Physica A 450 (2016) 435–443

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Hypergeometric connotations of quantum equations

A. Plastino *, M.C. Rocca

La Plata National University and Argentina's National Research Council, (IFLP-CCT-CONICET)-C. C. 727, 1900 La Plata, Argentina

HIGHLIGHTS

- We show that both the Schrödinger and Klein–Gordon equations can be derived from the confluent hypergeometric differential equation.
- Also a non linear Klein–Gordon equation can be analogously derived.
- The latter coincides with one advanced by Nobre, Rego-Monteiro, and Tsallis.

ARTICLE INFO

Article history:
Received 30 May 2015
Received in revised form 23 December 2015
Available online 20 January 2016

Keywords:
Schrödinger equation
Klein–Gordon equation
Hypergeometric functions

ABSTRACT

We show that the Schrödinger and Klein–Gordon equations can both be derived from a hypergeometric differential equation. The same applies to non linear generalizations of these equations.

1. Introduction

In this paper we uncover the rather surprising fact that the Schrödinger and Klein–Gordon equations can both be derived from a hypergeometric differential equation. The same applies to nonlinear generalizations of these equations, such as the ones recently proposed, in ad hoc fashion, by Nobre, Rego-Monteiro, and Tsallis (NRT) [1] (see Appendix B).

The paper is organized as follows. Section 2.1 deals with the hypergeometric Schrödinger derivation, Section 2.2 with the Klein–Gordon one, and Section 3 with the nonlinear Schrödinger equation, that turns out to be different from the NRT equation. Section 4 illustrates the latter equation with a wave packet example. Section 5 hypergeometrically derives a nonlinear Klein–Gordon equation, that does coincide with the NRT one. Some conclusions are drawn in Section 6, and special details are given in Appendices A–C.

2. Hypergeometric derivations: Schrödinger and Klein–Gordon equations

We start our considerations by showing that both Schrödinger's and Klein–Gordon's equations can be derived from the differential equation satisfied by the confluent hypergeometric function's ϕ without further ado.

* Corresponding author.
E-mail address: plastino@fisica.unlp.edu.ar (A. Plastino).

http://dx.doi.org/10.1016/j.physa.2016.01.022

2.1. Schrödinger's wave equation

We encounter in Ref. [2] a differential equation that can be solved by appeal to hypergeometric functions, namely,

$$U''(z) + U'(z) + \left[\frac{\lambda}{z} + \left(\frac{1}{4} - \mu^2\right)\frac{1}{z^2}\right]U(z) = 0, \tag{2.1}$$

and a solution is

$$U(z) = z^{\frac{1}{2}+\mu} e^{-z} \phi\left(\frac{1}{2} + \mu - \lambda, 2\mu + 1; z\right), \tag{2.2}$$

where ϕ is the confluent hypergeometric function. Appealing to the change of variables

$$2\mu + 1 = b; \qquad \frac{1}{2} + \mu - \lambda = a. \tag{2.3}$$

Eqs. (2.1) and (2.2) become

$$U''(z) + U'(z) + \left[\left(\frac{b}{2} - a\right)\frac{1}{z} - \left(\frac{b^2 - 2b}{4}\right)\frac{1}{z^2}\right]U(z) = 0, \tag{2.4}$$

$$U(z) = z^{\frac{b}{2}} e^{-z} \phi(a, b.z). \tag{2.5}$$

Introducing solution (2.5) into (2.4) we ascertain that the differential equation satisfied by ϕ is

$$z\phi''(a, b; z) + (b - z)\phi'(a, b; z) - a\phi(a, b; z) = 0. \tag{2.6}$$

For the special instance $a = b$ we obtain (see Ref. [3])

$$\phi(a, a; z) = e^z, \tag{2.7}$$

and (2.6) now becomes (for $a = b$)

$$z\phi''(a, a; z) + (a - z)\phi'(a, a; z) - a\phi(a, a; z) = 0. \tag{2.8}$$

At this point we make a *critical choice* and set z

$$z = \frac{i}{\hbar}(px - Et) \quad \text{and} \quad E = \frac{p^2}{2m}. \tag{2.9}$$

At this stage we have introduced some physical information into the abstract mathematical formalism, according to the invariance respected by z (Galilean here). A different choice is made in Ref. [4], that leads to the Fokker–Planck equation. Accordingly, in our formalism the entire physical content is given by the critical choice of z, while the rest is just mathematics.

Now, according to (2.9) one has

$$\frac{\partial z}{\partial t} = -\frac{iE}{\hbar}; \qquad \frac{\partial z}{\partial x} = \frac{ip}{\hbar}. \tag{2.10}$$

Then, on account of (2.7) one has

$$e^{\frac{i}{\hbar}(px-Et)} = \phi\left[a, a; \frac{i}{\hbar}(px - Et)\right] \equiv \phi. \tag{2.11}$$

The associated confluent hypergeometric differential equation is now

$$\frac{i}{\hbar}(px - Et)\phi'' + \left[a - \frac{i}{\hbar}(px - Et)\right]\phi' - a\phi = 0,$$

$$\phi' = \frac{d\phi}{dz}; \qquad \phi'' = \frac{d^2\phi}{dz^2}. \tag{2.12}$$

We see from (2.11) that, since

$$\frac{\partial \phi}{\partial t} = \phi' \frac{\partial z}{\partial t}; \qquad \frac{\partial \phi}{\partial x} = \phi' \frac{\partial z}{\partial x}, \tag{2.13}$$

then

$$\phi'' = -\frac{\hbar^2}{p^2}\frac{\partial^2 \phi}{\partial x^2} \qquad \phi' = \frac{i\hbar}{E}\frac{\partial \phi}{\partial t} \equiv \phi, \tag{2.14}$$

so that, in the differential equation (2.12) the two a-terms cancel each other and the factor $\frac{\mathrm{i}}{\hbar}(px - Et)$ can be simplified. Accordingly, (2.12) adopts the appearance

$$-\frac{\hbar^2}{p^2}\frac{\partial^2\phi}{\partial x^2} - \frac{\mathrm{i}\hbar}{E}\frac{\partial\phi}{\partial t} = 0, \tag{2.15}$$

or, equivalently,

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2}. \tag{2.16}$$

Since $H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ we can finally write

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = H_0\phi, \tag{2.17}$$

i.e., Schrödinger's free particle equation. For an arbitrary Hamiltonian H, (2.17) may be generalized by inference, *not deduction*, to

$$\mathrm{i}\hbar\frac{\partial\phi}{\partial t} = H\phi, \tag{2.18}$$

the usual Schrödinger equation (SE). Thus, we do deduce Schrödinger's wave equation (free particle) directly from the hypergeometric differential equation (HDE), but cannot do the same with (2.18), that is merely a "reasonable" generalization. Let us insist: the "quantumness" is inserted into the HDE via the choice (2.9).

2.2. *Klein–Gordon's equation*

We start with the *critical physical choice*

$$z = \mathrm{i}(kx - \omega t); \qquad \omega^2 = k^2c^2 + \frac{m^2c^4}{\hbar^2}, \tag{2.19}$$

(Lorentz invariance) and write

$$e^{\mathrm{i}(kx-\omega t)} = \phi\,[a, a; \mathrm{i}(kx - \omega t)] = \phi. \tag{2.20}$$

The operating confluent hypergeometric differential equation is here

$$\mathrm{i}(kx - \omega t)\phi'' + [a - \mathrm{i}(kx - \omega t)]\,\phi' - a\phi = 0. \tag{2.21}$$

We now perform similar manipulations of partial derivatives as in the preceding section and end up with the identities

$$\phi'' = -\frac{1}{k^2}\frac{\partial^2\phi}{\partial x^2} \qquad \phi' = \frac{\mathrm{i}}{\omega}\frac{\partial\phi}{\partial t} = \phi, \tag{2.22}$$

getting for (2.21)

$$-\frac{1}{k^2}\frac{\partial^2\phi}{\partial x^2} - \frac{\mathrm{i}}{\omega}\frac{\partial\phi}{\partial t} = 0. \tag{2.23}$$

Since

$$\frac{\partial\phi}{\partial t} = \frac{\mathrm{i}}{\omega}\frac{\partial^2\phi}{\partial t^2}, \tag{2.24}$$

(2.23) becomes

$$-\frac{1}{k^2}\frac{\partial^2\phi}{\partial x^2} + \frac{1}{\omega^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{2.25}$$

Using now the equality

$$\frac{\partial^2\phi}{\partial x^2} = -k^2\phi, \tag{2.26}$$

and a little algebra we arrive at

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^2c^2}{\hbar^2}\phi = 0, \tag{2.27}$$

the desired Klein–Gordon equation.

3. A nonlinear Schrödinger equation

The above procedure can be generalized by appealing to the more general hypergeometric function F. This F-treatment has been worked out by us in Ref. [5], *only for the Schrödinger case and without reference* to the present confluent instance, originally developed in Section 2. We reconsider this nonlinear generalization here for completeness's sake, in order to better appreciate the workings of our hypergeometric approach. See more details in Appendix A.

As shown in Ref. [5], one can write

$$F(-\alpha, \gamma; \gamma; -z) = (1+z)^{\alpha}. \tag{3.1}$$

It follows that for Tsallis' imaginary q-exponential function (see its definition in Refs. [5,4,3,2,1]) one has

$$\left[1 + \frac{\mathrm{i}}{\hbar}(1-q)(px - Et)\right]^{\frac{1}{1-q}} = F\left[\frac{1}{q-1}, \gamma; \gamma; \frac{\mathrm{i}}{\hbar}(q-1)(px - Et)\right] \equiv F, \tag{3.2}$$

where $E = \frac{p^2}{2m}$. According to Ref. [6], F satisfies

$$z(1-z)F''(\alpha, \beta; \gamma; z) + [\gamma - (\alpha+\beta+1)z]F'(\alpha, \beta; \gamma; z) - \alpha\beta F(\alpha, \beta; \gamma; z) = 0. \tag{3.3}$$

After following developments detailed in Appendix A one arrives at

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]^q = H_0\left[\frac{F(x,t)}{F(0,0)}\right]. \tag{3.4}$$

Generalizing to arbitrary H we have

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\left[\frac{\psi(x,t)}{\psi(0,0)}\right]^q = H\left[\frac{\psi(x,t)}{\psi(0,0)}\right]. \tag{3.5}$$

Differences and similarities between these equations and those obtained by Nobre, Rego-Monteiro, and Tsallis (NRT) [1] are fully discussed in Ref. [5]. Some NRT details are reviewed in Appendix B.

4. The *q*-Gaussian wave packet

We pass now to discuss an important solution of (3.5): the wave packet. Details of the ordinary case are summarized in Appendix C. The pertinent analysis for the NRT equation has been given in Ref. [7]. Setting $\psi(0,0) = 1$ one has

$$\mathrm{i}\hbar \frac{\partial \psi^q}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}. \tag{4.1}$$

Following Ref. [7], we propose as a solution

$$\psi = \left\{1 + (q-1)\left[a(t)x^2 + b(t)x + c(t)\right]\right\}^{\frac{1}{1-q}}, \tag{4.2}$$

where a, b, and c are temporal functions to be determined. From $\psi(0,0) = 1$ one has $c(0) = 0$. Deriving ψ^q with respect to time we find

$$\frac{\partial \psi^q}{\partial t} = -q\left[\dot{a}(t)x^2 + \dot{b}(t)x + \dot{c}(t)\right]\psi^{q-1}. \tag{4.3}$$

We then look for the second ψ-derivative with respect to x:

$$\frac{\partial^2 \psi}{\partial x^2} = \left[2(q+1)a^2x^2 + 2(q+1)abx + b^2 - 2(q-1)ac - 2a\right]\psi^{q-1}. \tag{4.4}$$

Introducing these two results into (4.1) we are led to a nonlinear system for a, b, and c

$$imq\dot{a} = \hbar(q+1)a^2, \tag{4.5}$$

$$imq\dot{b} = \hbar(q+1)ab, \tag{4.6}$$

$$2imq\dot{c} = \hbar\left[qb^2 - 2(q-1)ac - 2a\right]. \tag{4.7}$$

This system's solution is given by

$$a(t) = \frac{mq}{\mathrm{i}\hbar(q+1)t + mq\alpha}, \tag{4.8}$$

$$b(t) = \frac{1}{\beta}\frac{1}{[\mathrm{i}\hbar(q+1)t + mq\alpha]}, \tag{4.9}$$

$$c(t) = (mq\alpha)^{\frac{1-q}{1+q}}\left(\frac{1}{q-1} - \frac{1}{4m^2q^2\beta^2\alpha}\right)[\mathrm{i}\hbar(q+1)t + mq\alpha]^{\frac{q-1}{q+1}} + \frac{1}{4mq\beta^2[\mathrm{i}\hbar(q+1)t + mq\alpha]} + \frac{1}{1-q}, \tag{4.10}$$

where α and β are constants to be fixed according to initial or boundary conditions for (4.1).

It is straightforward to prove that Eqs. (4.8)–(4.10) are transformed into Eqs. (C.8)–(C.10) of Appendix C for $q \to 1$. Thus the q-Gaussian wave packet transforms into the Gaussian wave packet when $q \to 1$. Since our Eq. (3.4) is different from NRT's nonlinear one, so are also their associated wave packet solutions.

5. A hypergeometric-generated non-linear Klein–Gordon equation

We derive now in hypergeometric fashion a nonlinear KG equation satisfied by the q-exponential function. Let $z = \mathrm{i}(q-1)(kx-\omega t)$ (critical physical choice).

Then,

$$1-z = [1+\mathrm{i}(1-q)(kx-\omega t)]^{\frac{1}{1-q}} = F\left[\frac{1}{q-1}, \gamma; \gamma; \mathrm{i}(q-1)(kx-\omega t)\right] \equiv F. \tag{5.1}$$

Recourse to the equalities

$$F'' = -\frac{1}{k^2(q-1)^2}\frac{\partial^2 F}{\partial x^2}, \tag{5.2}$$

$$F' = -\frac{1}{\mathrm{i}\omega(q-1)}\frac{\partial F}{\partial t} = -\frac{F^{(1-q)}}{\omega^2 q(q-1)}\frac{\partial^2 F}{\partial t^2}, \tag{5.3}$$

$$F = \frac{\mathrm{i}}{\omega}F^{(1-q)}\frac{\partial F}{\partial t}, \tag{5.4}$$

allow one to obtain, via (A.3):

$$\frac{z(1-z)}{k^2(q-1)^2}\frac{\partial^2 F}{\partial x^2} + \left[\gamma - \left(\frac{q}{q-1}+\gamma\right)z\right]\frac{1}{\mathrm{i}\omega(q-1)}\frac{\partial F}{\partial t} + \frac{\mathrm{i}\gamma}{\omega(q-1)}F^{(1-q)}\frac{\partial F}{\partial t} = 0, \tag{5.5}$$

or, equivalently,

$$\frac{z(1-z)}{k^2(q-1)^2}\frac{\partial^2 F}{\partial x^2} + \left[\gamma(1-z) - \left(\frac{qz}{q-1}\right)\right]\frac{1}{\mathrm{i}\omega(q-1)}\frac{\partial F}{\partial t} + \frac{\mathrm{i}\gamma}{\omega(q-1)}F^{(1-q)}\frac{\partial F}{\partial t} = 0. \tag{5.6}$$

Now, from (5.1), one has $1-z = F^{(1-q)}$ so that (5.6) adopts the appearance

$$\frac{zF^{(1-q)}}{k^2(q-1)}\frac{\partial^2 F}{\partial x^2} + \left[\gamma F^{(1-q)} - \left(\frac{qz}{q-1}\right)\right]\frac{1}{\mathrm{i}\omega}\frac{\partial F}{\partial t} + \frac{\mathrm{i}\gamma}{\omega}F^{(1-q)}\frac{\partial F}{\partial t} = 0, \tag{5.7}$$

and, simplifying terms,

$$\frac{zF^{(1-q)}}{k^2(q-1)}\frac{\partial^2 F}{\partial x^2} + \frac{iqz}{\omega(q-1)}\frac{\partial F}{\partial t} = 0, \tag{5.8}$$

entailing

$$\frac{F^{(1-q)}}{k^2}\frac{\partial^2 F}{\partial x^2} + \frac{iq}{\omega}\frac{\partial F}{\partial t} = 0. \tag{5.9}$$

Using here (5.3) we reach

$$\frac{F^{(1-q)}}{k^2}\frac{\partial^2 F}{\partial x^2} - \frac{F^{(1-q)}}{\omega^2}\frac{\partial^2 F}{\partial t^2} = 0, \tag{5.10}$$

which, simplifying the common factor F^{1-q} in (5.10), yields

$$\frac{1}{k^2}\frac{\partial^2 F}{\partial x^2} - \frac{1}{\omega^2}\frac{\partial F}{\partial t} = 0. \tag{5.11}$$

Since, additionally, one has

$$\frac{\partial^2 F}{\partial x^2} = -k^2 q F^{(2q-1)}, \tag{5.12}$$

some algebra leads to

$$\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} - \frac{\partial^2 F}{\partial x^2} + \frac{qm^2c^2}{\hbar^2}F^{(2q-1)} = 0. \tag{5.13}$$

If ϕ is given by

$$\phi(x,t) = A\,[1 + \mathrm{i}(1-q)(kx-\omega t)]^{\frac{1}{1-q}}\,, \tag{5.14}$$

we find, via (5.13),

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{\phi(x,t)}{\phi(0,0)}\right] - \frac{\partial^2}{\partial x^2}\left[\frac{\phi(x,t)}{\phi(0,0)}\right] + \frac{qm^2c^2}{\hbar^2}\left[\frac{\phi(x,t)}{\phi(0,0)}\right]^{(2q-1)} = 0, \tag{5.15}$$

that in n dimensions becomes

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right] - \nabla^2\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right] + \frac{qm^2c^2}{\hbar^2}\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right]^{(2q-1)} = 0, \tag{5.16}$$

where

$$\phi(\vec{x},t) = A\left[1 + \mathrm{i}(1-q)(\vec{k}\cdot\vec{x}-\omega t)\right]^{\frac{1}{1-q}}. \tag{5.17}$$

Eq. (5.16) and its two dimensional case (5.15) coincides with the nonlinear Klein–Gordon equation advanced by NRT in Ref. [1].

6. Conclusions

In this work we have uncovered a rather surprising fact. *Both the Schrödinger and Klein–Gordon equations can be derived from a hypergeometric differential equation*. To this end, the choice of the independent variable in the differential equation proves to be critical, because physical information is in this way provided. See, as an example, (2.9).

The same procedure can be applied to nonlinear generalizations of these equations, such as the ones recently proposed by Nobre, Rego-Monteiro, and Tsallis (NRT) [1]. The wave packet solution of *our* nonlinear SE (see Section 3) has been developed as an example of it.

Acknowledgment

We acknowledge very fruitful discussion with Prof. A.R. Plastino.

Appendix A. Non linear Schrödinger eq. of Ref. [5] (review)

The procedure of Section 2 can be generalized by appealing to the more general hypergeometric function F. This F-treatment has been previously developed by us in Ref. [5], *only for the Schrödinger case and without reference* to the confluent instance. We review this generalization here for completeness's sake.

As shown in Ref. [5], one can write

$$F(-\alpha,\gamma;\gamma;-z) = (1+z)^{\alpha}. \tag{A.1}$$

It follows that for Tsallis' imaginary q-exponential function (see its definition in Refs. [5,4,3,2,1]) one has

$$\left[1 + \frac{\mathrm{i}}{\hbar}(1-q)(px-Et)\right]^{\frac{1}{1-q}} = F\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right] \equiv F, \tag{A.2}$$

where $E = \frac{p^2}{2m}$. According to Ref. [6], F satisfies

$$z(1-z)F''(\alpha,\beta;\gamma;z) + [\gamma - (\alpha+\beta+1)z]F'(\alpha,\beta;\gamma;z) - \alpha\beta F(\alpha,\beta;\gamma;z) = 0. \tag{A.3}$$

For the particular case (A.2), this last expression adopts the appearance

$$\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\left[1 - \frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]F'' + \left[\gamma - \left(\frac{1}{q-1}+\gamma+1\right)\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]F' - \frac{\gamma}{q-1}F = 0. \tag{A.4}$$

Accordingly, one can deduce a relationship between $\dot{F}$ (time) and F' (space) (see Ref. [5] for details)

$$F'\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right] = \frac{\mathrm{i}\hbar}{(q-1)E}\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]. \tag{A.5}$$

In similar fashion one obtains [6]

$$F''\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]=-\frac{\hbar^2}{(q-1)^2p^2}\frac{\partial^2}{\partial x^2}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]. \tag{A.6}$$

Replacing (A.5) and (A.6) into (A.4), the latter becomes

$$\begin{aligned}&-\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\left[1-\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]\frac{\hbar^2}{(q-1)^2p^2}\frac{\partial^2 F}{\partial x^2}\\&+\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]\frac{\mathrm{i}\hbar}{(q-1)E}\frac{\partial F}{\partial t}-\frac{\gamma}{q-1}F=0,\end{aligned} \tag{A.7}$$

that, in turn, can be rewritten as

$$\begin{aligned}&-\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\left[1-\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]\times\frac{\hbar^2}{(q-1)m^2}\frac{\partial^2 F}{\partial x^2}\\&+\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]\mathrm{i}\hbar\frac{\partial F}{\partial t}-\gamma EF=0.\end{aligned} \tag{A.8}$$

Also, we obtain from (A.2)

$$\begin{aligned}-\gamma EF\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]&=-\mathrm{i}\hbar\gamma\left\{F\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right]\right\}^{(1-q)}\\&\quad\times\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{\mathrm{i}}{\hbar}(q-1)(px-Et)\right].\end{aligned} \tag{A.9}$$

Using n (A.9), (A.8) adopts the appearance

$$-\frac{\hbar^2}{2m(q-1)}\left[1-F^{(1-q)}\right]F^{(1-q)}\frac{\partial^2}{\partial x^2}F+\mathrm{i}\hbar\left\{\gamma+\left(\frac{1}{q-1}+\gamma+1\right)\left[F^{(1-q)}-1\right]\right\}\frac{\partial}{\partial t}F-\mathrm{i}\hbar\gamma F^{(1-q)}\frac{\partial}{\partial t}F=0, \tag{A.10}$$

and, after simplifying,

$$-\frac{\hbar^2}{2m}F^{(1-q)}\frac{\partial^2}{\partial x^2}F-\mathrm{i}\hbar q\frac{\partial}{\partial t}F=0, \tag{A.11}$$

that can be recast as

$$\mathrm{i}\hbar q\frac{\partial}{\partial t}F=F^{(1-q)}H_0F, \tag{A.12}$$

where H_0 is the free particle Hamiltonian. Note that for $q = 1$ things'properly reduce to Schrödinger's wave equation. If, instead of (A.2) we have

$$F(x,t)=A\left[1+\frac{1}{\hbar}(1-q)(px-Et)\right]^{\frac{1}{1-q}}, \tag{A.13}$$

then $F(0,0) = A$ and (A.12) becomes

$$\mathrm{i}\hbar q\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]=\left[\frac{F(x,t)}{F(0,0)}\right]^{(1-q)}H_0\left[\frac{F(x,t)}{F(0,0)}\right], \tag{A.14}$$

or, equivalently,

$$\mathrm{i}\hbar q\left[\frac{F(x,t)}{F(0,0)}\right]^{(q-1)}\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]=H_0\left[\frac{F(x,t)}{F(0,0)}\right], \tag{A.15}$$

that can in turn be written as

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]^{q}=H_0\left[\frac{F(x,t)}{F(0,0)}\right]. \tag{A.16}$$

Generalizing to arbitrary H we have

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left[\frac{\psi(x,t)}{\psi(0,0)}\right]^{q}=H\left[\frac{\psi(x,t)}{\psi(0,0)}\right]. \tag{A.17}$$

Differences and similarities between these equations and those obtained by Nobre, Rego-Monteiro, and Tsallis (NRT) [1] are fully discussed in Ref. [5].

Appendix B. The NRT equation

We review here the NRT equation for the free particle in order to make this paper self-contained. It reads [1]

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\psi(\vec{x},t)}{\psi(0,0)}\right] = H_0\left[\frac{\psi(\vec{x},t)}{\psi(0,0)}\right]^{2-q}. \tag{B.1}$$

Setting $\phi = \psi^{2-q}$ we are led to

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right]^{\frac{1}{1-q}} = H_0\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right], \tag{B.2}$$

and generalizing to arbitrary H one writes

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right]^{\frac{1}{1-q}} = H\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right], \tag{B.3}$$

that in ψ terms becomes

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\psi(\vec{x},t)}{\psi(0,0)}\right] = H\left[\frac{\psi(\vec{x}x,t)}{\psi(0,0)}\right]^{2-q}. \tag{B.4}$$

Appendix C. The ordinary Gaussian wave packet

We want to tackle

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}, \tag{C.1}$$

via the Gaussian packet

$$\psi(x,t) = e^{-[a(t)x^2+b(t)x+c(t)]}, \tag{C.2}$$

with the initial condition $\psi(0,0) = 1$, entailing $c(0) = 0$. After a time derivative we get

$$\frac{\partial\psi(x,t)}{\partial t} = -[\dot{a}(t)x^2 + \dot{b}(t)x + \dot{c}(t)]\psi(x,t). \tag{C.3}$$

The spatial second derivative yields

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = -[4a^2(t)x^2 + 4a(t)b(t)x + b^2(t) - 2a(t)]\psi(x,t). \tag{C.4}$$

Replacing (C.3) and (C.4) into (C.1) we obtain

$$\dot{a} = \frac{2\hbar}{im}a^2, \tag{C.5}$$

$$\dot{b} = \frac{2\hbar}{im}ab, \tag{C.6}$$

$$\dot{c} = \frac{\hbar}{2im}(b^2 - 2a), \tag{C.7}$$

whose solution reads

$$a(t) = \frac{m}{2i\hbar t + m\alpha}, \tag{C.8}$$

$$b(t) = \frac{1}{\beta}\frac{m}{[2i\hbar t + m\alpha]}, \tag{C.9}$$

$$c(t) = \frac{1}{4m\beta^2[2i\hbar t + m\alpha]} + \frac{1}{2}\ln(2i\hbar t + m\alpha) - \frac{1}{4m^2\beta^2\alpha} - \frac{1}{2}\ln(m\alpha), \tag{C.10}$$

where for c one has $c(0) = 0$.

References

[1] F.D. Nobre, M.A. Rego-Monteiro, C. Tsallis, Phys. Rev. Lett. 106 (2011) 140601.
[2] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Vol. 9.202, Academic Press, 1965, p. 1057.
[3] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Vol. 9.215, Academic Press, 1965, p. 1059.
[4] A. Plastino, A. Rocca, arXiv:1511.00027.
[5] A. Plastino, M.C. Rocca, 2015. arxiv:1505.01334.
[6] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, Vol. 9.151, Academic Press, 1965, p. 1045.
[7] S. Curilef, A.R. Plastino, A. Plastino, Physica A 392 (2013) 2631.