Mixed estimates for singular integrals on weighted Hardy spaces

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Abstract

In this paper we give quantitative bounds for the norms of different kinds of singular integral operators on weighted Hardy spaces H_w^p, where $0 < p \leq 1$ and w is a weight in the Muckenhoupt A_∞ class. We deal with Fourier multiplier operators, Calderón–Zygmund operators of homogeneous type which are particular cases of the first ones, and, more generally, we study singular integrals of convolution type. In order to prove mixed estimates in the setting of weighted Hardy spaces, we need to introduce several characterizations of weighted Hardy spaces by means of square functions, Littlewood–Paley functions and the grand maximal function. We also establish explicit quantitative bounds depending on the weight w when switching between the H_w^p-norms defined by the Littlewood–Paley–Stein square function and its discrete version, and also by applying the mixed bound $A_q - A_\infty$ result for the vector-valued extension of the Hardy–Littlewood maximal operator given in Buckley (Trans Am Math Soc 340(1):253–272, 1993).

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1 Introduction

Boundedness of singular integrals in weighted norms arises in the analysis of PDE's. It is a classical result that many operators in harmonic analysis are bounded on $L_w^p(\mathbb{R}^n)$, $1 < p < \infty$, if the weight w belongs to the so-called A_p Muckenhoupt class. The sharp dependence of the corresponding $L_w^p(\mathbb{R}^n)$ operator norms in terms of the A_p characteristic of w has been studied for different operators in the last few years. We recall that a weight w, that is, a non-negative locally integrable function, belongs to the Muckenhoupt A_p class for $1 < p < \infty$ if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the axes. In the case $p = 1$ we say that $w \in A_1$ if

$$Mw \leq kw \quad a.e.$$

and we define $[w]_{A_1} = \inf \{k > 0 : Mw \leq kw \ a.e.\}$. For $p = \infty$, we say that $w \in A_\infty$ if

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty,$$

where $w(Q) = \int_Q w$. We call $[w]_{A_p}$ the A_p constant or characteristic of the weight w. It is well-known that there exists a constant that depends on the dimension n such that $[w]_{A_\infty} \leq c_n[w]_{A_p}$, for $p \geq 1$. All the above constants can be defined over balls instead of cubes, and they are the same up to a dimensional constant.

In [1], Buckley proved that for the Hardy–Littlewood maximal operator M and $1 < p < \infty$

$$\|M\|_{L_w^p(\mathbb{R}^n)} \leq C[w]_{A_p}^{\frac{1}{p-1}} \tag{1.1}$$

and this result is sharp in the sense that the right-hand side of (1.1) cannot be replaced by $\varphi([w]_{A_p})$ for any positive nondecreasing function φ growing more slowly than $t^{\frac{1}{p-1}}$. In the same article [1], it was shown that for the convolution Calderón–Zygmund singular integral operators T and $1 < p < \infty$,

$$\|T\|_{L_w^p(\mathbb{R}^n)} \leq C[w]_{A_p}^{\frac{p}{p-1}}$$

and the best power of $[w]_{A_p}$ is at least $\max \left\{1, \frac{1}{p-1}\right\}$. In the case $p = 2$ and $T = H$ is the Hilbert transform, Petermichl and Pott improved it to the power $\frac{3}{2}$ (see [28]) and, later on, Petermichl in [25] obtained the best possible linear dependence for $p \geq 2$. For the case $1 < p < 2$, that dependence of the $[w]_{A_p}$ constant is a consequence of the sharp extrapolation proved in [5]. For the Riesz transforms, in [26], the author

obtained the same linear bound. From [29], we also know that the linear dependence on the A_p constant is the sharpest for the Ahlfors–Beurling operator.

Then, proving a linear dependence on the A_2 constant for general Calderón–Zygmund operators became an important problem in the area that was finally solved by Hytönen in [15]. This result was improved in [13] by means of what it is now called mixed $A_p - A_\infty$ estimates, as the ones obtained in that paper. Namely, for a Calderón–Zygmund operator T, Hytönen and Pérez showed in [13] that

$$\|T\|_{L_w^2(\mathbb{R}^n)} \leq c \, [w]_{A_2}^{\frac{1}{2}} \max \left\{ [w]_{A_\infty}, [w^{-1}]_{A_\infty} \right\}^{\frac{1}{2}}.$$

Afterwards, this was extended for $p > 1$ in [11].

When considering $p = 1$, it is well-known that the above operators are unbounded in L_w^1, for $w \in A_1$, but a weak type estimate can be stated (see [21] and [13]) with the following mixed bound for the norm of the Calderón–Zygmund operator T

$$\|T\|_{L_w^1 \to L_w^{1,\infty}} \leq C[w]_{A_1}(1 + \log[w]_{A_\infty}).$$

Given that the singular integrals we shall be dealing with are not bounded for $p = 1$, a natural goal is to define a subspace of L_w^1 whose image under any singular integral is in L_w^1. With this purpose in mind, the Hardy space H_w^1 is introduced. Next, one can extend the definition for any $p > 0$, obtaining the space H_w^p, where $w \in A_q$ for some $q > 1$ (for more details, see the following sections). It is well-known that for $p > 1$ these spaces coincide with L_w^p. Then, the case of interest is, actually, $0 < p \leq 1$. Hardy spaces give rise to the so-called endpoints estimates for the L_w^p boundedness of singular integrals, and this fact has motivated us to study the boundedness of singular integrals in H_w^p for Muckenhoupt weights.

There are several ways to define Hardy spaces. According to each definition, different norms arise and, in most cases, they are equivalent up to constants that depend on the weight w. The weighted Hardy spaces were introduced by García-Cuerva in his Ph.D. dissertation [9]. A complete book about these spaces is [30] where the authors define the weighted Hardy spaces by using the non-tangential maximal functions. They introduce these spaces for doubling weights, namely, that there exists a constant $C > 0$ such that $w(2Q) \leq Cw(Q)$ for any cube Q. Moreover, they study the boundedness of Calderón–Zygmund singular integrals on weighted Hardy spaces.

More recently, Lu and Zhu [22] apply a discrete version of Calderón's reproducing formula and Littlewood–Paley theory with weights to establish $H_w^p(\mathbb{R}^n) \to H_w^p(\mathbb{R}^n)$ ($0 < p \leq \infty$) boundedness results for singular integrals of convolution type when $w \in A_\infty$. Several authors have studied this problem requiring stronger hypotheses on the weight (see, for example, [18,20]).

The authors in [22, Theorem 1.1] show the boundedness of some sort of singular integrals in $H_w^p(\mathbb{R}^n)$ for $w \in A_\infty$. They claim that the bounds obtained in their article are quantitative but they fail in giving the explicit constant depending on the weight when switching from one H_w^p-norm to another (see [22, Proposition 2.1 and Corollary 2.1]); there is actually a dependence on the A_∞ constant of the weight w. Therefore, a difficulty in obtaining quantitative bounds for norms of operators in this setting is

that one needs to be very careful when interchanging the norms in $H_w^p(\mathbb{R}^n)$ since there will be constants relating them that might depend on the weight. When one is just concerned about the boundedness of an operator, the constant can be disregarded.

In this article we shall precise the quantitative bounds obtained in [22]. While they provide bounds for the norm of the singular integral operator, they are not completely quantitative and accurate. In this sense, we first study multiplier operators T_m where m is a Fourier multiplier, and obtain bounds for $\|T_m\|_{H_w^p \to H_w^p}$ by means of the norm defined through a square function, where $w \in A_\infty$. In this case we obtain estimates for the same range of p for which the boundedness of these operators in H_w^p is well-known (see [30, Chapter XI, Theorem 6]). As a particular case of multiplier operators, we have the Calderón–Zygmund singular integrals of homogeneous type $Tf = K * f$, where $K(x) = \frac{\Omega(x)}{|x|^n}$, provided that Ω is an homogeneous function of degree zero, infinitely differentiable on the unit sphere S^{n-1} and $\int_{S^{n-1}} \Omega = 0$. However, in Section 3 we are able to get quantitative estimates for those operators using the characterization of $H_w^p(\mathbb{R}^n)$ by the grand maximal function. This choice on the definition of the weighted Hardy space allows us to prove quantitative bounds for every $p \in (0, 1]$. Using estimates for the grand maximal function given by Lerner and Wilson [19,33,34] we get mixed bounds for the norm in $H_w^p(\mathbb{R}^n)$ when $w \in A_1$. Next, as a consequence of an extrapolation result from [3] we can extend this last result for $w \in A_\infty$.

Afterwards, we consider singular integrals of convolution type with more general kernels that include the homogeneous type operators mentioned above. We follow the ideas by Lu and Zhu [22] and use the mixed bound $A_q - A_\infty$ for the vector-valued Hardy–Littlewood maximal operator obtained in [2], which is an improvement of the bound for this operator used by the authors. Moreover, we take special care of the equivalence between $H_w^p(\mathbb{R}^n)$ norms in order to get the correct dependence on the weight w.

2 Quantitative bounds for multiplier operators with A_∞ weights

From now on, for simplicity, we will denote H_w^p and L_w^p instead of $H_w^p(\mathbb{R}^n)$ and $L_w^p(\mathbb{R}^n)$, respectively.

Set

$$\Gamma_\beta(x) = \left\{ (y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\beta \right\}, \quad \beta > 0.$$

Let $P(x) = \frac{c_n}{(1+|x|^2)^{\frac{n+1}{2}}}$ be the Poisson kernel and, for $t > 0$, set $P_t(x) = \frac{1}{t^n} P(\frac{x}{t})$. Let $f \in \mathscr{S}$ and define $f(x, t) = P_t * f(x)$ its Poisson integral. If we consider the maximal function

$$N_\beta f(x) = \sup_{(y,t) \in \Gamma_\beta(x)} |f(y, t)|,$$

the weighted Hardy space H_w^p can be defined in terms of N_β for $w \in A_\infty$ and any $\beta > 0$. This means, $f \in H_w^p$ if and only if $N_\beta f \in L_w^p$. The norm is defined in the

natural way $\|f\|_{H_w^p} := \|N_\beta f\|_{L_w^p}$. We may refer to H_w^p through the above definition, so the different equivalent norms used in our results will be related to it.

Let us denote by $\mathscr{S}$ the Schwartz class of functions and consider the subset $\hat{\mathscr{D}}_0 := \{f \in \mathscr{S} : \text{supp}(\hat{f}) \text{ is compact and } 0 \notin \text{supp}\hat{f}\}$. A bounded function m defined on $\mathbb{R}^n \setminus \{0\}$ is called a multiplier. A multiplier operator T is defined by means of the Fourier transform as

$$(Tf)\hat{\ } = m\hat{f}$$

for $f \in \hat{\mathscr{D}}_0(\mathbb{R}^n)$, so m is also called a Fourier multiplier. If T is bounded on H_w^p, since $\hat{\mathscr{D}}_0$ is dense in H_w^p for w a doubling weight ([30, Chapter VII]), T admits an extension as a continuous operator on H_w^p. We shall denote $T = T_m$ to indicate the corresponding Fourier multiplier of T.

In this section, we consider the following functions. We define the square function

$$S_\beta(f)(x) = \left(\int_{\Gamma_\beta(x)} |t\nabla f(y,t)|^2 dy \frac{dt}{t^{n+1}} \right)^{1/2},$$

for $\beta > 0$, and the Littlewood–Paley function

$$g_\mu^*(f)(x) = \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\frac{t}{t+|x-y|} \right)^{\mu n} |t\nabla f(y,t)|^2 dy \frac{dt}{t^{n+1}} \right)^{\frac{1}{2}}$$

for $\mu > 1$, where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n}, \frac{d}{dt})$.

Assume that $m \in C^{n+1}(\mathbb{R}^n \setminus \{0\})$ and for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq n+1$,

$$\sup_{0 < R < \infty} R^{2|\alpha|-n} \int_{R \leq |x| \leq 2R} |D^\alpha m(x)|^2 dx < \infty.$$

This condition is known in the literature as $m \in M(2, n+1)$. Then, it is well-known (see [31, p. 233]) that

$$S_1(T_m f)(x) \leq C g_{2+\frac{2}{n}}^*(f)(x), \quad x \in \mathbb{R}^n, \tag{2.1}$$

for any $f \in \hat{\mathscr{D}}_0$. This inequality will be the key ingredient in order to obtain quantitative bounds for $\|T_m\|_{H_w^p \to H_w^p}$.

As we said in the introduction, changing the H_w^p-norm for an equivalent one can lead to a different bound for the operator norm. Therefore, we shall fix a norm according to the operator we are dealing with. Since the L_w^p norms of N_β and S_β are equivalent by [10, Theorem 1], we can also characterize H_w^p using S_β, in particular, using S_1. Thus, in this section, we denote the weighted Hardy space by $\mathscr{H}_w^p$ and fix the norm $\|f\|_{\mathscr{H}_w^p} = \|S_1(f)\|_{L_w^p}$, $0 < p \leq 1$. Note that we choose to present this notation

because it seemed to us that it made the exposition clearer. However, these space are all equivalent.

To give a quantitative bound for $\|T_m\|_{\mathscr{H}_w^p \to \mathscr{H}_w^p}$, in view of the pointwise inequality (2.1), it will be enough to prove that $g_\mu^*(f)$ can be estimated by $S_1(f)$ in norm L_w^p for $0 < p \leq 1$. This result is in the literature, more precisely, it can be found in [30, Theorem 4], but we shall revisit its proof in order to give the explicit constant depending on the weight. Our result is the following.

Theorem 1 *Let $0 < p \leq 1$, $w \in A_\infty$ and $0 < r < \min\{\frac{p}{q_w}, 1\}$, where $q_w = \inf\{q > 1 : w \in A_q\}$. Then,*

$$\int_{\mathbb{R}^n} (g_\mu^*(f)(x))^p w(x) dx \leq C(n, p, r, \mu)[w]_{A_{\frac{p}{r}}} \int_{\mathbb{R}^n} (S_1(f)(x))^p w(x) dx \quad (2.2)$$

for each $\mu > \frac{2}{r}$.

Therefore, we have the following quantitative bound for multiplier operators on $\mathscr{H}_w^p$, by combining (2.1) and theorem above. Notice that we can choose r to verify $2 + \frac{2}{n} > \frac{2}{r}$, that is, $r > \frac{n}{n+1}$. Thus, for $\frac{n}{n+1} < r < \min\{\frac{p}{q_w}, 1\}$, both inequalities hold. Moreover, if $r > \frac{1}{2}$, they are valid for any dimension.

Theorem 2 *Let $\frac{n}{n+1} q_w < p \leq 1$, $w \in A_\infty$ and $0 < r < \min\{\frac{p}{q_w}, 1\}$. Then, if $m \in M(2, n+1)$,*

$$\|T_m\|_{\mathscr{H}_w^p \to \mathscr{H}_w^p} \leq C[w]_{A_{\frac{p}{r}}}^{\frac{1}{p}},$$

where the constant C does not depend on the weight.

Proof of Theorem 1 Let us introduce some notation and properties that we will use to prove (2.2).

First, notice that, since $w \in A_\infty$, from the definition of q_w there exists $r > \frac{n}{n+1}$ such that $w \in A_{\frac{p}{r}}$. Therefore, the weight satisfies the following doubling condition:

$$w(B(x, ar)) \leq c_d a^{\frac{np}{r}} w(B(x, r)), \quad \forall r > 0, a \geq 1.$$

c_d is called the doubling constant of w and $B(x, r)$ denotes the ball centered at x and radius r. It can be shown that

$$c_d = \|M\|_{L_w^{\frac{p}{r}} \to L_w^{\frac{p}{r}, \infty}}^{\frac{p}{r}} \approx C(n, p, r)[w]_{A_{\frac{p}{r}}}$$

which means that this weak type norm of the Hardy–Littlewood maximal operator is equivalent to the constant of the weight up to a constant that depends on n, r and p (see [1]). We shall also make use of the Hardy–Littlewood maximal operator associated to the measure $d\nu(x) = w(x)dx$ given by

$$M_w(f)(x) = \sup_{B \ni x} \frac{1}{w(B)} \int_B |f(x)| w(x) dx, \quad f \in L_{\text{loc}}^1(w),$$

where the supremum is taken over the balls B containing x. It is known that M_w is of weak type $(1, 1)$ respect to the measure $w(x)\,dx$ with constant C_n, only depending on n, not on w.

Let us proceed with the proof of (2.2). It is easy to see that

$$g_\mu^*(f)(x) \le C(\mu, n) \sum_{k=0}^\infty 2^{-k\mu n} (S_{2^k}(f)(x))^2$$

for every $\mu > 1$ (c.f. [30, p. 54]).

Since $\frac{p}{2} < 1$, we have

$$\int_{\mathbb{R}^n} (g_\mu^*(f)(x))^p w(x)dx \le C(\mu, n) \int_{\mathbb{R}^n} \left(\sum_{k=0}^\infty 2^{-k\mu n} (S_{2^k}(f)(x))^2 \right)^{\frac{p}{2}} w(x)dx$$

$$\le C \int_{\mathbb{R}^n} \sum_{k=0}^\infty 2^{-k\mu n \frac{p}{2}} (S_{2^k}(f)(x))^p w(x)dx$$

$$= C \sum_{k=0}^\infty 2^{-\frac{k\mu np}{2}} \int_{\mathbb{R}^n} (S_{2^k}(f)(x))^p w(x)dx.$$

We will show that the L_w^p-norm of S_{2^k} can be replaced by the L_w^p-norm of S_1 times a constant that depends on the weight w.

Fix $k \ge 0$. Using a similar notation to [30], we define the level sets

$$\mathscr{O}_s = \{x \in \mathbb{R}^n : S_1(f)(x) > s\}, \ \mathscr{O}_s' = \{x \in \mathbb{R}^n : S_{2^k}(f)(x) > s\}$$

and

$$\mathscr{U}_s = \{x \in \mathbb{R}^n : M_w(\chi_{\mathscr{O}_s})(x) > (2c_d(2^k)^{\frac{np}{r}})^{-1}\}.$$

Given $r, s > 0$, we can write

$$w(\mathscr{O}_{rs}') \le w(\mathscr{U}_s) + w(\mathscr{O}_{rs}' \cap \mathscr{U}_s^c).$$

From the weak type $(1, 1)$ of M_w,

$$w(\mathscr{U}_s) = w(\{x \in \mathbb{R}^n : M_w(\chi_{\mathscr{O}_s})(x) > (2c_d(2^k)^{\frac{np}{r}})^{-1}\}) \le C_n 2c_d 2^{\frac{knp}{r}} w(\mathscr{O}_s).$$

On the other hand,

$$w(\mathscr{O}_{rs}' \cap \mathscr{U}_s^c) = (rs)^{-2} \int_{\mathscr{O}_{rs}' \cap \mathscr{U}_s^c} (rs)^2 w(x)dx \le (rs)^{-2} \int_{\mathscr{U}_s^c} \left(S_{2^k}(f)(x)\right)^2 w(x)dx.$$

(2.3)

From [30, Chapter IV, Lemma 7], we have that $\Gamma_{2^k}(\mathscr{U}_s^c) \subset \Gamma_1(\mathscr{O}_s^c)$, where $\Gamma_\beta(E) :=$ $\cup_{x \in E} \Gamma_\beta(x)$, and also, from that result,

$$w(B(y, 2^k t) \cap \mathscr{U}_s^c) \leq 2c_d 2^{\frac{knp}{r}} w(B(y, t) \cap \mathscr{O}_s^c), \quad t > 0.$$

These two properties allow us to deduce the following bound for the integral on the right-hand side of (2.3):

$$\int_{\mathscr{U}_s^c} \left(S_{2^k}(f)(x)\right)^2 w(x)dx$$
$$= \int_0^\infty \int_{\Gamma_{2^k}(\mathscr{U}_s^c)} |t\nabla f(y, t)|^2 w(B(y, 2^k t) \cap \mathscr{U}_s^c) dy \frac{dt}{t^{n+1}}$$
$$\leq 2c_d 2^{\frac{knp}{r}} \int_0^\infty \int_{\Gamma_1(\mathscr{O}_s^c)} |t\nabla f(y, t)|^2 w(B(y, t) \cap \mathscr{O}_s^c) dy \frac{dt}{t^{n+1}}$$
$$= 2c_d 2^{\frac{knp}{r}} \int_{\mathscr{O}_s^c} (S_1(f)(x))^2 w(x)dx.$$

Thus, inserting this into (2.3),

$$w(\mathscr{O}_{rs}' \cap \mathscr{U}_s^c) \leq (rs)^{-2} 2c_d 2^{\frac{knp}{r}} \int_{\mathscr{O}_s^c} (S_1(f)(x))^2 w(x)dx$$
$$= 4c_d 2^{\frac{knp}{r}} (rs)^{-2} \int_0^\infty tw(\{t < S_1(f)(x) \leq s\})dt$$
$$\leq 4c_d 2^{\frac{knp}{r}} (rs)^{-2} \int_0^s tw(\mathscr{O}_t)dt.$$

Hence,

$$w(\mathscr{O}_{rs}') \leq 4c_d 2^{\frac{knp}{r}} (rs)^{-2} \int_0^s tw(\mathscr{O}_t)dt + C_n 2c_d 2^{\frac{knp}{r}} w(\mathscr{O}_s).$$

By [30, Chapter IV, Lemma 9] with $c_1 = 4c_d 2^{\frac{knp}{r}}$ and $c_2 = C_n 2c_d 2^{\frac{knp}{r}}$,

$$\int_{\mathbb{R}^n} (S_{2^k}(f)(x))^p w(x)dx \leq C(n, p, r)c_d 2^{\frac{knp}{r}} \int_{\mathbb{R}^n} (S_1(f)(x))^p w(x)dx$$

as we claimed.

Then,

$$\int_{\mathbb{R}^n} (g_\mu^*(f)(x))^p w(x)dx \leq C \sum_{k=0}^\infty 2^{-\frac{k\mu np}{2}} \int_{\mathbb{R}^n} (S_{2^k}(f)(x))^p w(x)dx$$
$$\leq C(n, p, r)c_d \int_{\mathbb{R}^n} (S_1(f)(x))^p w(x)dx \sum_{k=0}^\infty 2^{-\frac{k\mu np}{2}} 2^{\frac{knp}{r}}$$

$$\leq C(n, p, r)[w]_{A_{\frac{p}{r}}} \int_{\mathbb{R}^n} (S_1(f)(x))^p w(x) dx$$

where the series converges since $\mu > \frac{2}{r}$. The proof is now finished. $\quad\square$

Remark 1 Note that in each step we are using the optimal constant c_d for the doubling condition which give us a good constant in last theorem.

Remark 2 We also observe that the restriction on p, which translates in a restriction for r, coincides with the sufficient conditions given in [30, Chapter XI, Theorem 6] for the boundedness of T_m on $\mathscr{H}_w^p$.

3 Quantitative bounds for singular integrals of homogeneous type

In this section, we first obtain quantitative bounds for the norm of T on weighted Hardy spaces, when $w \in A_1$, for singular integrals of homogeneous type. Next, using an extrapolation argument, we will extend these bounds for $w \in A_\infty$.

We will need to introduce different functions in order to give the definition of the weighted Hardy spaces and several characterizations (see [35]).

For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\alpha > 0$, let

$$A_\alpha f(y, t) = \sup_{\varphi \in \mathscr{C}_\alpha} |f * \varphi_t(y)|,$$

where $\varphi_t(y) = t^{-n}\varphi(\frac{y}{t})$ and $\mathscr{C}_\alpha$ is the collection of functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ supported in the unit ball with integral zero, such that $|\varphi(y) - \varphi(y')| \leq |y - y'|^\alpha$ for every $y, y' \in \mathbb{R}^n$.

First, we define the square function of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$S_{\psi,\beta}(f)(x) = \left(\int_{\Gamma_\beta(x)} |f * \psi_t(y)|^2 \frac{dy\, dt}{t^{n+1}} \right)^{1/2}$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ is radial, non trivial, supported in the unit ball with $\int \psi = 0$ and such that

$$\int_0^\infty |\hat{\psi}(\xi t)|^2 \frac{dt}{t} = 1, \qquad \forall \xi \neq 0.$$

The intrinsic square function is defined as

$$g_{\alpha,\mu}^*(f)(x) = \left(\int_{\mathbb{R}_+^{n+1}} (A_\alpha(f)(y, t))^2 \left(\frac{t}{t + |x - y|} \right)^{\mu n} \frac{dy\, dt}{t^{n+1}} \right)^{1/2}, \qquad \alpha, \mu > 0,$$

and the Littlewood–Paley function is given by

$$g_{\phi,\mu}^*(f)(x) = \left(\int_{\mathbb{R}_+^{n+1}} |f * \phi_t(y)|^2 \left(\frac{t}{t + |x - y|} \right)^{\mu n} \frac{dy\, dt}{t^{n+1}} \right)^{1/2}, \quad \mu > 0,$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$, is radial, with support contained in the ball $\{x : |x| \leq 1/2\}$ and $\int \phi = 0$.

Finally, we define the grand maximal function G_N for $f \in \mathscr{S}'$, the set of tempered distributions. Given a natural number N, let $\mathscr{A}_N$ denote the set of all functions $\eta \in C^\infty(\mathbb{R}^n)$ with supp $\eta \subset \{|x| \leq 1\}$ and

$$\left\| \sum_{|\beta| \leq N} |D^\beta \eta| \right\|_\infty \leq 1$$

where β is a multi-index. Then

$$G_N(f)(x) := \sup_{t>0} \sup_{\eta \in \mathscr{A}_N} |f * \eta_t(x)|.$$

It is well-known that one can define the weighted Hardy space for $0 < p < \infty$, as the set

$$\mathscr{H}_w^p = \left\{ f \in \mathscr{S}' : G_N(f) \in L_w^p \right\}. \tag{3.1}$$

The norm is given by $\|f\|_{\mathscr{H}_w^p} := \|G_N(f)\|_{L_w^p}$. Note that this definition of H_w^p is possible according to [24,36]. We fix the norm given by G_N for the results of the present section.

We will deal with homogeneous type operators. Let us consider Ω an homogeneous function of degree 0, infinitely differentiable on the unit sphere with

$$\int_{|x|=1} \Omega(x)\, d\sigma(x) = 0.$$

We consider the following singular integral operator given by

$$Tf(x) := \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y)}{|y|^n} f(x - y)\, dy.$$

These type of operators are known as homogeneous singular integrals and they are particular cases of multiplier operators. More precisely, if Ω is as above, there exists a multiplier (a bounded function outside the origin) $m \in C^\infty(S^{n-1})$ which is homogeneous of degree 0 (see [31, Chapter III, Theorem 6]). Thus, under some extra assumptions on Ω, one can have $m \in M(2, n + 1)$ (see [17] or [30]) and T will be in the conditions of the previous section. We will show now that with no other hypotheses on Ω than the given above, we get mixed estimates and reach all the values of $p \in (0, 1]$.

We state the following pointwise estimate that holds for singular integrals and it can be found in [33] (see also [19]). It relates the square function of the homogeneous singular integral applied to f with the Littlewood–Paley function of f.

Theorem 3 [33] *Let us consider an homogeneous singular integral operator T as defined above.*

Let $\phi \in C^\infty(\mathbb{R}^n)$ be radial, $\operatorname{supp}(\phi) \subset \{|x| \le 1/2\}$, satisfying that there exist positive constants c and ξ such that

$$\int_s^\infty |\hat{\phi}(t, 0, \ldots, 0)|^2 \frac{dt}{t} \ge c(1+s)^{-\xi} \tag{3.2}$$

for every $s > 0$, and that $\int P(x)\phi(x)dx = 0$ for every polynomial P of degree $\le 2n$. Let $\psi = \phi * \phi$. There exists a constant $C(\beta, \mu, \phi)$ and $\mu > 3$ such that for all $\beta > 0$

$$S_{\psi,\beta}(Tf) \le C(\beta, \mu, \phi)g_{\phi,\mu}^*(f). \tag{3.3}$$

We will use the following result given by Lerner [19] in order to obtain bounds for the norms on weighted Hardy spaces.

Proposition 1 [19] *Let ψ be as in the definition of the square function and also verifying (3.2). Then there exist $N \in \mathbb{N}$ depending on ξ and n, and β depending on n such that for any weight w and for any f with $G_N(f) \in L_w^p$,*

$$\|G_N(f)\|_{L_w^p} \le c[w]_{A_\infty}^{\frac{1}{2}} \|S_{\psi,\beta}(f)\|_{L_w^p} \tag{3.4}$$

where c does not depend on either f or w.

Remark 3 This result was obtained by Wilson in [33] but in [19] the author showed the bound in terms of $[w]_{A_\infty}$ which was defined in the Introduction. Inequality (3.4) with f instead of $G_N(f)$ is essentially contained in [33].

We will obtain a mixed $A_1 - A_\infty$ estimate for the norm $\|T\|_{\mathscr{H}_w^p \to \mathscr{H}_w^p}$ when $0 < p \le 1$. For that, we shall use the following weighted inequality.

Lemma 1 [34] *Let $0 < p \le 1$. There exists $\widetilde{\mu} = \tilde{\mu}(p) > 0$ such that for any $\mu > \widetilde{\mu}$ and any weight w we have*

$$\left(\int_{\mathbb{R}^n} g_{\phi,\mu}^*(f)^p w(x)\,dx\right)^{\frac{1}{p}} \le C(n, N, p)\left(\int_{\mathbb{R}^n} |G_N(f)|^p M w(x)\,dx\right)^{\frac{1}{p}} \tag{3.5}$$

where the constant does not depend on w.

Remark 4 If we consider $w \in A_1$, by the lemma above we obtain

$$\left(\int_{\mathbb{R}^n} g_{\phi,\mu}^*(f)^p w(x)\,dx\right)^{\frac{1}{p}} \le C(n, N, p)[w]_{A_1}^{\frac{1}{p}}\left(\int_{\mathbb{R}^n} |G_N(f)|^p w(x)\,dx\right)^{\frac{1}{p}} \tag{3.6}$$

where the constant does not depend on w.

Theorem 4 *If $0 < p \leq 1$, $w \in A_1$ and T is a singular integral as in Theorem* 3, *then*

$$\|T\|_{\mathscr{H}^p_w \to \mathscr{H}^p_w} \leq C[w]^{\frac{1}{p}}_{A_1}[w]^{\frac{1}{2}}_{A_\infty},$$

where the constant C does not depend on the weight.

Proof Let ϕ and ψ be as in Theorem 3. Then, for this choice of ψ, from Proposition 1 there exists $N \in \mathbb{N}$ such that

$$\|G_N(Tf)\|_{L^p_w} \leq C[w]^{\frac{1}{2}}_{A_\infty}\|S_{\psi,\beta}(Tf)\|_{L^p_w}$$

where the constant C does not depend on w. Now, using the pointwise inequality (3.3) and last remark, with $\mu > \max\{3, \tilde{\mu}\}$, for $\tilde{\mu}$ as in Lemma 1, we have

$$\|S_{\psi,\beta}(Tf)\|_{L^p_w} \leq C(\beta,\mu,\phi)\|g^*_{\phi,\mu}(f)\|_{L^p_w} \leq C(\beta,\mu,\phi,n,N,r)[w]^{\frac{1}{p}}_{A_1}\|G_N(f)\|_{L^p_w},$$

so we obtain the desired inequality.   □

Now we will extend inequality (3.6) to the case of A_∞ weights.

Proposition 2 *If $0 < p \leq 1$, $w \in A_\infty$. Let $0 < r < \min\{\frac{p}{q_w}, 1\}$. Then, there exists $\tilde{\mu} = \tilde{\mu}(p) > 0$ such that for any $\mu > \tilde{\mu}$, we have*

$$\left(\int_{\mathbb{R}^n} g^*_{\phi,\mu}(f)^p w(x)\,dx\right)^{\frac{1}{p}} \leq C(n,N,p,r)[w]^{\frac{1}{p}}_{A_{\frac{p}{r}}}[w]^{\frac{1}{r}-\frac{1}{p}}_{A_\infty}\left(\int_{\mathbb{R}^n} |G_N(f)|^p w(x)\,dx\right)^{\frac{1}{p}}. \tag{3.7}$$

Proof We will give the proof in order to obtain the explicit constant in the inequality (3.7). However, the result is an immediate consequence of [3, Theorem 2.2].

We first notice that, since $w \in A_\infty$, from the definition of r we have that $w \in A_{\frac{p}{r}}$. We denote $q = \frac{p}{r}$.

We are going to use Rubio de Francia's algorithm (see [9]). Define the operator $\mathscr{R}$, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, by

$$\mathscr{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|^k_{L^{q'}_{w^{1-q'}}}},$$

where M^k is the Hardy–Littlewood maximal operator iterated k times if $k \geq 1$ and, for $k = 0$ is just the identity operator. It is immediate from the definition of $\mathscr{R}$ that

(a) $h(x) \leq \mathscr{R}h(x)$,

(b) $\|\mathscr{R}h\|_{L^{q'}_{w^{1-q'}}} \leq 2\|h\|_{L^{q'}_{w^{1-q'}}}$,

(c) $M(\mathscr{R}h)(x) \leq 2\|M\|_{L^{q'}_{w^{1-q'}}}\mathscr{R}h(x)$, meaning that $\mathscr{R}h \in A_1$ with constant independent of h.

We write

$$\|g_{\phi,\mu}^*(f)\|_{L_w^p}^r = \| \left(g_{\phi,\mu}^* f \right)^r \|_{L_w^q} = \sup_h \int_{\mathbb{R}^n} \left(g_{\phi,\mu}^* f(x) \right)^r h(x) w(x) dx,$$

where the supremum is taken over all non-negative functions $h \in L_w^{q'}$ with $\|h\|_{L_w^{q'}} = 1$. For any of such functions h, we have that $hw \in L_{w^{1-q'}}^{q'}$ with $\|hw\|_{L_{w^{1-q'}}^{q'}} = \|h\|_{L_w^{q'}} = 1$.

Let us fix $h \in L_w^{q'}$ with $\|h\|_{L_w^{q'}} = 1$. Then, from (a) and (c) with hw and (3.6), we have

$$\int_{\mathbb{R}^n} \left(g_{\phi,\mu}^* f(x) \right)^r h(x) w(x) dx \leq \int_{\mathbb{R}^n} \left(g_{\phi,\mu}^* f(x) \right)^r \mathscr{R}(hw)(x) dx$$

$$\leq C(n, N, r)[\mathscr{R}(hw)]_{A_1} \int_{\mathbb{R}^n} |G_N f(x)|^r \mathscr{R}(hw)(x) dx.$$

Note that, if $F \in L_w^p$ and $\mathscr{R}(hw) \in L_{w^{1-q'}}^{q'}$, $0 < w < \infty$ almost everywhere in the set where $f^r \mathscr{R}(hw) > 0$. Then, by applying Hölder's inequality with q and q', we get

$$\int_{\mathbb{R}^n} |F(x)|^r \mathscr{R}(hw)(x) dx = \int_{\mathbb{R}^n} |F(x)|^r w(x)^{\frac{1}{q}} \mathscr{R}(hw)(x) w(x)^{-\frac{1}{q}} dx$$

$$\leq \|F\|_{L_w^p}^r \|\mathscr{R}(hw)\|_{L_{w^{1-q'}}^{q'}}$$

$$\leq \|F\|_{L_w^p}^r 2\|hw\|_{L_{w^{1-q'}}^{q'}} = 2\|F\|_{L_w^p}^r$$

where we have used (b) and that $\|hw\|_{L_{w^{1-q'}}^{q'}} = 1$.

Particularly, we can apply the inequality above to $F = G_N(f)$, which we assume in L_w^p since, otherwise, there is nothing to prove. Thus, we get

$$\int_{\mathbb{R}^n} \left(g_{\phi,\mu}^* f(x) \right)^r h(x) w(x) dx \leq C[\mathscr{R}(hw)]_{A_1} \left(\int_{\mathbb{R}^n} |G_N f(x)|^p w(x) dx \right)^{\frac{r}{p}}.$$

In other words, we have proved that

$$\left(\int_{\mathbb{R}^n} g_{\phi,\mu}^*(f)^p w(x) \, dx \right)^{\frac{1}{p}} \leq C[\mathscr{R}(hw)]_{A_1}^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} |G_N(f)|^p w(x) \, dx \right)^{\frac{1}{p}}.$$

Now, we need to estimate $[\mathscr{R}(hw)]_{A_1}$ in terms of the weight w. According to (c), we have that $[\mathscr{R}(hw)]_{A_1} \leq 2\|M\|_{L_{w^{1-q'}}^{q'}}$. But, the $A_{q'} - A_\infty$ mixed estimate for the Hardy–Littlewood maximal operator is known, see [13, Corollary 1.10]. Hence,

$$[\mathscr{R}(hw)]_{A_1} \le c_n q \left([w^{1-q'}]_{A_{q'}} [w]_{A_\infty} \right)^{\frac{1}{q'}} = c_n \frac{p}{r} [w]_{A_q}^{\frac{1}{q}} [w]_{A_\infty}^{\frac{1}{q'}} = c_n \frac{p}{r} [w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} [w]_{A_\infty}^{1-\frac{r}{p}},$$

where we have used that $[w]_{A_q} = [w^{1-q'}]_{A_{q'}}^{q-1}$. Therefore,

$$[\mathscr{R}(hw)]_{A_1}^{\frac{1}{r}} \le c_n^{\frac{1}{r}} \left(\frac{p}{r} \right)^{\frac{1}{r}} [w]_{A_{\frac{p}{r}}}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{r} - \frac{1}{p}}$$

which completes the proof. □

Remark 5 An interesting open problem is to find the optimal value of q such that $w \in A_q$ in terms of the constant $[w]_{A_\infty}$. It was proved in [14] that $q > e^{c_n[w]_{A_\infty}}$, with $[w]_{A_q} \le e^{e^{c_n[w]_{A_\infty}}}$, where c_n is a dimensional constant. These type of embedding results has been studied in [7,16,23,27,32], and, for the one-dimesional case, they are sharper.

By reproducing the proof of Theorem 4 with (3.7) instead of (3.6), we obtain the following quantitative mixed bound for $\|T\|_{\mathscr{H}_w^p \to \mathscr{H}_w^p}$.

Theorem 5 *If $0 < p \le 1$, $w \in A_\infty$ and T is a singular integral of homogeneous type, then*

$$\|T\|_{\mathscr{H}_w^p \to \mathscr{H}_w^p} \le C[w]_{A_{\frac{p}{r}}}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{2} + \frac{1}{r} - \frac{1}{p}},$$

where the constant C does not depend on the weight.

Remark 6 Note that in last theorem the quantitative constant is bigger than the one proved in Theorem 2. However, for the operators of homogeneous type considered in this section the bounds can be obtained for every $p \in (0, 1]$.

4 Quantitative bounds for singular integrals of convolution type

In this section we obtain a quantitative bound for the $H_w^p \to H_w^p$ boundedness of singular integrals of convolution type. The idea is to use a Calderón reproducing formula and to improve the bound for the vector valued maximal Hardy–Littlewood operator given in [22, (2.2)]. In fact we use an $A_q - A_\infty$ mixed estimate obtained in [2] which is a sharp bound.

Let ψ be a Schwartz function on $\mathbb{R}^n$ which satisfies

$$\int_{\mathbb{R}^n} \psi(x) x^\alpha \, dx = 0$$

for all multi-indexes α and

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^{-j}\xi)|^2 = 1$$

for all $\xi \neq 0$.

Definition 1 A function f defined on $\mathbb{R}^n$ is said to be in $\mathscr{S}_M(\mathbb{R}^n)$ where M is a positive integer, if there exists $C > 0$ such that f satisfies the following conditions:

(1) For $|\alpha| \leq M - 1$,

$$|D^\alpha f(x)| \leq C \frac{1}{(1 + |x|)^{n+M+|\alpha|}};$$

(2) For $|x - x'| \leq \frac{1}{2}(1 + |x|)$ and $|\gamma| = M$

$$|D^\gamma f(x) - D^\gamma f(x')| \leq C \frac{|x - x'|}{(1 + |x|)^{n+2M}};$$

(3) For $|\alpha| \leq M - 1$

$$\int_{\mathbb{R}^n} f(x) x^\alpha \, dx = 0.$$

If $f \in \mathscr{S}_M(\mathbb{R}^n)$ its norm is then defined by

$$\|f\|_{\mathscr{S}_M(\mathbb{R}^n)} := \inf \{C : (1) \text{ and } (2) \text{ hold}\}.$$

To simplify notation we denote $\mathscr{S}_M(\mathbb{R}^n)$ by $\mathscr{S}_M$. For $f \in (\mathscr{S}_M)'$, the dual space of $\mathscr{S}_M$, define the Littlewood–Paley-Stein square function of f by

$$g(f)(x) = \left(\sum_{j \in \mathbb{Z}} |\psi_j * f(x)|^2 \right)^{\frac{1}{2}}$$

where $\psi_j(x) = 2^{-jn} \psi(2^{-j}x)$, with ψ as above.

For convenience, in this section we will modify the definition of the weighted Hardy space introduced in (3.1).

Definition 2 Let $0 < p < \infty$, $w \in A_\infty$. Let $M = [(\frac{2q_w}{p} - 1)n] + 1$, where $[\cdot]$ denotes the integer function . We define the weighted Hardy spaces as follows

$$\mathbb{H}_w^p = \{f \in (\mathscr{S}_M)' : g(f) \in L_w^p\}$$

and the norm of f in $\mathbb{H}_w^p$ is defined by

$$\|f\|_{\mathbb{H}_w^p} = \|g(f)\|_{L_w^p(\mathbb{R}^n)}.$$

Remark 7 In [6], the authors showed that the spaces H_w^p and $\mathbb{H}_w^p$, coincide (see [6, Theorem 2.1 and Remark 2.1]). Moreover, they proved (see [6, Proposition 3.2]) that the space $\mathscr{S}_M$ is dense in $\mathbb{H}_w^p$ for any A_∞ weight and $0 < p < \infty$.

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be supported in the unit ball and satisfying the same conditions as ψ for a fixed $M_0 \geq M$, where M is the same as in the definition of $\mathbb{H}_w^p$.

For $j \in \mathbb{Z}$, we will consider the set $\mathscr{D}_j$ of dyadic cubes Q of $\mathbb{R}^n$ with sides parallel to the coordinate axes and side-length $\ell(Q) = 2^{-j}$.

We define the discrete Littlewood–Paley function

$$g_d(f)(x) = \left(\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{D}_{j+N}} |\phi_j * f(x_Q)|^2 \chi_Q(x) \right)^{\frac{1}{2}}$$

where x_Q is a point in Q, $\phi_j(x) = 2^{-jn}\phi(2^j x)$ and N is some fixed large integer. We reserve the notation ϕ for g_d.

We will consider singular integrals of convolution type $Tf = K * f$ so that the kernel K satisfies

(1) For all multi-indexes α, and $x \neq 0$

$$|D^\alpha K(x)| \leq C_\alpha |x|^{-n-|\alpha|}; \tag{4.1}$$

(2) For any $0 < \varepsilon < N < \infty$

$$\left| \int_{\varepsilon < |x| < N} K(x)\, dx \right| \leq C. \tag{4.2}$$

The next two lemmas involving functions on $\mathscr{S}_M$ were proved in [22], and will be useful in the proof of Theorem 7.

Lemma 2 [22] *If $\psi, \varphi \in \mathscr{S}_M$, for any given positive integers a and b, there exists a constant C depending only on a and b, such that*

$$|\psi_i * \varphi_j(x)| \leq C \left(\frac{i}{j} \wedge \frac{j}{i} \right)^a \frac{(i \vee j)^b}{(i \vee j + |x|)^{n+b}}, \quad \forall x \in \mathbb{R}^n.$$

Lemma 3 [22] *Let $\psi \in \mathscr{S}_M$, $i, j \in \mathbb{Z}$ and let Q be a dyadic cube in $\mathscr{D}_{j+N}$. Set $a, b \in \mathbb{N}$. Then, for any $u, u^* \in Q$ and any r satisfying $\frac{n}{n+b} < r \leq 1$, we have*

$$\sum_{W \in \mathscr{D}_{i+N}} \frac{2^{-|j-i|a}|W|2^{-(j \wedge i)b}}{(2^{-j \wedge i} + |u - x_W|)^{n+b}}|\psi_i * f(x_W)|$$

$$\leq C2^{-|j-i|a}2^{(\frac{1}{r}-1)n(N+(i-j)_+)}M_r \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)|\chi_W \right)(u^*),$$

where $(i-j)_+ = \max\{i-j, 0\}$, x_W is a fixed point of the cube W, $M_r h = M(|h|^r)^{1/r}$ and C is a constant depending on the dimension n.

We will need the following discrete Calderón's reproducing formula, proved in [22, Theorem 2.1] (c.f. [8] or [12]).

Theorem 6 [22] *Suppose that ψ_j is as in the definition of g. Then for any $M \geq 1$, we can choose a large N depending on M and ψ such that the following discrete Calderón's reproducing identity*

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{D}_{j+N}} |Q| \widetilde{\psi}_j(x - x_Q) \psi_j * f(x_Q) \tag{4.3}$$

holds in $\mathscr{S}_M$ and in the dual space $(\mathscr{S}_M)'$, where $\widetilde{\psi}_j(\cdot - x_Q) \in \mathscr{S}_M$ for each Q and x_Q is any fixed point in Q.

Next, we give the aforementioned sharp weighted estimate for the vector-valued extension of the Hardy–Littlewood maximal operator given in [2].

Lemma 4 [2] *Let $1 < p, q < \infty$ and let w be a weight in A_p. We define the vector-valued Hardy–Littlewood maximal operator $\overline{M}_q$ as*

$$\overline{M}_q f(x) = \left(\sum_{j=1}^{\infty} M f_j(x)^q \right)^{\frac{1}{q}}.$$

Then,

$$\|\overline{M}_q f\|_{L_w^p} \leq C[w]_{A_p}^{\frac{1}{p}} \left([w]_{A_\infty}^{(\frac{1}{q} - \frac{1}{p})_+} + [\sigma]_{A_\infty}^{\frac{1}{p}} \right) \||f|_q\|_{L_w^p}$$

where

$$|f|_q(x) = \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{\frac{1}{q}}$$

and we denote $\sigma = w^{-\frac{1}{p-1}}$ the dual weight of w in A_p.

This result improved Theorem 1.12 from [4] since, as we stated in the introduction, $[w]_{A_\infty} \leq c_n[w]_{A_q}$, for c_n a dimensional constant.

We will use the following inequality between the norms of Littlewood–Paley square functions.

Note that in [22], the authors mention that the norms of g_d and g in L_w^p are equivalent, but this is true up to a constant that depends on the weight. For our purposes, the equivalence is not needed, so we just write here the inequality that it is useful for us with the explicit dependence on the weight.

Lemma 5 *Let $w \in A_\infty$ and $\frac{n}{n+M} < r < \min\left\{ \frac{p}{q_w}, 1 \right\}$, then*

$$\|g_d(f)\|_{L_w^p} \leq C_w \|g(f)\|_{L_w^p}$$

where $C_w = [w]_{A_{p/r}}^{1/p} \left([w]_{A_\infty}^{(\frac{r}{2} - \frac{r}{p})_+} + \left[w^{-\frac{r}{p-r}} \right]_{A_\infty}^{\frac{r}{p}} \right)^{\frac{1}{r}}$. In other words, $\|g_d(f)\|_{L_w^p} \leq C_w \|f\|_{\mathbb{H}_w^p}$.

Proof Let us recall that

$$g_d(f)(x) = \left(\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{D}_{j+N}} |\phi_j * f(x_Q)|^2 \chi_Q(x) \right)^{\frac{1}{2}}$$

where x_Q is a fixed point in Q. According to the Calderón discrete reproducing formula (4.3)

$$f(x) = \sum_{i \in \mathbb{Z}} \sum_{W \in \mathscr{D}_{i+N}} |W| \widetilde{\psi}_i(x - x_W) \psi_i * f(x_W)$$

Then,

$$(\phi_j * f)(x_Q) = \sum_{i \in \mathbb{Z}} \sum_{W \in \mathscr{D}_{i+N}} |W| (\phi_j * \widetilde{\psi}_i)(x_Q - x_W)(\psi_i * f)(x_W)$$

Now, due to Lemma 2 we have

$$|(\phi_j * \widetilde{\psi}_i)(x_Q - x_W)| \leq C 2^{-|j-i|a} \frac{2^{-(j \wedge i)b}}{(2^{-(j \wedge i)} + |x_Q - x_W|)^{n+b}}, \quad a, b \in \mathbb{N}.$$

Then, applying Lemma 3 with $a = b = M$, we get

$$|(\phi_j * f)(x_Q)| \leq C \sum_{i \in \mathbb{Z}} \sum_{W \in \mathscr{D}_{i+N}} |W| \frac{2^{-|j-i|M} 2^{-(j \wedge i)M}}{(2^{-(j \wedge i)} + |x_Q - x_W|)^{n+M}} |\psi_i * f(x_W)|$$

$$\leq C \sum_{i \in \mathbb{Z}} 2^{-|j-i|M} 2^{(\frac{1}{r} - 1)n(N + (i-j)_+)} M_r \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)| \chi_W \right)(u)$$

$$\leq C(N, M) \sum_{i \in \mathbb{Z}} 2^{-|j-i|M} 2^{(\frac{1}{r} - 1)n(i-j)_+} M_r \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)| \chi_W \right)(u)$$

where $u \in Q$. Then, by Hölder's inequality

$$|(\phi_j * f)(x_Q)|^2 \leq C \sum_{i \in \mathbb{Z}} \left(M_r \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)| \chi_W \right)(u) \right)^2 \left(\sum_{i \in \mathbb{Z}} 2^{-2|j-i|M + (\frac{1}{r} - 1)n(i-j)_+} \right).$$

Therefore, since $\sum_{Q \in \mathscr{D}_{j+N}} \chi_Q(x) = 1$ for each $j \in \mathbb{Z}$, and the above inequality is independent of Q,

$$
g_d(f)(x) = \left(\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathscr{D}_{j+N}} |\phi_j * f(x_Q)|^2 \chi_Q(x) \right)^{\frac{1}{2}}
$$

$$
\leq C \left[\sum_{i \in \mathbb{Z}} \left(M \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)| \chi_W \right)^r (u) \right)^{\frac{2}{r}} \right]^{\frac{r}{2} \cdot \frac{1}{r}}
$$

$$
\times \left(\sum_{i,j \in \mathbb{Z}} 2^{-2|j-i|M + (\frac{1}{r}-1)n(i-j)_+} \right)^{\frac{1}{2}}.
$$

It is easy to see that the double-series is convergent since $r > \frac{n}{n+M}$ and the sum only depends on M. Then, if we set $f_i = \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)| \chi_W \right)^r$, from Lemma 4 with $q = \frac{2}{r} > 1$ in $L_w^{\frac{p}{r}}$ ($\frac{p}{r} > 1$),

$$
\|g_d(f)\|_{L_w^p} \leq C[w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} \left([w]_{A_\infty}^{(\frac{r}{2}-\frac{r}{p})+} + [\sigma]_{A_\infty}^{\frac{r}{p}} \right) \left\| \left(\sum_{i \in \mathbb{Z}} \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)| \chi_W \right)^2 \right)^{\frac{r}{2}} \right\|_{L_w^{\frac{p}{r}}}
$$

$$
= C[w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} \left([w]_{A_\infty}^{(\frac{r}{2}-\frac{r}{p})+} + [\sigma]_{A_\infty}^{\frac{r}{p}} \right) \left\| \left(\sum_{i \in \mathbb{Z}} \left(\sum_{W \in \mathscr{D}_{i+N}} |\psi_i * f(x_W)| \chi_W \right)^2 \right)^{\frac{1}{2}} \right\|_{L_w^p}.
$$

Notice that the above inequality holds for any x_W in W, so we can take the infimum in each cube and then use that the dyadic cubes are disjoint in each level to get

$$
\|g_d(f)\|_{L_w^p} \leq C[w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} \left([w]_{A_\infty}^{(\frac{r}{2}-\frac{r}{p})+} + [\sigma]_{A_\infty}^{\frac{r}{p}} \right) \left\| \left(\sum_{i \in \mathbb{Z}} \left(\sum_{W \in \mathscr{D}_{i+N}} \inf_{x_W \in W} |\psi_i * f(x_W)| \chi_W \right)^2 \right)^{\frac{1}{2}} \right\|_{L_w^p}
$$

$$
\leq C[w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} \left([w]_{A_\infty}^{(\frac{r}{2}-\frac{r}{p})+} + [\sigma]_{A_\infty}^{\frac{r}{p}} \right) \left\| \left(\sum_{i \in \mathbb{Z}} |\psi_i * f|^2 \right)^{\frac{1}{2}} \right\|_{L_w^p}
$$

$$
= C[w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} \left([w]_{A_\infty}^{(\frac{r}{2}-\frac{r}{p})+} + [\sigma]_{A_\infty}^{\frac{r}{p}} \right) \|g(f)\|_{L_w^p}.
$$

The proof is now complete. □

We are now in position to state the main result of this section, which gives us a precise quantitative bound for the operator norm on $\mathbb{H}_w^p$ (c.f. [22, Theorem 1.1]).

Theorem 7 *Let T be a singular integral of convolution type with kernel K satisfying conditions* (4.1) *and* (4.2)*. Let* $w \in A_\infty$ *and* $0 < p < \infty$*. Then,*

$$\|Tf\|_{\mathbb{H}^p_w} \leq C[w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} \left([w]_{A_\infty}^{(\frac{r}{2}-\frac{r}{p})+} + [\sigma]_{A_\infty}^{\frac{r}{p}}\right) \|f\|_{\mathbb{H}^p_w}$$

where r satisfies $\frac{n}{n+M} < r < \min\left\{\frac{p}{q_w}, 1\right\}$.

Proof Since $L^2 \cap \mathbb{H}^p_w$ is dense in $\mathbb{H}^p_w$, by standard density arguments, we assume $f \in L^2 \cap \mathbb{H}^p_w$. The proof is very similar to the proof of Lemma 5. By Calderón's reproducing formula (4.3)

$$g(Tf)(x) = \left(\sum_{j\in\mathbb{Z}} \sum_{Q\in\mathscr{D}_{j+N}} |\psi_j * K * f(x)|^2 \chi_Q(x)\right)^{1/2}$$

$$= \left[\sum_{j,Q} \left|\psi_j * K * \left(\sum_{i\in\mathbb{Z}} \sum_{W\in\mathscr{D}_{i+N}} |W|\widetilde{\psi}_i(x-x_W)\psi_i * f(x_W)\right)\right|^2 \chi_Q(x)\right]^{1/2}$$

$$= \left[\sum_{j,Q} \left|\left(\sum_{i,W} \psi_j * K * \widetilde{\psi}_i(x-x_W)|W|\psi_i * f(x_W)\right)\right|^2 \chi_Q(x)\right]^{1/2}.$$

From the properties on K, $K * \widetilde{\psi}_i \in \mathscr{S}_M$, so we can apply the almost orthogonality property (Lemma 2) to $K * \widetilde{\psi}_i$ and ψ_j with $a = b = M$. Then, as in the proof of Lemma 5, we use Lemma 3, Hölder's inequality, the properties of the dyadic cubes and the restriction on r to get

$$g(Tf)(x) \leq C \left(\sum_{i\in\mathbb{Z}} \left(M\left(\sum_{W\in\mathscr{D}_{i+N}} |\psi_i * f(x_W)|\chi_W(x)\right)^r\right)^{\frac{2}{r}}\right)^{\frac{1}{2}},$$

and now the proof continues in the same way. Thus, from Lemma 4 we obtain the following quantitative bound

$$\|Tf\|_{\mathbb{H}^p_w} \leq C[w]_{A_{\frac{p}{r}}}^{\frac{r}{p}} \left([w]_{A_\infty}^{(\frac{r}{2}-\frac{r}{p})+} + [\sigma]_{A_\infty}^{\frac{r}{p}}\right) \|g(f)\|_{L^p_w} = \|f\|_{\mathbb{H}^p_w}.$$

$\square$

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