q-Fourier Transform and its inversion-problem

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Abstract

Tsallis' q-Fourier transform is not generally one-to-one. It is shown here that, if we eliminate the requirement that q be fixed, and let it instead "float", a simple extension of the F_q-definition, this procedure

restores the one-to-one character.

KEYWORDS: q-Fourier transform, generalization, ono to one character, statistical mechanics, nonextensive statistical mechanics.

1 Introduction

Nonextensive statistical mechanics (NEXT) [1, 2, 3], a current generalization of the BoltzmannGibbs (BG) one, is actively studied in diverse areas of Science. NEXT is based on a nonadditive (though extensive [4]) entropic information measure characterized by the real index q (with q = 1 recovering the standard BG entropy). It has been applied to variegated systems such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spinglasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], Fokker-Planck equation's applications [16], etc.

NEXT can be advantageously expressed via q-generalizations of standard mathematical concepts (the logarithm and exponential functions, addition and multiplication, Fourier transform (FT) and the Central Limit Theorem (CLT) [17, 22, 25]). The q-Fourier transform F_q exhibits the nice property of transforming q-Gaussians into q-Gaussians [17]. Recently, plane waves, and the representation of the Dirac delta into plane waves have been also generalized [18, 19, 21, 22].

A serious problem afflicts F_q. It is not generally one-to-one. A detailed example is discussed below. In this work we show that by recourse to a rather simple but efficient stratagem that consists in

- eliminating the requirement that q be fixed and instead

- let it"float",

one restores the one-to-one character.

2 Generalizing the q-Fourier transform

We define, following [17], a q-Fourier transform of $f(x) \in L^1(\mathbb{R})$, $f(x) \geq 0$ as

$$F(k,q) = [H(q-1) - H(q-2)] \times \int_{-\infty}^{\infty} f(x)\{1 + i(1-q)kx[f(x)]^{(q-1)}\}^{\frac{1}{1-q}}\, dx \tag{2.1}$$

where $H(x)$ is the Heaviside step function.

The only difference between this definition and that given in [17] is that q is not fixed and varies within the interval $[1,2)$. Herein lies the hard-core of our presentation. This simple change of perspective makes it is easy to find the inversion-formula for (2.1) by recourse to the inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\lim_{\epsilon \to 0^+} \int_{1}^{2} F(k,q)\delta(q-1-\epsilon)\, dq \right] e^{-ikx}\, dk. \tag{2.2}$$

As a consequence, we see that this q-Fourier transform is one-to-one, unlike what happens in [23],[24]. In the next section we give an illustrative example.

3 Example

As an illustration we discuss the example given by Hilhorst in Ref. ([22]). Let $f(x)$ be

$$f(x) = \begin{cases} \left(\frac{\lambda}{x}\right)^{\beta} \; ; \; x \in [a,b] \; ; \; 0 < a < b \; ; \; \lambda > 0 \\ \\ 0 \; ; \; x \text{ outside [a, b].} \end{cases} \tag{3.1}$$

The corresponding q-Fourier transform is

$$F(k,q) = \lambda^{\beta} \int\limits_a^b x^{-\beta} \{1 + i(1-q)k\lambda^{\beta(q-1)} x^{1-\beta(q-1)}\}^{\frac{1}{1-q}} \, dx. \tag{3.2}$$

Effecting the change of variables

$$y = x^{1-\beta(q-1)},$$

we have for (3.2)

$$F(k,q) = [H(q-1) - H(q-2)] \times$$

$$\frac{\lambda^{\beta}}{1-\beta(q-1)} \int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \{1 + i(1-q)k\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}} \, dy. \tag{3.3}$$

Now, (3.3) can be rewritten in the useful form

$$F(k,q) = [H(q-1) - H(q-2)]\times$$

$$\left\{\left\{H(q-1) - H\left[q - \left(1+\frac{1}{\beta}\right)\right]\right\}\times\right.$$

$$\frac{\lambda^\beta}{1-\beta(q-1)} \int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1-q)k\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\ dy +$$

$$\left\{H\left[q - \left(1+\frac{1}{\beta}\right)\right] - H(q-2)\right\}\times$$

$$\left.\frac{\lambda^\beta}{\beta(q-1)-1} \int\limits_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \{1 + i(1-q)k\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\ dy\right\}. \qquad (3.4)$$

Taking into account that the involved integrals are defined in a finite interval, we can cast (3.4) as

$$F(k,q) = [H(q-1) - H(q-2)]\times$$

$$\left\{\left\{H(q-1) - H\left[q - \left(1+\frac{1}{\beta}\right)\right]\right\}\times\right.$$

$$\frac{\lambda^\beta}{1-\beta(q-1)} \lim_{\epsilon\to 0^+} \int\limits_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \{1 + i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\ dy +$$

$$\left\{H\left[q - \left(1+\frac{1}{\beta}\right)\right] - H(q-2)\right\}\times$$

$$\left.\frac{\lambda^\beta}{\beta(q-1)-1} \lim_{\epsilon\to 0^+} \int\limits_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \{1 + i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)} y\}^{\frac{1}{1-q}}\ dy\right\}. \qquad (3.5)$$

We now use results of the Integral's table [26] to evaluate (3.5) and get

$$\lim_{\epsilon\to 0^+}\int_{a^{1-\beta(q-1)}}^{\infty} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}}\{1+i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy =$$

$$\frac{(q-1)[1-\beta(q-1)]a^{\frac{q-2}{q-1}}}{(2-q)[(1-q)i(k+i0)\lambda^{\beta}]^{\frac{1}{q-1}}}\times$$

$$F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.$$

$$\left.-\frac{1}{(1-q)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\right), \tag{3.6}$$

and

$$\lim_{\epsilon\to 0^+}\int_{0}^{a^{1-\beta(q-1)}} y^{\frac{\beta(2-q)}{\beta(q-1)-1}}\{1+i(1-q)(k+i\epsilon)\lambda^{\beta(q-1)}y\}^{\frac{1}{1-q}}\,dy =$$

$$\frac{[\beta(q-1)-1]a^{1-\beta}}{\beta-1}\times$$

$$F\left(\frac{1}{q-1},\frac{\beta-1}{\beta(q-1)-1},\frac{\beta q-2}{\beta(q-1)-1};\right.$$

$$\left.(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right), \tag{3.7}$$

where $F(a,b,c;z)$ is the hypergeometric function. Thus we obtain for $F(k,q)$

$$F(k,q)=[H(q-1)-H(q-2)]\times$$

$$\left\{\left\{H(q-1)-H\left[q-\left(1+\frac{1}{\beta}\right)\right]\right\}\right\}\times$$

$$\frac{(q-1)\lambda^{\beta}}{(2-q)[(1-q)i(k+i0)\lambda^{\beta}]^{\frac{1}{q-1}}}\times$$

$$\left\{a^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1},\frac{2-q}{(q-1)[1-\beta(q-1)]},\frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.\right.$$

$$\frac{1}{(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}\Bigg) -$$

$$b^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1}, \frac{2-q}{(q-1)[1-\beta(q-1)]}, \frac{1}{q-1}+\frac{\beta(2-q)}{1-\beta(q-1)};\right.$$

$$\left.\frac{1}{(q-1)i(k+i0)\lambda^{\beta(q-1)}b^{1-\beta(q-1)}}\right)\Bigg\} +$$

$$\left\{H\left[q-\left(1+\frac{1}{\beta}\right)\right]-H(q-2)\right\}\frac{\lambda^\beta}{\beta-1}\times$$

$$\left\{a^{1-\beta}F\left(\frac{1}{q-1}, \frac{\beta-1}{\beta(q-1)-1}, \frac{\beta q-2}{\beta(q-1)-1};\right.\right.$$

$$\left.(q-1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}\right) -$$

$$b^{1-\beta}F\left(\frac{1}{q-1}, \frac{\beta-1}{\beta(q-1)-1}, \frac{\beta q-2}{\beta(q-1)-1};\right.$$

$$\left.\left.(q-1)i(k+i0)\lambda^{\beta(q-1)}b^{1-\beta(q-1)}\right)\right\}\Bigg\}. \tag{3.8}$$

As we can see from (3.8), $F(k,q)$ is dependent of a and b, and, as consequence, one-to-one as has been shown in Section 2.

However, and this is the crucial issue, if we **fix** q and select $\beta = 1/(q-1)$ (3.8) simplifies and adopts the appearance

$$F(k,q) = \lambda^{\frac{1}{q-1}}\frac{q-1}{2-q}\left[H(q-1)-H(q-2)\right]\times$$

$$\left[a^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)i(k+i0)\lambda\right) -\right.$$

$$\left.b^{\frac{q-2}{q-1}}F\left(\frac{1}{q-1}, \frac{2-q}{q-1}, \frac{2-q}{q-1}; (q-1)i(k+i0)\lambda\right)\right]. \tag{3.9}$$

With the help of the result given in [27] for

$$F(-a,b,b,-z) = (1+z)^a,$$

we obtain for (3.9):

$$F(k,q) = \lambda^{\frac{1}{q-1}} \frac{q-1}{2-q} \left[H(q-1) - H(q-2)\right] \left(a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}}\right) \left[1+(1-q)ik\lambda\right]^{\frac{1}{1-q}} . \tag{3.10}$$

Using now the expression for λ of [22], i.e.,

$$\lambda = \left[\left(\frac{q-1}{2-q}\right)\left(a^{\frac{q-2}{q-1}} - b^{\frac{q-2}{q-1}}\right)\right]^{1-q},$$

we have, finally,

$$F(k,q) = \left[H(q-1) - H(q-2)\right] \left[1+(1-q)ik\lambda\right]^{\frac{1}{1-q}} , \tag{3.11}$$

which is the result given by Hilthorst in [22], demonstrating that $F(k,q)$ is not one-to-one. As a conclusion we can say that for fixed q the q-Fourier transform is NOT one-to-one. On the contrary, as we have shown in section 2, when q is NOT fixed, the q-Fourier transform is indeed one-to-one.

Conclusions

In the present communication we have discussed the NOT one-to-one nature of the q-Fourier transform F_q. We have shown that, if we eliminate the requirement that q be fixed and let it"float" instead, such simple extension of the F_q-definition restores the desired one-to-one character.

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