. Therefore, we need to first analyze the asymptotic behavior of the interior solution (4.22) with respect to $p \to \infty$, and then find the proper values of the logarithmic derivatives.

The limit $\varepsilon \to 0$ of Eq. (4.22) relies on well-known properties of the Bessel functions of large order [24]. We will start with Debye's asymptotic expansion,

$$J_p(p \sec \beta) \overset{(p\to\infty)}{\sim} \sqrt{\frac{2}{\pi p \tan \beta}} \left\{ \cos\left[ p \tan \beta - p\beta - \frac{\pi}{4} \right] + O(p^{-1}) \right\}, \tag{4.37}$$

where the argument will become

$$y = p \sec \beta = \tilde{\eta}^{1/2} z_2. \tag{4.38}$$

For our analysis, a few relations between these variables are in order; in particular,

$$y = p\sqrt{\frac{\lambda}{\lambda_l^{(*)}}}\,[1 + O(\varepsilon)] = p\sqrt{1 + \frac{\Theta_l^2}{\lambda_l^{(*)}}}\,[1 + O(\varepsilon)], \tag{4.39}$$

and

$$\tan \beta = \sqrt{(y/p)^2 - 1} = \frac{\Theta_l}{(\lambda_l^{(*)})^{1/2}}. \tag{4.40}$$

Then the logarithmic derivative (4.30) becomes

$$\mathscr{L}^{(<)}(\Theta_l) = -p\,\frac{\Theta_l}{(\lambda_l^{(*)})^{1/2}} \tan\left\{p\left[\frac{\Theta_l}{(\lambda_l^{(*)})^{1/2}} - \tan^{-1}\left(\frac{\Theta_l}{(\lambda_l^{(*)})^{1/2}}\right)\right] - \frac{\pi}{4}\right\} + O(1). \tag{4.41}$$

Equation (4.41) displays an anomalous behavior as $p \to \infty$ in that the tangent function will oscillate wildly from $-\infty$ to $\infty$, thus rendering the regularized problem ill-defined. The cure for this behavior is afforded by the renormalization of the coupling constant $\lambda = \lambda(\varepsilon)$, which implies a corresponding renormalization of $\Theta_l = \Theta_l(\varepsilon)$ (from Eq. (3.6)), in such a way that $\Theta_l \to 0$ as $\varepsilon \to 0$. Then this amounts to the realization of the limiting critical coupling $\lambda \to \lambda_l^{(*)} + 0^+$. A more careful analysis of this limiting behavior is afforded by

$$\mathscr{F} = \tan\beta - \beta = \frac{\beta^3}{3} + O(\beta^5) = \frac{\Theta_l^3}{3(\lambda_l^{(*)})^{3/2}} + O(\Theta_l^5), \tag{4.42}$$

so that the finiteness of the argument of the tangent implies that $p\Theta_l^3$ should be finite; then

$$\Theta_l = O(p^{-1/3}) \tag{4.43}$$

and

$$y = p + p^{1/3}x, \tag{4.44}$$

where $x$ is a finite variable.

On the other hand, using the small-argument behavior of the modified Bessel function of the second kind [24], Eq. (2.15), it follows that

$$K_{\varepsilon/4}(z^{2/\varepsilon}) \overset{(z\to 0)}{\sim} \left[\frac{2}{\varepsilon}\left(\frac{1}{\sqrt{z}} - \sqrt{z}\right) - \frac{1}{2}(\gamma - \ln 2)\left(\frac{1}{\sqrt{z}} + \sqrt{z}\right)\right][1 + O(\varepsilon^2, z^{4/\varepsilon})], \tag{4.45}$$

so that, from Eqs. (4.28), (4.31), and (4.43), the logarithmic derivative becomes

$$\mathscr{L}^{(>)}(\Theta_l) = \frac{2}{\varepsilon}(\ln \Theta_l - c)^{-1}\,[1 + O(\varepsilon^{2/3})], \tag{4.46}$$

where

$$c = \ln 2 - \gamma. \tag{4.47}$$

Equation (4.46) implies that, as $\varepsilon \to 0$,

$$\frac{w^{(>)}(z_2)}{\sqrt{z_2}} \overset{(\varepsilon \to 0)}{\sim} \frac{\varepsilon}{2}(\ln \Theta_l - c)\frac{d}{dz}\left[\frac{w^{(>)}(z)}{\sqrt{z}}\right]\Bigg|_{z = z_2}, \tag{4.48}$$

so that continuity of the wave function implies that Eq. (4.32) is indeed correct to zeroth order.

The interior logarithmic derivative can now be evaluated from the asymptotic expansion [24]

$$J_p(p + p^{1/3}x) \overset{(p \to \infty)}{\sim} \frac{2^{1/3}}{p^{1/3}} A_i(-2^{1/3}x) + O(p^{-1}), \tag{4.49}$$

where $Ai$ is the Airy function of the first kind [24]. The zeroth order approximation to the logarithmic derivative amounts to $w(1) = 0$ or $\mathscr{L}^{(<)} = \infty$, so that the value of $x$, to this order, is provided by the zeros of the Airy function,

$$Ai(-2^{1/3}x^{(0)}_{n_r}) = 0. \tag{4.50}$$

More precisely, if $a_{n_r}$ is the $n_r$th negative zero of $Ai$, then

$$C_{n_r} = x^{(0)}_{n_r} = 2^{-1/3}a_{n_r}, \tag{4.51}$$

where we have identified the leading contribution in the asymptotic formula for the zeros of the Bessel function of large order [38],

$$j_{p,\,n_r} = p + C_{n_r}p^{1/3} + O(p^{-2/3}); \tag{4.52}$$

for example, $C_1 = 1.8558$, $C_2 = 3.2447$, etc. It should be pointed out that, from the asymptotic form of the Airy functions, which amounts to a WKB integration of Eq. (4.20), an approximate—but extremely accurate—formula can be derived for these coefficients, namely [35],

$$C_{n_r} \approx \frac{(3\pi)^{2/3}}{2}\left(n_r - \frac{1}{4}\right)^{2/3}. \tag{4.53}$$

Then, for the next order,

$$x_{n_r} = C_{n_r} + \delta_{n_r}, \tag{4.54}$$

with $\delta_{n_r} \ll C_{n_r}$, one can expand the Airy function in a Taylor series in the neighborhood of $x^{(0)}_{n_r}$, so that Eq. (4.30) becomes

$$\mathscr{L}^{(<)}(\Theta_l) = -2^{1/3} p^{2/3} \frac{Ai'(-2^{1/3} x)}{Ai(-2^{1/3} x)} [1 + O(p^{-2/3})]$$

$$= \frac{p^{2/3}}{\delta_{n_r}} [1 + O(\delta_{n_r}, p^{-2/3})]. \tag{4.55}$$

The equality of the logarithmic derivatives, Eqs. (4.46) and (4.55), gives

$$\delta_{n_r} = p^{-1/3} (\lambda_l^{(*)})^{1/2} [-c + \ln \Theta_l][1 + O(\varepsilon^{1/3})], \tag{4.56}$$

whence

$$y_{n_r} = p\{1 + C_{n_r} p^{-2/3} + (\lambda_l^{(*)})^{1/2} [-c + \ln \Theta_l] \, p^{-1} + O(p^{-4/3})\}. \tag{4.57}$$

Therefore, from Eqs. (4.28) and (4.38), the correct replacement for Eq. (4.34) is

$$\tilde{\eta}_{n_r} = \left(\frac{y}{z_2}\right)^2 = p^2\{1 + 2C_{n_r} p^{-2/3} - 2(\lambda_l^{(*)})^{1/2} \, cp^{-1} + O(p^{-4/3})\}. \tag{4.58}$$

Finally, from Eq. (4.8), the expansion of $\mathscr{I}(\varepsilon)$ about $\varepsilon = 0^+$ is

$$\mathscr{I}(\varepsilon) = 1 + \frac{\varepsilon}{2} (\gamma + \ln \pi) + O(\varepsilon^2), \tag{4.59}$$

which combined with Eqs. (4.16) and (4.58), gives the asymptotically exact expression for the eigenvalue function of Eqs. (1.4) and (4.17), namely,

$$\Xi_{n_r l}(\varepsilon) = \frac{1}{\lambda_l^{(*)}} \{1 - 2p^{-2/3} C_{n_r} + p^{-1} (\lambda_l^{(*)})^{1/2}$$

$$\times [\ln 4\pi - \gamma + 2(\lambda_l^{(*)})^{-1/2}] + O(p^{-4/3})\} \tag{4.60}$$

$$= \frac{1}{\lambda_l^{(*)}} \left\{1 - 2^{1/3} C_{n_r} (\lambda_l^{(*)})^{-1/3} \, \varepsilon^{2/3} \right.$$

$$\left. + \frac{\varepsilon}{2} [\ln 4\pi - \gamma + 2(\lambda_l^{(*)})^{-1/2}] + O(\varepsilon^{4/3}) \right\}, \tag{4.61}$$

which is of the form [2]

$$\Xi_n(\varepsilon) = [L_n(\varepsilon)]^{-1} \left[ 1 + \frac{\varepsilon}{2} \mathcal{G}_n(\varepsilon) \right], \tag{4.62}$$

with a constant critical coupling function

$$L_{n_r l} = \lambda_l^{(*)} = \left( l + \frac{D_0}{2} - 1 \right)^2 \tag{4.63}$$

(independent of $n_r$), and with

$$\mathcal{G}_{n_r l}(\varepsilon) = -2^{4/3} C_{n_r} (\lambda_l^{(*)})^{-1/3} \, \varepsilon^{-1/3} + [\ln 4\pi - \gamma + 2(\lambda_l^{(*)})^{-1/2}] + O(\varepsilon^{1/3}). \tag{4.64}$$

From Eq. (4.64), we see that

$$\mathcal{G}_{(\mathrm{gs})}(\varepsilon) = -2^{4/3} C_1 (\lambda_{(\mathrm{gs})}^{(*)})^{-1/3} \, \varepsilon^{-1/3} + [\ln 4\pi - \gamma + 2(\lambda_{(\mathrm{gs})}^{(*)})^{-1/2}] + O(\varepsilon^{1/3}), \tag{4.65}$$

where

$$\lambda_{(\mathrm{gs})}^{(*)} = \lambda_0^{(*)} = \nu_0^2 = (D_0/2 - 1)^2 \tag{4.66}$$

is the "principal coupling constant," i.e., the critical coupling for the ground state ($l = 0$).

Equations (1.4) and (4.62)–(4.64) provide the regularized energies

$$\begin{aligned} |E_{n_r l}| &= \mu^2 \left( \frac{\lambda}{\lambda_l^{(*)}} \right)^{2/\varepsilon} \exp[\mathcal{G}_{n_r l}(\varepsilon)] \\ &= \mu^2 \left( \frac{\lambda}{\lambda_l^{(*)}} \right)^{2/\varepsilon} \exp\{ -2^{4/3} C_{n_r} (\lambda_l^{(*)})^{-1/3} \, \varepsilon^{-1/3} + [\ln 4\pi - \gamma + 2(\lambda_l^{(*)})^{-1/2}]\}. \end{aligned} \tag{4.67}$$

Ground-state renormalization can be implemented from Eq. (4.67), by means of the regularized coupling [2]

$$\lambda(\varepsilon) = [\Xi_{(\mathrm{gs})}(\varepsilon)]^{-1} \left[ 1 + \frac{\varepsilon}{2} \, g^{(0)} + o(\varepsilon) \right], \tag{4.68}$$

which now becomes

$$\begin{aligned} \lambda(\varepsilon) = \lambda_{(\mathrm{gs})}^{(*)} \Big\{ & 1 + 2^{1/3} (\lambda_{(\mathrm{gs})}^{(*)})^{-1/3} \, \varepsilon^{2/3} C_1 \\ & + \frac{\varepsilon}{2} [g^{(0)} - (\ln 4\pi - \gamma) - 2(\lambda_{(\mathrm{gs})}^{(*)})^{-1/2}] \Big\} + o(\varepsilon), \end{aligned} \tag{4.69}$$

with an arbitrary finite part $g^{(0)}$, so that, for the ground state,

$$E_{(\mathrm{gs})} = -\mu^2 \exp[\, g^{(0)}] \rightsquigarrow -\mu^2 \tag{4.70}$$

(cf. Eq. (2.24)), while for any regularized bound state

$$|E_{n_r l}| = \mu^2 \left( \frac{\lambda_{(\mathrm{gs})}}{\lambda_l^{(*)}} \right)^{2/\varepsilon} \exp\{ -2^{4/3} [\, C_{n_r} (\lambda_l^{(*)})^{-1/3} - C_1 (\lambda_{(\mathrm{gs})}^{(*)})^{-1/3}]\, \varepsilon^{-1/3}$$
$$+ g^{(0)} + 2[(\lambda_l^{(*)})^{-1/2} - (\lambda_{(\mathrm{gs})}^{(*)})^{-1/2}]\}. \tag{4.71}$$

Equation (4.69) explicitly shows that

$$\lambda(\varepsilon) \xrightarrow{(\varepsilon \to 0)} \lambda_{(\mathrm{gs})}^{(*)} + 0^+; \tag{4.72}$$

in other words, when the system is renormalized, the coupling becomes critically strong with respect to the ground state (which, as we will see below, has $l = 0$).

Equations (4.61)–(4.72) are easily interpreted. As expected, the critical coupling defined by Eq. (3.4) from the unregularized theory becomes the critical coupling for the regularized theory, Eq. (4.63), with respect to dimensional transmutation. We already know, from the criteria of the general theory of dimensional transmutation [2], that Eq. (4.61) implies the existence of a ground state alone. However, we will now illustrate the general arguments for this particular problem.

The value of the critical coupling depends on the angular momentum quantum number $l$ but is independent of the radial quantum number $n_r$. This, combined with the form of $\mathcal{G}_{n_r l}(\varepsilon)$ in Eq. (4.64), imposes very stringent conditions on the existence of bound states, as we shall see next.

The dependence of $\lambda_l^{(*)}$ with respect to $l$ requires that *only* $l = 0$ states (if any) be allowed, as implied by the following argument. First, only a finite number of states can exist with different angular momentum numbers $l$. In effect, let us assume the existence of a given state with angular momentum $l_0$; as the coupling is critically strong, then $\lambda = (l_0 + \nu_0)^2$, so that the potential is "weak" and has no bound states for all $l > l_0$—for weak coupling, the unregularized theory suffices. In other words, the only allowed states are those with $0 \leqslant l \leqslant l_0$. Let us now see that $l_0 \neq 0$ would lead to a contradiction; in effect, if $l_0 > 0$, then a state with $l < l_0$ would potentially exist as it would correspond to the strong regime; however, as $l \neq l_0$, the coupling would not be critical with respect to $l$, rendering the theory ill-defined. In other words, if the ground state exists, it should have $l = 0$ (as expected) and all states with $l > 0$ are forbidden.

Therefore, the ground state is characterized by the quantum numbers

$$(\mathrm{gs}) \equiv (n_r = 1,\, l = 0), \tag{4.73}$$

and only hypothetical states with $l = 0$ survive the renormalization process as bound states.

The next question is whether states with $l=0$ but $n_r \neq 0$ can survive renormalization. Of course, the ratio of Eqs. (4.70) and (4.71) provides the answer: starting with the ground-state energy, any other such state would have an energy overwhelmingly suppressed by an exponential factor,

$$\left|\frac{E_{n_r0}}{E_{(\mathrm{gs})}}\right| = \exp[-2^{4/3}(C_{n_r} - C_1)(\lambda^{(*)}_{(\mathrm{gs})})^{-1/3}\,\varepsilon^{-1/3}] \xrightarrow{(\varepsilon \to 0,\, n_r > 1)} 0. \tag{4.74}$$

Furthermore, for these states, the wave function (4.25) has an ill-defined limit, so that they cease to exist when $\varepsilon \to 0$. In short, the singular nature of the potential is responsible for the destruction of all candidates for a renormalized bound state, except for the limit of the ground state of the regularized theory—which, upon renormalization, becomes the ground state of the system and acquires the finite energy value (4.70).

Finally, the ground-state wave function can also be explicitly derived. From Eqs. (4.25), (4.70), and (A14), it follows that the ground state wave function is given by

$$\Psi_{(\mathrm{gs})}(\mathbf{r}) = \sqrt{\Gamma(\nu+1)\left(\frac{\mu^2}{\pi}\right)^{\nu+1}}\,\frac{K_0(\mu r)}{(\mu r)^{\nu}}, \tag{4.75}$$

which is similar to that of the two-dimensional delta-function potential, with which it coincides when $D_0 = 2$.

## C. *Scattering Sector for the Inverse Square Potential*

For the scattering sector of the theory we will match the interior and exterior solutions by means of the parameter

$$\omega(\Theta_l) = \frac{\varepsilon}{2}\,\mathscr{L}(\Theta_l) = \Theta_l \frac{d \ln \mathscr{C}_{\varepsilon/4}(\Theta_l)}{d\Theta_l}. \tag{4.76}$$

In particular, the exterior scattering problem is described by the second line in Eqs. (4.24) and (4.25), so that Eq. (4.31) can now be rewritten as

$$\omega^{(>)}(\Theta_l) = \Theta_l \frac{d \ln \mathscr{C}_{\varepsilon/4}(\Theta_l)}{d\Theta_l}, \tag{4.77}$$

which is explicitly given by

$$\omega^{(>)}(\Theta_l) = \frac{\tilde{A}^{(+)}_l \Theta_l H^{(1)\prime}_{\varepsilon/4}(\Theta_l) + \tilde{A}^{(-)}_l \Theta_l H^{(2)\prime}_{\varepsilon/4}(\Theta_l)}{\tilde{A}^{(+)}_l H^{(1)}_{\varepsilon/4}(\Theta_l) + \tilde{A}^{(-)}_l H^{(2)}_{\varepsilon/4}(\Theta_l)}. \tag{4.78}$$

Equation (4.78) can be evaluated from the small-argument behavior of the Hankel functions [24],

$$H_p^{(1,2)}(z) \overset{(z\to 0)}{\sim} \pm\frac{e^{\mp ip\pi/2}}{\pi i}\left[e^{\mp ip\pi/2}\Gamma(-p)\left(\frac{z}{2}\right)^{p} + e^{\pm ip\pi/2}\Gamma(p)\left(\frac{z}{2}\right)^{-p}\right][1+O(z^2)], \tag{4.79}$$

so that

$$\omega_l = \omega^{(>)}(\Theta_l) = \frac{2i}{\pi}\frac{\tilde{A}_l^{(+)} - \tilde{A}_l^{(-)}}{(1+iQ_l)\,\tilde{A}_l^{(+)} + (1-iQ_l)\,\tilde{A}_l^{(-)}}, \tag{4.80}$$

where

$$Q_l = \frac{2}{\pi}(-c + \ln \Theta_l), \tag{4.81}$$

with $c$ defined in Eq. (4.47). Then the scattering matrix can be derived from the asymptotic expansion of the Hankel functions as $r\to\infty$, which according to Eq. (4.25) is of the form

$$u(r) \overset{(r\to\infty,\,\varepsilon\to 0)}{\sim} \sqrt{r}\left[A_l^{(+)}H^{(1)}_{l+\nu_0}\left(kr + (l+\nu_0)\frac{\pi}{2}\right)\right.$$

$$\left. + A_l^{(-)}H^{(2)}_{l+\nu_0}(kr)\left(kr + (l+\nu_0)\frac{\pi}{2}\right)\right], \tag{4.82}$$

where the coefficients for the $s$ wave ($l=0$) are related via

$$A_0^{(\pm)} = \tilde{A}_0^{(\pm)} e^{\pm i(D_0-2)\pi/4}; \tag{4.83}$$

then the $s$-wave scattering matrix elements—from Eqs. (4.80), (4.83), and (B12)—are given by

$$S_0^{(D_0)}(k) = e^{i\nu_0\pi}\tilde{S}_0^{(D_0)}(k), \tag{4.84}$$

with $\tilde{S}_0^{(D_0)}(k) = \tilde{A}_0^{(+)}/\tilde{A}_0^{(-)}$ of the form

$$\tilde{S}_0^{(D_0)}(k) = -\frac{1 + i(2\omega^{-1}_{(\mathrm{gs})}/\pi - Q_{(\mathrm{gs})})}{1 - i(2\omega^{-1}_{(\mathrm{gs})}/\pi - Q_{(\mathrm{gs})})}, \tag{4.85}$$

and where $\omega_{(\mathrm{gs})} = \omega_0$ and $Q_{(\mathrm{gs})} = Q_0$.

In order to obtain an explicit expression for the scattering matrix in terms of the energy, we need to evaluate the coefficient $\omega_{(\mathrm{gs})}$ in Eq. (4.85). This program can be implemented through the expansion of the dual energy $\tilde{\eta}$, defined in Eq. (4.13), which should now be derived again for the scattering sector of the theory. With that purpose in mind, making use of Eqs. (4.8), (4.16), and

$$\left|\frac{E}{\mu^2}\right|^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2}\ln\left|\frac{E}{\mu^2}\right| + O(\varepsilon^2), \tag{4.86}$$

the required expression becomes (for $l=0$)

$$\tilde{\eta} = p^2 \frac{\lambda}{\lambda^{(*)}_{(\mathrm{gs})}} \left\{ 1 + (\lambda^{(*)}_{(\mathrm{gs})})^{1/2} \left[ \ln \pi + \gamma + 2(\lambda^{(*)}_{(\mathrm{gs})})^{-1/2} - \ln\left|\frac{E}{\mu^2}\right|\right] p^{-1} + O(p^{-2}) \right\}. \tag{4.87}$$

In addition, we have already renormalized the bound-state sector of the theory, with the result that the coupling constant should behave as dictated by Eq. (4.69), whence

$$\tilde{\eta}^{1/2} = p\left\{1 + C_1 p^{-2/3} + \frac{\varepsilon}{4}\left[-2c + g^{(0)} - \ln\left|\frac{E}{\mu^2}\right|\right] + o(\varepsilon)\right\}, \tag{4.88}$$

from which it follows that the variable $y = \tilde{\eta}^{1/2} z_2$, defined by Eq. (4.38), is of the form (4.44), with

$$x = C_1 + \left[(-c + \ln \Theta_{(\mathrm{gs})}) - \frac{1}{2}\ln\left|\frac{E}{E_{(\mathrm{gs})}}\right|\right] (\lambda^{(*)}_{(\mathrm{gs})})^{1/2} p^{-1/3}, \tag{4.89}$$

and $c$ defined in Eq. (4.47). Then the logarithmic derivative (4.30) can be derived for the scattering sector using an argument similar to the one employed in Eq. (4.55), i.e., with $\delta = x - C_1$,

$$\begin{aligned} \mathscr{L}^{(<)}(\Theta_l) &= -2^{1/3} p^{2/3} \frac{Ai'(-2^{1/3}x)}{Ai(-2^{1/3}x)} [1 + O(p^{-2/3})] \\ &= \frac{p^{2/3}}{x - C_1} [1 + O(p^{-2/3})]. \end{aligned} \tag{4.90}$$

As a consequence, we conclude that the parameter defined in Eq. (4.76) becomes

$$\omega_{(\mathrm{gs})} = \omega^{(<)}(\Theta_{(\mathrm{gs})}) = \left[(-c + \ln \Theta_{(\mathrm{gs})}) - \frac{1}{2}\ln\left|\frac{E}{E_{(\mathrm{gs})}}\right|\right]^{-1}. \tag{4.91}$$

TABLE I

Phase Shift $\delta_0^{(D_0)}(k)$, Scattering Matrix $S_0^{(D_0)}(k)$, and Partial Scattering Amplitude $a_0^{(D_0)}(k)$ for the Inverse Square Potential, as a Function of the Geometric Dimension $D_0$ of Position Space

| $D_0$ | $\tan\delta_0^{(D_0)}(k)$ | $S_0^{(D_0)}(k)$ | $a_0^{(D_0)}(k)\,k^{(D_0-1)/2}$ |
|---|---|---|---|
| 1 (mod 4) | $\frac{\pi-L}{\pi+L}$ | $\frac{(1+i)\,\pi+(1-i)\,L}{(1-i)\,\pi+(1+i)\,L}$ | $\frac{\pi-L}{(1-i)\,\pi+(1+i)\,L}$ |
| 2 (mod 4) | $\frac{\pi}{L}$ | $\frac{L+i\pi}{L-i\pi}$ | $\frac{\pi}{L-i\pi}$ |
| 3 (mod 4) | $\frac{L+\pi}{L-\pi}$ | $\frac{(-1+i)\,\pi+(1+i)\,L}{-(1+i)\,\pi+(1-i)\,L}$ | $\frac{L+\pi}{-(1+i)\,\pi+(1-i)\,L}$ |
| 4 (mod 4) | $-\frac{L}{\pi}$ | $\frac{\pi-iL}{\pi+iL}$ | $-\frac{L}{\pi+iL}$ |

*Note.* The shorthand $L=\ln(k^2/|E_{(gs)}|)$ is used in this table.

Finally, from Eqs. (4.81), (4.84), (4.85), and (4.91), the S-matrix reads

$$S_0^{(D_0)}(k)=e^{i\nu_0\pi}\,\frac{\ln(k^2/|E_{(gs)}|)+i\pi}{\ln(k^2/|E_{(gs)}|)-i\pi}, \tag{4.92}$$

while the phase shifts are implicitly given by the expression

$$\tan\left(\delta_0^{(D_0)}(k)-\frac{\pi\nu_0}{2}\right)=\frac{\pi}{\ln(k^2/|E_{(gs)}|)}, \tag{4.93}$$

and the partial scattering amplitude reads

$$a_0^{(D_0)}(k)=\frac{1}{2ik^{(D_0-1)/2}}\,\frac{\ln(k^2/|E_{(gs)}|)(e^{i\nu_0\pi}-1)+i\pi(e^{i\nu_0\pi}+1)}{\ln(k^2/|E_{(gs)}|)-i\pi}. \tag{4.94}$$

Equations (4.92), (4.93), and (4.94) are remarkably similar to Eqs. (2.37), (2.38), and (2.39) for the two-dimensional delta function potential. Table I summarizes the main results of the scattering for an arbitrary number of dimensions $D_0$.

The above analysis refers to $l=0$. For all other values $l>0$, the coupling will be weak, so that the phase shifts will be given by the values of Eq. (3.12), with the condition that $\lambda=\lambda_{(gs)}^{(*)}$; then,

$$\delta_l^{(D_0)}|_{l\neq 0}=\left[(l+\nu_0)-\sqrt{l(l+2\nu_0)}\right]\frac{\pi}{2}, \tag{4.95}$$

and

$$S_l^{(D_0)}(k)|_{l\neq 0} = (-1)^l \exp(i\nu_0\pi) \exp[-i\pi \sqrt{l(l+2\nu_0)}], \tag{4.96}$$

which are scale-invariant expressions.

A number of consequences follow from Eqs. (4.92) and (4.93):

1. The unique pole of the scattering matrix (4.92) corresponds to the unique bound state.

2. The phase shifts can be renormalized using a floating renormalization scale $\mu$, as an alternative to ground-state renormalization. Then the $s$-wave phase shift reads simply

$$\frac{1}{\tan(\delta_0^{(D_0)}(k) - \pi\nu_0/2)} = \frac{1}{\tan(\delta_0^{(D_0)}(\mu) - \pi\nu_0/2)} + \frac{1}{\pi} \ln\left(\frac{k}{\mu}\right)^2. \tag{4.97}$$

3. Equations (4.94), (4.95), (B8), and (B9) give the differential scattering cross section

$$f_k^{(D_0)}(\cos\theta) = \frac{N_{\nu_0}}{2ik^{(D_0-1)/2}} \times \frac{\ln(k^2/|E_{(\mathrm{gs})}|)(e^{i\nu_0\pi} - 1) + i\pi(e^{i\nu_0\pi} + 1)}{\ln(k^2/|E_{(\mathrm{gs})}|) - i\pi} + \overset{\triangledown}{f}_k^{(D_0)}(\cos\theta), \tag{4.98}$$

where

$$\overset{\triangledown}{f}_k^{(D_0)}(\cos\theta) = \frac{N_{\nu_0}}{2ik^{(D_0-1)/2}} \sum_{l=1}^{\infty} \left(\frac{l}{\nu_0} + 1\right) \times \{(-1)^l e^{i\nu_0\pi} e^{i\pi\sqrt{l(l+2\nu_0)}} - 1\}\, C_l^{(\nu_0)}(\cos\theta). \tag{4.99}$$

4. All the relevant quantities are logarithmic with respect to the energy and agree with the predictions of generalized dimensional analysis [2].

5. Equations (4.92), (4.93), and (4.94) relate the bound-state and scattering sectors of the theory and show that the inverse square potential is renormalizable.

## V. CONCLUSIONS

In this paper, we have uncovered a number of remarkable analogies between the two-dimensional delta-function and inverse square potentials. In addition to displaying their characteristic transmuting behavior, with all the ensuing implications, we have explicitly seen that they share the following properties:

(i) Unusual boundary conditions at the origin.

(ii) Characteristic critical couplings (coincident for $D_0 = 2$) that determine the possible regimes of each potential.

(iii) Only one bound state in the renormalized theory, even though this is achieved through different mechanisms (the delta-function potential always generates a unique state, while the inverse square potential annihilates all the regularized excited states by exponential suppression).

(iv) Almost identical ground-state wave functions (up to the normalization constant).

(v) Similar $s$-wave scattering matrix elements—they are proportional, differing only upon an extra $D_0$-dependent phase factor for the inverse square potential.

(vi) Characteristic logarithmic behavior of $s$-wave scattering quantities.

It should be noticed that the bound-state sectors look essentially identical in both theories, while the $s$-wave scattering sectors are identical for $D = 2$ and almost identical for $D \neq 2$.

However, there a number of differences as well. Due to its zero-range nature, the two-dimensional delta function only scatters $s$ waves, while the inverse square potential, due to its infinite range, scatters all other angular-momentum channels in a scale-invariant (energy-independent) way.

Moreover, these results are independent of the regularization technique; in particular, they are in perfect agreement with the $D_0$-dimensional generalization [35] of the cutoff-renormalization method of Ref. [16].

Finally, the techniques used in this paper could be easily generalized. For example, one could consider the generalized inverse square potential

$$V(\mathbf{r}) = -\lambda \frac{v(\Omega^{(D)})}{r^2}, \tag{5.1}$$

with a dimensionless function $v(\Omega^{(D)})$ that depends on the $D$-dimensional solid angle $\Omega^{(D)}$; the angular part of the solution could be properly modified, but the basic scaling relationships would remain the same. In principle, this strategy could be conveniently used for the dipole potential [31, 32] and for other forms of angular dependence in Eq. (5.1).

## APPENDIX A

### *Rotational Invariance: Central Potentials in D Dimensions*

In this appendix we will enforce the condition of rotational invariance and use the notation and definitions of hyperspherical coordinates as introduced in Ref. [2].

As usual, a central potential $V(r)$ is defined to be rotationally invariant; thus, its functional form is independent of the angular variables in $D$-dimensional hyperspherical coordinates. Its associated symmetry group is $SO(D)$, with a quantum mechanical representation generated by the $D(D-1)/2$ generalized angular momentum operators $x_i p_k - x_k p_i$ (with $i<k$). The quadratic Casimir operators

$$L_j^2 = \sum_{k>i=j}^{D-1} (x_i p_k - x_k p_i)^2$$

$$= -\hbar^2 \sum_{k=j}^{D-1} \left( \prod_{i=j}^{k-1} \sin^2 \theta_i \right)^{-1} \left[ \frac{\partial^2}{\partial \theta_k^2} + (D-k-1) \cot \theta_k \frac{\partial}{\partial \theta_k} \right] \tag{A1}$$

(with $j=1, ..., D-1$) commute with all the generators of the Lie algebra and with all possible rotations. Each $L_j^2$ represents the "total" $D$-dimensional angular momentum squared in the subspace spanned by the Cartesian coordinates $(x_j, ..., x_D)$, and is characterized by the properties that it commutes with the Hamiltonian of any central potential and that it satisfies the eigenvalue equation

$$L_j^2 \, |E, L\rangle = l_j(l_j + D - j - 1) \, \hbar^2 \, |E, L\rangle, \tag{A2}$$

where the eigenstates $|E, L\rangle$ are labeled by the energy and by the collective index of generalized angular momentum quantum numbers

$$L = (l \equiv l_1, l_2, ..., l_{D-2}, m \equiv l_{D-1}). \tag{A3}$$

The eigenstates $|E, L\rangle$ lead to the generalization of the usual 3D factorization of the position wave function, which can be written as [39]

$$\Psi_{EL}(r, \Omega^{(D)}) = \langle r, \Omega^{(D)} \,|\, E, L\rangle = R_{El}(r) \; Y_L(\Omega^{(D)}), \tag{A4}$$

in terms of the hyperspherical harmonics $Y_L(\Omega^{(D)}) = \langle \Omega^{(D)} \,|\, L\rangle$. Notice that the peculiar number $m$ is associated with rotations on the $(x_{D-1}, x_D)$ plane, which are characterized by the azimuthal angle $\phi \equiv \theta_{D-1}$; the corresponding operator is usually chosen to be

$$L_{D-1} = x_{D-1} p_D - x_D p_{D-1} = \frac{\hbar}{i} \frac{\partial}{\partial \phi}, \tag{A5}$$

instead of $L_{D-1}^2$, with eigenvalues $m\hbar$ (both positive and negative). In addition, the generalized angular momentum quantum numbers satisfy the constraints

$$0 \leqslant |m| \leqslant l_{D-2} \leqslant l_{D-3} \leqslant \cdots \leqslant l_3 \leqslant l_2 \leqslant l. \tag{A6}$$

Moreover, the angular part of the Laplacian [36],

$$\Delta_{\Omega^{(D)}} = \sum_{j=1}^{D-1} \left[ \left( \prod_{k=1}^{j-1} \sin^2\theta_k \right) \sin^{D-j-1}\theta_j \right]^{-1} \frac{\partial}{\partial\theta_j} \left( \sin^{D-j-1}\theta_j \frac{\partial}{\partial\theta_j} \right), \tag{A7}$$

from Eq. (A1), can be simply written in terms of the total angular momentum $L^2 = L_1^2$, namely,

$$\Delta_{\Omega_D} = -\frac{L^2}{\hbar^2}, \tag{A8}$$

so that Eq. (A2) leads to

$$\Delta_{\Omega^{(D)}} Y_L(\Omega^{(D)}) = -l(l+D-2)\, Y_L(\Omega^{(D)}). \tag{A9}$$

In particular, the solutions of the angular part of Laplace's equation that are independent of all angles but $\theta_1$ are the hyperspherical harmonics with $l_2 = \cdots = l_{D-2} = m = 0$, for which Eq. (A9) reduces to the ultraspherical differential equation, with regular solutions given by the Gegenbauer (or ultraspherical) polynomials $C_l^{(\nu)}(\cos\theta_1)$, defined in terms of its generating function (see Refs. [24, 40])

$$(1-2tz+z^2)^{-\nu} = \sum_{l=0}^{\infty} C_l^{(\nu)}(t)\, z^l. \tag{A10}$$

This is all the background needed for our work; the reader may consult Ref. [39] for general proofs and Refs. [41, 42] for explicit expressions of the hyperspherical harmonics.

Equation (A9) justifies a posteriori the use of the notation $R_{El}(r)$ introduced in Eq. (A4), which assumes that $R(r)$ depends only on the angular momentum quantum number $l$ (but not on $l_2, ..., l_{D-2}, m$), in addition to the energy $E$. In effect, after isolating the angular factor that is common to all central potentials, the wave function $R_{El}(r)$ satisfies the equation

$$\left[ \Delta_r^{(D)} - \frac{l(l+D-2)}{r} + V(r) \right] R_{El}(r) = E R_{El}(r), \tag{A11}$$

with a radial Laplacian

$$\Delta_r^{(D)} = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) \tag{A12}$$

$$= \frac{1}{r^{(D-1)/2}} \frac{d^2}{dr^2} [r^{(D-1)/2}] + \frac{(D-1)(D-3)}{4r^2}. \tag{A13}$$

As a result, the function

$$u_{El}(r) = R_{El}(r)\, r^{(D-1)/2} \tag{A14}$$

satisfies a one-dimensional Schrödinger equation

$$\left[ -\frac{d^2}{dr^2} + V(r) + \frac{\Lambda_{l,D}}{r^2} \right] u_{El}(r) = E u_{El}(r), \tag{A15}$$

in which the centrifugal barrier is characterized by the effective coupling constant

$$\Lambda_{l,D} = l(l+D-2) + (D-1)(D-3)/4 \tag{A16}$$

$$= (l+\nu)^2 - 1/4, \tag{A17}$$

where $\nu$ is the variable defined in Eq. (2.3). In our work, Eq. (A17) is most useful, particularly because, for the potentials of interest in this work, it provides a straightforward connection with a family of Bessel differential equations.

Equation (A17) depends on the number of dimensions of the space only through the combination $l+\nu$; this amounts to the remarkable phenomenon known as interdimensional dependence [43]: the solutions for any two problems related via $l+\nu = l' + \nu'$, with $\nu \neq \nu'$ (i.e., $D \neq D'$) are identical.

A remark about notation is in order. In most contexts, it is customary to drop the subscript $E$ labeling the wave functions in Eqs. (A4), (A11), (A14), and (A15), i.e., to write $R_{El}(r) \equiv R_l(r)$ and $u_{El}(r) \equiv u_l(r)$. Alternatively, for the bound-state sector, one often writes $R_{n_r l}(r)$, $u_{n_r l}(r)$, and $E_{n_r l}$, where $n_r$ is the radial quantum number.

Finally, in order to have a well-defined problem, boundary conditions are needed both at infinity and at the origin. At infinity, the bound-state solutions should go to zero in order to ensure their integrability, while the scattering solutions are subject to the usual requirements [44]. On the other hand, the boundary condition at $r=0$ requires further analysis.

In fact, the boundary condition at the origin is the basis for the classification of potentials into the regular and singular families, as discussed in Appendix C. In this framework, regular and semi-regular potentials are characterized by the limit

$$r^2 V(r) \xrightarrow{r \to 0} 0, \tag{A18}$$

so that, near the origin, both the potential and total energy terms are negligible and the limiting form of the wave function becomes asymptotically a solution of the radial part of Laplace's equation, i.e.,

$$R_l(r) \overset{(r\to 0)}{\sim} \frac{u_l(r)}{r^{(D-1)/2}} = \{ r^l, r^{-(l+2\nu)} \}, \tag{A19}$$

where { , } stands for linear combination. In Eq. (A19), the first component is acceptable but the second one should be discarded because $\nabla_D^2[Y_L(\Omega_D)/r^{l+D-2}]$ is proportional to the multipole density of order, $l$, i.e., it involves derivatives of the delta function $\delta^{(D)}(\mathbf{r})$ of order $l$. Therefore, for regular potentials, there is a criterion for the selection between the two linearly independent solutions of Eq. (A15), and this provides the boundary condition

$$R_l(r) \propto r^l. \tag{A20}$$

In practice, it is sufficient to consider the weaker boundary condition

$$u_l(0) = 0. \tag{A21}$$

As discussed in Subsection III.B, the source of the unusual properties of critical (dimensionally transmuting) and singular potentials is the "loss" of the boundary condition (A21).

## APPENDIX B

### *Rotational Invariance: Partial-Wave Analysis in D Dimensions*

For our work, we need another aspect of rotational invariance in $D$ dimensions: the expansion in $D$-dimensional partial waves, which is developed next. A spherical wave state $|E, L\rangle$ is represented by a solution of the radial Schrödinger equation for a free particle; then Eq. (A11), when $V(r) = 0$, has solutions proportional to the Bessel functions $Z_{l+\nu}(kr)$, with $\nu = D/2 - 1$, where $Z = J$, or $N$, or $H^{(1,2)}$; explicitly,

$$R_l(r) \propto z_{l,\nu} = \sqrt{\frac{\pi}{2}}\, x^{-\nu} Z_{l+\nu}(x), \tag{B1}$$

where the ultraspherical Bessel functions admit the asymptotic expansions

$$z_{l,\nu}(x) \overset{(x\to\infty)}{\sim} x^{-(D-1)/2}\,\tau\left(x + \gamma_D - l\frac{\pi}{2} - \frac{\pi}{2}\right), \tag{B2}$$

in which $\tau(\xi)$ is the corresponding trigonometric function (i.e., $\cos\xi$ for $j(\xi)$, $\sin\xi$ for $n(\xi)$, and $\exp(\pm i\xi)$ for $h^{(1,2)}(\xi)$), and $\gamma_D = (3-D)\,\pi/4$ (from Ref. [2]). In particular, the regular free-particle solution of Eq. (A11) (based on its behavior at the origin) is provided by $j_{l,\nu}(kr)$. On the other hand, the angular part leads to the Gegenbauer (or ultraspherical) polynomials $C_l^{(\nu)}(\cos\theta)$ defined through Eq. (A10). In short, the regular solution is of the form $j_{l,D/2-1}(kr)\, C_l^{(D/2-1)}(\cos\theta)$.

The partial-wave expansion for central potentials in $D$ dimensions then proceeds in complete analogy to the three-dimensional case [45]. This can be accomplished by considering the transition from a plane-wave state $|\mathbf{k}\rangle$, represented by the wave function $e^{ikr\cos\theta}$ (we will choose $\theta \equiv \theta_1$ for the sake of simplicity), to a spherical-wave state $|E, L\rangle$. The coefficients of the transition can be found from the identity (Ref. [24, p. 363])

$$e^{ix\cos\theta} = \Gamma(\nu)\left(\frac{x}{2}\right)^{-\nu} \sum_{l=0}^{\infty} (l+\nu)\, i^l J_{l+\nu}(x)\, C_l^{(\nu)}(\cos\theta), \tag{B3}$$

with $x = kr$; this implies that

$$e^{ikr\cos\theta} = N_\nu \sum_{l=0}^{\infty} \left(\frac{l}{\nu}+1\right) i^l j_{l,\nu}(kr)\, C_l^{(\nu)}(\cos\theta), \tag{B4}$$

which is the $D$-dimensional generalization of Rayleigh's formula, with

$$N_\nu = 2^\nu \Gamma(\nu+1)\sqrt{\frac{2}{\pi}}. \tag{B5}$$

Equation (B4) straightforwardly reduces to the familiar result for $D=3(\nu=1/2)$. The case $D=2(\nu=0)$ appears to be singular, but also reproduces the known results [46] when the following replacements are made: (i) $C_l^{(0)}(t) = \lim_{\nu\to 0} C_l^{(\nu)}(t)/\nu$ (Ref. [24]); (ii) $C_l^{(0)}(\cos\theta) = 2\cos(l\theta)/l$ for $l \neq 0$ (this is proportional to a Chebyshev polynomial, Ref. [24]) but $C_0^{(0)}(\cos\theta) = 1$; and (iii) most often, the sum is extended from $m = -\infty$ to $m = \infty$, with $m = \pm l$, and the factor of 2 in (ii) is removed (otherwise this factor is kept as the Neumann number, $\varepsilon_l = 2$ for $l \neq 0$, but $\varepsilon_0 = 1$).

As the transition operator $T$ commutes with the generalized angular momentum operators, it has a diagonal form in the angular momentum eigenbasis, with elements $T_l^{(D)}(k)$; thus, one can expand its matrix elements $\langle \mathbf{k}|\, T\, |\mathbf{k}'\rangle$ in terms of $T_l^{(D)}(k)$, with

$$T_l^{(D)}(k) = -\frac{S_l^{(D)}(k) - 1}{2\pi i}, \tag{B6}$$

where the scattering matrix elements

$$S_l^{(D)}(k) = \exp[2i\delta_l^{(D)}(k)] = \frac{1 + i\tan\delta_l^{(D)}(k)}{1 - i\tan\delta_l^{(D)}(k)} \tag{B7}$$

are usually expressed in terms of the scattering phase shifts $\delta_l^{(D)}(k)$. Then the corresponding scattering amplitude $f_k^{(D)}(\cos\theta)$ admits a straightforward expansion

$$f_k^{(D)}(\cos\theta) = N_\nu \sum_{l=0}^{\infty} \left( \frac{l}{\nu} + 1 \right) a_l^{(D)}(k)\, C_l^{(\nu)}(\cos\theta), \tag{B8}$$

with an $l$th partial-wave amplitude

$$a_l^{(D)}(k) = -\frac{\pi}{k^{(D-1)/2}}\, T_l^{(D)}(k) = \frac{\exp[2i\delta_l^{(D)}(k)] - 1}{2ik^{(D-1)/2}} \tag{B9}$$

that provides the asymptotic form of the wave function

$$\Psi(\mathbf{r}) \overset{(r\to\infty)}{\sim} \frac{N_\nu}{(kr)^{(D-1)/2}} \sum_{l=0}^{\infty} \left( \frac{l}{\nu} + 1 \right) \exp\{ i[\delta_l^{(D)}(k) + l\pi/2]\}$$

$$\times \sin[kr + \gamma_D - l\pi/2 + \delta_l^{(D)}(k)]\, C_l^{(\nu)}(\cos\theta). \tag{B10}$$

As a practical matter, the scattering matrix can be computed directly from the asymptotic expansion of the exact solution to the problem, i.e.,

$$R_l(r) \equiv R_{El}(r) \overset{(r\to\infty)}{\sim} A_l^{(+)} h_{l,\nu}^{(1)}(kr) + A_l^{(-)} h_{l,\nu}^{(2)}(kr), \tag{B11}$$

which yields

$$S_l^{(D)}(k) = \frac{A_l^{(+)}}{A_l^{(-)}}. \tag{B12}$$

Finally, the total scattering cross section can be obtained directly by integration of $|f_k^{(D)}(\cos\theta)|^2$, with the result

$$\sigma_D(k) = \frac{2\Omega_{D-1}}{k^{D-1}} \sum_{l=0}^{\infty} (2l + 2\nu)\, \frac{\Gamma(l+2\nu)}{l!}\, \sin^2 \delta_l^{(D)}(k). \tag{B13}$$

## APPENDIX C

### *Duality Transformation for Power-Law Potentials*

In this appendix, which we have adapted from Ref. [37], we will consider the class of central power-law potentials

$$V(r) = \operatorname{sgn}(\beta)\, \lambda r^{\beta}, \tag{C1}$$

where the sign is chosen so that $\lambda > 0$ corresponds to attractive potentials. According to their behavior at the origin, they can be classified into the following categories.

1. Regular potentials: $\beta \geqslant 0$.
2. Semi-regular potentials: $-2 < \beta < 0$.
3. Critically singular potential: $\beta = -2$.
4. Strictly singular potentials: $\beta < -2$.

What physically characterizes the singular potentials is that the centrifugal barrier fails to be the dominant term at the origin; mathematically, the Hamiltonian loses its self-adjoint character [34]. In particular, the critical potential is the one that generates dimensional transmutation.

The central theme of this appendix is the existence of a remarkable duality transformation $\mathscr{D}$, which relates the potentials in the regular and semi-regular families; in particular, $\mathscr{D}$ establishes a one-to-one correspondence between the exponents $\beta$ in the intervals $(-2, 0]$ and $[0, \infty)$. As we will see below, if

$$\mathscr{D}(\operatorname{sgn}(\beta)\, \lambda r^{\beta}) = \operatorname{sgn}(\tilde{\beta})\, \tilde{\lambda} r^{\tilde{\beta}}, \tag{C2}$$

then the corresponding relation between exponents reads

$$(\beta + 2)(\tilde{\beta} + 2) = 4. \tag{C3}$$

Equations (C2) and (C3) exhibit the following properties: (i) $\mathscr{D}$ is idempotent; (ii) $-2 < \beta < 0$ if and only if $0 < \tilde{\beta} < \infty$, so that $\operatorname{sgn}(\beta) = -\operatorname{sgn}(\tilde{\beta})$; (iii) the Coulomb potential ($\beta = -1$) is the dual of the harmonic oscillator ($\beta = 2$); (iv) its limiting exponents ($\beta = -2$ and $\beta = \infty$) define the inverse square potential as the dual of the infinite hyperspherical potential; and (v) the constant potential or free-particle case is self-dual.

Let us now show that Eq. (C3) represents the only nontrivial duality transformation implemented by a scale transformation for arbitrary power-law potentials. In general, under the transformation

$$\begin{cases} r = f(\varrho) \\ u(r) = \tilde{u}(\varrho)\; g(\varrho), \end{cases} \tag{C4}$$

from a radial wave function $u(r)$ to $\tilde{u}(\varrho)$, the Schrödinger equation

$$\left[\frac{d^2}{dr^2} + E - V(r) - \frac{(l + \nu^2 - 1/4)}{r^2}\right] u_l(r) = 0 \tag{C5}$$

is mapped into a transformed equation of the same form, without first-order derivative term, if and only if

$$[g(\varrho)]^2 \propto f'(\varrho), \tag{C6}$$

where the prime, as usual, denotes the derivative; then

$$\left\{\frac{d^2}{d\varrho} + [f'(\varrho)]^2 [E - V(f(\varrho))] - \mathcal{L}_1^2(\varrho)(l+\nu)^2 + \mathcal{L}_2'(\varrho) - \mathcal{L}_2^2(\varrho)\right\} \tilde{u}(\varrho) = 0, \tag{C7}$$

where

$$\mathcal{L}_j(\varrho) = \frac{1}{j!}\left(\frac{d \ln}{d\varrho}\right)^j f(\varrho) \tag{C8}$$

(for $j = 1, 2$). Even though Eqs. (C4) and (C7) are fairly general and have many applications, the scale transformation

$$\begin{cases} r = \varrho^{\alpha} \\ u(r) = \tilde{u}(\varrho)\, \varrho^{(\alpha-1)/2}, \end{cases} \tag{C9}$$

leading to

$$\left\{\frac{d^2}{d\varrho^2} + \alpha^2 \varrho^{2(\alpha-1)}[E - V(\varrho^{\alpha})] - \frac{\alpha^2(l+\nu)^2 - 1/4}{\varrho^2}\right\} \tilde{u}(\varrho) = 0, \tag{C10}$$

can immediately provide the desired connection for the power-law potentials of Eq. (C1). In effect, unless the trivial transformation $\alpha = 1$ is allowed, the only way of bringing the middle term in Eq. (C10) to the required transformed form, $\tilde{E} - \tilde{V}(\varrho)$, is to perform the exchange

$$-V(r) = -\operatorname{sgn}(\beta)\, \lambda r^{\beta} \to \tilde{E} \tag{C11}$$

$$E \to -\tilde{V}(\varrho) = -\operatorname{sgn}(\tilde{\beta})\, \tilde{\lambda} \varrho^{\tilde{\beta}}; \tag{C12}$$

this “crossing” implies that the exponents be constrained by

$$\alpha = -\frac{\tilde{\beta}}{\beta} = \frac{2}{\beta+2}, \tag{C13}$$

which is equivalent to the anticipated duality relation (C3). In conclusion, the required coordinate and wave function substitutions are

$$\begin{cases} r = \varrho^{2/(\beta+2)} \\ u(r) = \tilde{u}(\varrho)\, \varrho^{-\beta/2(\beta+2)}, \end{cases} \tag{C14}$$

which transform Eq. (C5) into

$$\left[ \frac{d^2}{d\varrho^2} + \tilde{E} - \text{sgn}(\tilde{\beta})\, \tilde{\lambda} \varrho^{\tilde{\beta}} - \frac{(\tilde{l}+\tilde{\nu})^2 - 1/4}{\varrho^2} \right] \tilde{u}_l(\varrho) = 0, \tag{C15}$$

with a dual energy eigenvalue

$$\tilde{E} = -\text{sgn}(\beta)\, \lambda \alpha^2, \tag{C16}$$

dual coupling

$$\tilde{\lambda} = \text{sgn}(\beta)\, E \alpha^2, \tag{C17}$$

and dual angular momentum quantum number $\tilde{l}$, such that

$$\tilde{l} + \tilde{\nu} = \alpha(l + \nu), \tag{C18}$$

where $\alpha$ is explicitly given by Eq. (C13), and we also allow for the possibility of a change in the number of dimensions, according to $\tilde{\nu} = \tilde{D}/2 - 1$. In particular, Eq. (C15) shows that the dual potential is, simply,

$$\tilde{V}(\varrho) = -E\alpha^2 \varrho^{\tilde{\beta}}. \tag{C19}$$

Finally, the dimensionless form of Eq. (C15) can be obtained by introducing an inverse length scale from the original problem through the energy, i.e.,

$$\kappa = \sqrt{|E|}; \tag{C20}$$

then the resulting transformation involves the dimensionless variables

$$\begin{cases} z = \kappa^{1/\alpha} \varrho \\ w(z) = \kappa^{-(D+1-1/\alpha)/2} \tilde{u}(\kappa^{-1/\alpha} z), \end{cases} \tag{C21}$$

in terms of which

$$\begin{cases} \kappa r = z^{2/(\beta+2)} \\ \kappa^{-D/2} u(r) = w(z)\, z^{-\beta/2(\beta+2)}, \end{cases} \tag{C22}$$

and the transformed Schrödinger equation becomes

$$\left[ \frac{d^2}{dz^2} - \text{sgn}(\beta)\, \lambda \alpha^2 \kappa^{-2/\alpha} + \sigma \alpha^2 z^{\tilde{\beta}} - \frac{(\tilde{l}+\tilde{\nu})^2 - 1/4}{z^2} \right] w_l(z) = 0, \tag{C23}$$

where $\sigma = \text{sgn}(E)$. Two important remarks immediately follow from Eq. (C23): (i) the ensuing dimensionless energy equation,

$$\tilde{\eta} = -\text{sgn}(\beta)\, \lambda \alpha^2 \, |E|^{-1/\alpha}, \tag{C24}$$

is the basis for the relation between the energy eigenvalue problems of the corresponding dual potentials (for example, for $\beta = -1$, $\tilde{\beta} = 2$, $\alpha = 2$, it provides the well-known connection between the Coulomb potential and the harmonic oscillator, in spaces of any number of dimensions); and (ii) the dimensionless dual potential

$$\tilde{\mathscr{V}}(z) = -\sigma \alpha^2 z^{\tilde{\beta}} \tag{C25}$$

satisfies the sign relation $\text{sgn}(\tilde{\mathscr{V}}) = -\text{sgn}(E)$, which, combined with Eq. (C24), leads to a one-to-one correspondence between the bound-state sectors and between the scattering sectors of the dual potentials.

For our work, the transformation in the "neighborhood" of the inverse square potential amounts to $\beta = -(2-\varepsilon) < 0$, with $\varepsilon \ll 1$, which implies that $\tilde{\beta} = 4/\varepsilon - 2 \gg 1$ and $\alpha = 2/\varepsilon$, thus converting Eq. (C23) into

$$\left[ \frac{d^2}{dz^2} + \frac{4}{\varepsilon^2} \lambda \kappa^{-\varepsilon} + \sigma \frac{4}{\varepsilon^2} z^{4/\varepsilon - 2} - \frac{(\tilde{l} + \tilde{\nu})^2 - 1/4}{z^2} \right] w_l(z) = 0, \tag{C26}$$

where the potential energy term $-\sigma 4 z^{4/\varepsilon - 2}/\varepsilon^2$ is seen to behave as an infinite hyperspherical potential well in the limit $\varepsilon = 0^+$, for $E < 0$ (i.e., $\sigma = -1$).

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