Classical Electromagnetic Field Theory in the Presence of Magnetic Sources *

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(Received 6 November 2002)

Using two new well-defined four-dimensional potential vectors, we formulate the classical Maxwell field theory in a form which has manifest Lorentz covariance and $SO(2)$ duality symmetry in the presence of magnetic sources. We set up a consistent Lagrangian for the theory. Then from the action principle we obtain both Maxwell's equation and the equation of motion of a dyon moving in the electromagnetic field.

PACS: 03.50.De, 14.80.Hv, 11.10.Ef

Recently, there has been an increasing interest in the study of electromagnetic (EM) duality symmetry, because it plays a fundamental role in superstring and brane theory.[1,2] From Maxwell's equations we know that general EM duality implies the existence of magnetic source [magnetic charge (monopole) and currents]. However, when considering the quantum dynamics of particles carrying both electric and magnetic charges (dyons), one faces the lack of a naturally defined classical field theory despite of the fact that a consistent quantum field theory does exist.[3] This issue was analysed by many researchers.[5−9] in recent contributions. In our previous paper,[14] we presented an alternative formulation of electric-magnetic field theory in the presence of magnetic source. The advantages of our formulation are as follows. First, we introduce two new potential vectors that have no singularities and we do not need to use the concept of Dirac string. Secondly, from the present paper we can set up a consistent Lagrangian theory from which we can obtain all the information of classical electromagnetic field theory which returns to the usual Maxwell field theory when only electric source is considered. Thirdly, it has manifest Lorentz covariant and $SO(2)$ duality symmetry. Finally, it seems that our formulation can be quantized directly, which will be reported in a forthcoming article.

In this Letter, we present the details of the construction of a Lagrangian for the EM field theory in the formulation of Ref. [14]. From the action principle we expect to obtain the Maxwell equation as well as the equation of motion of a dyon moving in the electromagnetic field. We also explain why our formalism has manifestly $SO(2)$ duality symmetry.

Let us first give a brief review the formulation of the two four-vector potentials of the electromagnetic field in the presence of magnetic source.[14] Besides the usual definition of four-dimensional potential which we called A^1_μ, i.e.,

$$A^1_\mu = (\phi_1, -\boldsymbol{A}_1), \text{ or } A^{\mu 1} = (\phi_1, \boldsymbol{A}_1), \tag{1}$$

we also introduce

$$A^2_\mu = (\phi_2, -\boldsymbol{A}_2), \text{ or } A^{\mu 2} = (\phi_2, \boldsymbol{A}_2), \tag{2}$$

where ϕ_1 and $\boldsymbol{A}_1$ are the usual electric scalar potential and magnetic vector potential in electrodynamics, while the newly introduced potential ϕ_2 is the scalar potential associated with the magnetic field and $\boldsymbol{A}_2$ is a vector potential associated with the electric field. It should be emphasized that these two four-potentials have no singularities around the magnetic charges (monopoles). Using these potentials, the electric field strength $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ are then expressed by

$$\boldsymbol{E} = -\nabla\phi_1 - \frac{\partial \boldsymbol{A}_1}{\partial t} + \nabla\times\boldsymbol{A}_2, \tag{3}$$

$$\boldsymbol{B} = \nabla\phi_2 + \frac{\partial \boldsymbol{A}_2}{\partial t} + \nabla\times\boldsymbol{A}_1. \tag{4}$$

In the magnetic source free case, ϕ_2 and $\boldsymbol{A}_2$ are expected to be zero, so the above equation returns to the usual magnetic source free case.

Now we introduce two field tensors

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu, \quad I = 1, 2. \tag{5}$$

Then, choosing Lorentz gauge $\partial^\mu A^I_\mu = 0$, Maxwell's equation in the case of existing both electric and magnetic sources,

$$\nabla\cdot\boldsymbol{E} = \rho_e, \quad \nabla\times\boldsymbol{B} = \boldsymbol{j}_e + \frac{\partial \boldsymbol{E}}{\partial t}, \tag{6}$$

$$\nabla\cdot\boldsymbol{B} = \rho_m, \quad \nabla\times\boldsymbol{E} = -\boldsymbol{j}_m - \frac{\partial \boldsymbol{B}}{\partial t}, \tag{7}$$

* Supported by the Natural Science Foundation of Zhejiang Province under Grant Nos 102011 and 102028, by the Consejo Nacional de Investigaciones Cient´ificas y Técnicas (CONICET), Argentina.

can be recast as

$$\partial^{\mu} F^{I}_{\mu\nu} = g^{II'} J^{I'}_{\nu}, \tag{8}$$

where

$$g^{II'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$J^1_\mu = J^e_\mu = (\rho_e, -\boldsymbol{j}_e), \quad J^2_\mu = J^m_\mu = (\rho_m, -\boldsymbol{j}_m).$$

In this formulation, the currents are manifestly conserved:

$$\partial^\nu J^I_\nu \propto \partial^\nu \partial^\mu F^I_{\mu\nu} = 0.$$

We will see that the index I is the $SO(2)$ index, so our formulation described above has manifestly $SO(2)$ duality symmetry which is related to the general gauge transformation $A^I_\mu \to A^I_\mu + \partial_\mu \chi^I$. The fields $\boldsymbol{E}$, $\boldsymbol{B}$, the field tensors in Eq. (5) and Maxwell's Eqs. (8) are all invariant under the transformations.

Let us also stress that in the above expressions, neither $F^1_{\mu\nu}$ nor $F^2_{\mu\nu}$ has the same matrix form as the usual electromagnetic tensor. From Eq. (5) together with Eqs. (3) and (4), we can find

$$E_i = F^1_{0i} + {}^*F_{0i}{}^2,$$
$$B_i = {}^*F_{0i}{}^1 - F^2_{0i}.$$

Thus, it is convenient to define a new field tensor as

$$F_{\mu\nu} = F^1_{\mu\nu} + {}^*F^2_{\mu\nu}, \tag{9}$$
$$\widetilde{F}_{\mu\nu} = {}^*F^1_{\mu\nu} - F^2_{\mu\nu}, \tag{10}$$

where $\widetilde{F}_{\mu\nu}$ is exactly the Hodge star dual of $F_{\mu\nu}$. As we see in the following, using these new field tensors we can easily express the duality symmetry in a compact fashion. It is easy to see that $F_{\mu\nu}$ is the analogue to the usual electromagnetic tensor defined in classical electrodynamics, because they have exactly the same matrix form in terms of the field strengths. Since the vector potentials in our formalism have no singularities, one has $\partial^\mu {}^*F^I_{\mu\nu} = 0$, so Maxwell's equations can also be written as

$$\partial^\mu F_{\mu\nu} = \partial^\mu F^1_{\mu\nu} = J^1_\nu,$$
$$\partial^\mu \widetilde{F}_{\mu\nu} = -\partial^\mu F^2_{\mu\nu} = J^2_\nu. \tag{11}$$

From $F_{\mu\nu}$ or $\widetilde{F}_{\mu\nu}$ defined above, we can easily build a Lagrangian such that the Maxwell Eq. (11) can be derived from the action principle.

The $SO(2)$ duality symmetry of electromagnetic field theory has been discussed in the literature.[6–8,10] In our previous paper,[14] we explained in detail why the general duality symmetry is the $SO(2)$ symmetry, but there still exists something that is not very clear. For example, under the general dual transformation for $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$, i.e.,

$$\begin{pmatrix} F'_{\mu\nu} \\ \widetilde{F}'_{\mu\nu} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ \widetilde{F}_{\mu\nu} \end{pmatrix}, \tag{12}$$

why should the same transformation hold simultaneously for $J^{\mu 1}$ and $J^{\mu 2}$? One can make the same question concerning the dual transformations of $(\boldsymbol{E}, \boldsymbol{B})$ and $(q, g), (\boldsymbol{J}_e, \boldsymbol{J}_m)$, etc. Why must all these dual transformations be the same? In our formulation, we can shed light on this issue. Thus, we would first like to give answers to these questions and then explain in details why our formulation has manifestly $SO(2)$ duality symmetry, i.e., we can see that the index I is the $SO(2)$ index.

Let us first solve Maxwell's equation in our formalism. It is easy to check that the potential functions defined in the above section satisfy the differential equations:[14]

$$\frac{\partial^2}{\partial t^2}\phi_1 - \nabla^2 \phi_1 = \rho_e,$$
$$\frac{\partial^2}{\partial t^2}\boldsymbol{A}_1 - \nabla^2 \boldsymbol{A}_1 = \boldsymbol{j}_e$$
$$\frac{\partial^2}{\partial t^2}\phi_2 - \nabla^2 \phi_2 = -\rho_m,$$
$$\frac{\partial^2}{\partial t^2}\boldsymbol{A}_2 - \nabla^2 \boldsymbol{A}_2 = -\boldsymbol{j}_m. \tag{13}$$

In the static case, i.e., when the sources do not depend on time t, we can write

$$\rho_I = \rho_I(\boldsymbol{x}), \quad \boldsymbol{J}_I = \boldsymbol{J}_I(\boldsymbol{x}), \quad I = 1, 2,$$

where $I = 1$ and 2 represent $I = e$ and m, respectively. Then, as is exactly performed in the standard classical electrodynamics (magnetic source free case),[11] the solution of Eq. (13) is given by

$$\phi_I = \frac{1}{4\pi} g^{II'} \int \frac{\rho_{I'}(\boldsymbol{x}')}{r} \mathrm{d}^3 x' \tag{14}$$
$$\boldsymbol{A}_I(\boldsymbol{x}) = \frac{1}{4\pi} g^{II'} \int \frac{\boldsymbol{J}_{I'}(\boldsymbol{x}')}{r} \mathrm{d}^3 x', \tag{15}$$

where $r = |\boldsymbol{x} - \boldsymbol{x}'|$, then from Eqs. (3) and (4) we find that the field strengths have the following representation

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi} \int \rho_e(\boldsymbol{x}') \frac{\boldsymbol{r}}{r^3} \mathrm{d}^3 x' + \frac{1}{4\pi} \int \boldsymbol{J}_m(\boldsymbol{x}') \times \frac{\boldsymbol{r}}{r^3} \mathrm{d}^3 x, \tag{16}$$
$$\boldsymbol{B}(\boldsymbol{x}) = \frac{1}{4\pi} \int \rho_m(\boldsymbol{x}') \frac{\boldsymbol{r}}{r^3} \mathrm{d}^3 x' - \frac{1}{4\pi} \int \boldsymbol{J}_e(\boldsymbol{x}') \times \frac{\boldsymbol{r}}{r^3} \mathrm{d}^3 x. \tag{17}$$

Now we can give the answer to the question mentioned above under Eq. (12). Because $E_i = F_{0i}$, $B_i = \widetilde{F}_{0i}$, we know that if $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ have a transformation given by Eq. (12), which leads to the field strengths $\boldsymbol{E}$ and $\boldsymbol{B}$ having the same transformation, and because the field strengths are related to the sources by Eqs. (16) and (17), the sources ρ_e, ρ_m and $\boldsymbol{J}_e, \boldsymbol{J}_m$ must change in the same way. The same transformation must be satisfied by the four-dimensional currents $J^{\mu 1}$ and $J^{\mu 2}$. That is why once one chooses the transformation for $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ given by Eq. (12), then the corresponding field strengths and the sources must obey the same transformation. If we impose that Maxwell's Eqs. (6) and (7) are invariant under these transformations of field strengths and the sources, we obtain $a = d$ and $b = -c$. Moreover, if we also impose that the energy density $\frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)$ and the Poynting vector $\boldsymbol{E} \times \boldsymbol{B}$ are invariant under this transformation, we obtain $a^2 + b^2 = 1$. It is then natural to introduce an angle α such that $a = \cos\alpha$ and $b = \sin\alpha$. Hence the general duality transformation matrix coincides with the general rotation matrix in two dimensions. Thus it becomes apparent that the general electromagnetic duality symmetry is the $SO(2)$ symmetry. Under the special case, $\alpha = \pi/2$, transformation (12) coincides with the replacement $F_{\mu\nu} \to \widetilde{F}_{\mu\nu}$, $\widetilde{F}_{\mu\nu} \to -F_{\mu\nu}$ and the same replacements must be taken simultaneously, i.e., $\boldsymbol{E} \to \boldsymbol{B}, \boldsymbol{B} \to -\boldsymbol{E}$, $\rho_e \to \rho_m$, $\rho_m \to -\rho_e$ and $\boldsymbol{J}_e \to \boldsymbol{J}_m$, $\boldsymbol{J}_m \to -\boldsymbol{J}_e$, etc. This corresponds to the usual special electromagnetic duality symmetry.

Now we would like to point out that the index I is the $SO(2)$ index. Under the general dual transformation, i.e., $SO(2)$ transformation, from the above discussion we know that the sources ρ_I and $\boldsymbol{J}_I$ change into

$$\rho_{I'} = R(\alpha)_{II'}\rho^{I'}$$
$$\boldsymbol{J'}_I = R(\alpha)_{II'}\boldsymbol{J}^{I'},$$

where $R(\alpha)$ is the $SO(2)$ rotation matrix. Then from Eqs. (14) and (15), we know that the potentials $A^I_\mu = (\phi_I, -\boldsymbol{A}_I)$ should have the same $SO(2)$ transformation, that is to say, the index I of potential A^I_μ is the $SO(2)$ index. Therefore, our formulation has manifestly $SO(2)$ duality symmetry.

Now we would like to give a Lagrangian for the electromagnetic field which gives right Maxwell's equation in the presence of magnetic source. We will also see that from this Lagrangian, one can deduce the right Lorentz force formula.

The Lagrangian of the field is given by

$$L = -\frac{1}{4}(F_{\mu\nu})^2 - (A^1_\mu + {}^*A^2_\mu)J^{\mu 1} + ({}^*A^1_\mu - A^2_\mu)J^{\mu 2}, \tag{18}$$

where ${}^*A^I_\mu$ is defined by

$${}^*A^I_\mu = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\int_P^x \partial^\alpha A^{\beta I}\mathrm{d}x^\nu. \tag{19}$$

From a simple calculation we find

$$\frac{\partial F^2}{\partial(\partial_\mu A^1_\nu)} = \frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A^1_\nu)} = 4F^{\mu\nu},$$
$$\frac{\partial F^2}{\partial(\partial_\mu A^2_\nu)} = \frac{\partial(F_{\alpha\beta}F^{\alpha\beta})}{\partial(\partial_\mu A^2_\nu)} = 4\widetilde{F}^{\mu\nu}.$$

It is noticed that ${}^*A^I_\mu$ is related to the derivative of A^I_α, and if we take into account the conservation conditions of the currents, then the Euler–Lagrange equation of the Lagrangian defined in Eq. (18) gives Maxwell's Eq. (11).

For simplicity, we consider here one dyon with electric charge q and magnetic charge g, which moves in the electromagnetic field (the extension to the multi-dyon system can be easily carried out). From the action of the system we expect to obtain both field Eq. (11) and the equation of motion of the dyon. The action of this system consists of three parts, i.e,

$$S = S_p + S_I + S_F, \tag{20}$$

where

$$S_p = -m\int_{\lambda_1}^{\lambda_2}\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda \tag{21}$$

is the free action of the dyon, and

$$\begin{aligned} S_I =& q\int_{\lambda_1}^{\lambda_2}(A^1_\mu + {}^*A^2_\mu)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\mathrm{d}\lambda \\ &- g\int_{\lambda_1}^{\lambda_2}({}^*A^1_\mu - A^2_\mu)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\mathrm{d}\lambda \\ =& \int_\Omega (A^1_\mu + {}^*A^2_\mu)J^{\mu 1}\mathrm{d}^4x \\ &- \int_\Omega ({}^*A^1_\mu - A^2_\mu)J^{\mu 2}d^4x \end{aligned} \tag{22}$$

is the term of interaction between the dyon and the electromagnetic field around it. $J^{\mu I}$ in the above equation are the currents for one particle dyon which have the form

$$J^{\mu 1} = q\int\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\delta^4(x - x(\tau))\mathrm{d}\tau,$$
$$J^{\mu 2} = g\int\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\delta^4(x - x(\tau))\mathrm{d}\tau.$$

The last term of the action

$$S_F = -\frac{1}{4}\int_\Omega F_{\mu\nu}F^{\mu\nu}\mathrm{d}^4x \tag{23}$$

is nothing but the action of the electromagnetic field.

Let us now vary the potentials as

$$A^I_\mu \Rightarrow A^I_\mu + \epsilon_I B^I_\mu(x), \quad B^I_\mu|_\Omega = 0.$$

We can check that

$$\frac{\partial S_f}{\partial\epsilon_1}|_{\epsilon_1=\epsilon_2=0} = \int_\Omega \partial_\mu F^{\mu\nu}B^1_\nu\mathrm{d}^4x,$$

$$\frac{\partial S_f}{\partial \epsilon_2}|_{\epsilon_1=\epsilon_2=0} = \int_\Omega \partial^*_\mu F^{\mu\nu} B^2_\nu \mathrm{d}^4x.$$

Noticing that $\frac{\partial\ ^*A\mu^I}{\partial\epsilon^{I'}} = 0$ and $\frac{\partial S_p}{\partial \epsilon_I} = 0$, we then have

$$0 = \frac{\partial S}{\partial \epsilon_1}|_{\epsilon_1=\epsilon_2=0} = \int_\Omega (\partial_\mu F^{\mu\nu} - J^{\nu 1}) B^1_\mu \mathrm{d}^4x, \quad (24)$$

$$0 = \frac{\partial S}{\partial \epsilon_2}|_{\epsilon_1=\epsilon_2=0} = \int_\Omega (\partial_\mu \widetilde{F}^{\mu\nu} - J^{\nu 2}) B^2_\mu \mathrm{d}^4x. \quad (25)$$

Because B^1_μ and B^2_μ are arbitrary, then from Eqs. (24) and (25), we obtain Maxwell's Eq. (11) again.

Furthermore, if we change the coordinate of the dyon in the form

$$x^\mu \Rightarrow x^\mu + \epsilon_0 y^\mu, \quad \text{with} \quad y^\mu(\lambda_1) = y^\mu(\lambda_2) = 0, \quad (26)$$

we find

$$\frac{\partial S}{\partial \epsilon_0}\Big|_{\epsilon_0=0} = \int_{\tau_1}^{\tau_2} \Big(-m\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} + qF^\alpha_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} - g\widetilde{F}^\alpha_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\Big) y_\alpha \mathrm{d}\tau. \quad (27)$$

Since y_α is arbitrary, then from $\frac{\partial S}{\partial \epsilon_0}|_{\epsilon_0=0} = 0$ we obtain

$$m\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} = qF^\alpha_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} - g\widetilde{F}^\alpha_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}. \quad (28)$$

This is just the equation of motion of a dyon moving in the electromagnetic field. From this equation we can find that the Lorentz force the dyon acquires in the magnetic field can be represented in terms of field strengths as

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + g(\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E}). \quad (29)$$

We would like to stress that the general Lorentz force has also the $SO(2)$ electromagnetic duality symmetry.

In summary, we have used the formulation of Ref. [14] to explain why the classical electromagnetic field theory in the presence of a magnetic source has exactly the $SO(2)$ duality symmetry. Then we find a proper Lagrangian formulation for the theory, and lastly we have used the action principle of the system of dyons to derive both Maxwell's equation and the equation of motion for the dyon. From this equation of motion we have obtained the general Lorentz force for a dyon moving in the electromagnetic field.

As a consistency check of our formulation we see that for $g = 0$ and $J^{\mu 2} = 0$ (no magnetic sources), from Eqs. (14) and (15) we can set $A^2_\mu = 0$, and so $F_{\mu\nu} = F^1_{\mu\nu} +^* F^2_{\mu\nu} \Rightarrow F^1_{\mu\nu}$. This means that our formulation contains standard electrodynamics as a particular case. For $q = 0$ and $J^{\mu 1} = 0$ (no electric sources), one has $A^1_\mu = 0$, and then $F_{\mu\nu} = F^1_{\mu\nu} +^* F^2_{\mu\nu} \Rightarrow^* F^2_{\mu\nu}$, and the Lagrangian becomes:

$$L = -\frac{1}{4}(^*F^2_{\mu\nu})^2 - A^2_\mu J^{\mu 2} = \frac{1}{4}(F^2_{\mu\nu})^2 - A^2_\mu J^{\mu 2}.$$

Thus in this case the formulation is completely parallel to the magnetic source free case.

We would like to thank S. Carneiro for his kind and valuable comments and for letting us know Ref. [5]. Part of this work was carried out during K. Li's research visiting to Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

References

[1] Kiritsis K 1999 *Preprint* hep-th/9911525 and references therein
[2] Schwarz J H and Seiberg N 1999 *Rev. Mod. Phys.* **71** S112
[3] Brandt R A, Neri F and Zwanziger D 1979 *Phys. Rev.* D **19** 1153
[4] Dirac P A M 1931 *Proc. R. Soc. London* A **133** 60
[5] Cardoso de Mello, Carneiro S and Nemes M C 1996 *Phys. Lett.* B **384** 197
Carneiro S 1998 *J. High Energy Phys.* **07** 022
[6] Lechner K and Marchetti P A 2000 *Nucl. Phys.* B **569** 529
[7] Gamberg L and Milton K A 2000 *Phys. Rev.* D **61** 75013
[8] Méndez J 1999 *Preprint* hep-th/9911188
[9] Galvao C A P and Mignaco J A 2000 *Preprint* hep-th/0002182
[10] Donev S 2000 *Preprint* hep-th/0006208
[11] Jackson J D 1999 *Classical Electrodynamics* (Singapore: Wiley)
[12] Felsager B 1998 *Geometry, Particles and Fields* (New York: Springer)
[13] Itzykson C and Zuber J B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
[14] Li K and Naón Carlos M 2001 *Mod. Phys. Lett.* A **16** 1671
[15] Shi Z Q and Ni G J 2002 *Chin. Phys. Lett.* **19** 1427
[16] Huang S Z and Ruan T N 2002 *Chin. Phys. Lett.* **19** 1676