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Dimensional Transmutation and Dimensional Regularization in Quantum Mechanics

I. General Theory

Horacio E. Camblong

Department of Physics, University of San Francisco, San Francisco, California 94117-1080

and

Luis N. Epele, Huner Fanchiotti, and Carlos A. García Canal

Laboratorio de Física Teórica Departmento de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina

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DEDICATED TO C. G. BOLLINI AND J. J. GIAMBIAGI

This is the first in a series of papers addressing the phenomenon of dimensional transmutation in nonrelativistic quantum mechanics within the framework of dimensional regularization. Scale-invariant potentials are identified and their general properties are derived. A strategy for dimensional renormalization of these systems in the strong-coupling regime is presented, and the emergence of an energy scale is shown, both for the bound-state and scattering sectors. Finally, dimensional transmutation is explicitly illustrated for the two-dimensional delta-function potential. © 2001 Academic Press

1. INTRODUCTION

It is well known that, for various models of quantum field theory, a mass scale emerges spontaneously through the renormalization procedure, even when the original theory has no dimensional parameters. This phenomenon, called dimensional transmutation, was first analyzed in the 1973 seminal work of Coleman and Weinberg [1], where the scalar field of massless scalar electrodynamics was shown to develop a nonzero but arbitrary expectation value; as a consequence, the particles of the theory acquire nonzero physical masses [2, 3]. In short, the Coleman–Weinberg mechanism induces radiative corrections to the Higgs potential, thereby suggesting the relevance of dimensional transmutation for the generation of particle masses [3, 4].

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The main goal of this paper is to present a thorough investigation of dimensional transmutation in nonrelativistic quantum mechanics, with a threefold purpose in mind: (i) at the conceptual level, to show that quantum field theory is not a prerequisite for its existence; (ii) mathematically, to characterize the class of scale-invariant potentials, as well as the subclass of potentials that display dimensional transmutation; and (iii) at the practical level, to develop useful tools for the treatment of a certain class of singular quantum-mechanical potentials.

In particular, our work offers additional insight into two problems that have been extensively studied in the literature: the two-dimensional delta-function potential [2, 5, 6] and the inverse square potential [7–12]. Parenthetically, the family of delta-function potentials that is discussed in this article is actually included in a larger class of singular potentials whose apparent phenomenological usefulness has been recognized for a long time, since the introduction of pseudopotentials [13] in the early days of quantum mechanics [14] and with subsequent applications of the zero-range potential in nuclear physics [15], condensed matter physics [16], statistical mechanics [17], atomic physics [18], and particle physics [5]. Likewise, the inverse square potential is related to the dipole potential, which has found applications in molecular physics [19]. Even though earlier research on the subject had relied solely on traditional quantum-mechanical techniques, this situation has changed in recent years, with the introduction of numerous applications of quantum field-theoretic tools and renormalization theory to the same problems. Specifically, among the many applications related directly or indirectly to our singular potentials, the following are worth mentioning: (i) the mathematical formulation of the theory of pseudopotentials [20], which started with the works of Ref. [21] and includes the technique of self-adjoint extensions [6, 22, 23]; (ii) the study of the nonrelativistic limit of the ϕ^4 theory, with the concomitant question of its triviality [24]; (iii) the basic conceptual understanding of quantum field theory mechanisms in the simpler framework of quantum mechanics, including standard regularization and renormalization of the singular potentials mentioned above [6, 12, 25–30], anomalies [6, 26, 29], renormalization group analysis [12, 25, 28, 31–33], and effective field theory approach [30, 34, 35]; (iv) the analysis of $(2+1)$-dimensional theories, including gravity [36], as well as Chern–Simons theory [37], the Aharonov–Bohm effect [23, 38], and the dynamics of anyons [39]; (v) new applications of contact potentials in condensed matter physics, e.g., for the quantum Hall effect [40]; and (vi) the modern formulation, using effective field theory [41], of the nucleon–nucleon potential [42], which has led to a plethora of quantum-mechanical pseudopotentials [30, 43]. Our paper naturally follows the trend set by this extensive bibliography.

The main focus of our work will be on the concept of *dimension*, a term that has been extensively used in the physics literature to describe two conceptually distinct ideas. The meaning to which dimensional transmutation refers is that which relates to measurement and which characterizes the class of physical quantities that exhibit a certain type of power-law behavior (dimensional homogeneity) with respect to a given choice of fundamental quantities [44]. In what follows, we will use the term

dimensionality to denote the exponents of the associated homogeneous behavior for any given physical quantity [45]. At first sight, dimensional transmutation is paradoxical, because it produces a scale in a problem devoid of dimensional parameters, in apparent violation of Buckingham's Π theorem of orthodox dimensional analysis [44]. However, this paradox can be ultimately resolved by invoking the dimensional arbitrariness intrinsic in the renormalization framework [46].

The other widely used acceptation of dimension refers to a geometric concept, a property of the space where events take place (e.g., space-time in relativistic physics). In this paper we will extensively exploit the trivial connection between these two concepts, that is, that the dimensionality of an element of volume in a given space, expressed in terms of units of length, is equal to its geometric dimension. This connection has been largely used in the renormalization of relativistic quantum field theories, where it is further reinforced by the implementation of dimensional regularization [47, 48]. Correspondingly, we will use dimensional regularization as a natural technique that renders obvious the spontaneous generation of a dimensional scale for a scale-invariant theory.

Our paper is organized as follows. In Section II we discuss the meaning of dimensional transmutation in terms of dimensional analysis and renormalization theory. Section III is devoted to the concept of scale-invariant potentials in nonrelativistic quantum mechanics, where generic properties related to scaling and scale symmetry are derived. Section IV establishes a general framework for the regularization of scale-invariant potentials based upon dimensional continuation; this procedure is later extended to a renormalization scheme in Section V, where dimensional transmutation is shown to arise in the strong-coupling regime. Finally, an application of the theory is illustrated in Section VI, which deals with some aspects of the two-dimensional delta-function potential. The conclusions of our analysis are presented in Section VII. The appendices deal with the necessary results regarding D-dimensional Euclidean spaces, Green's functions, and scattering.

Additional applications for rotationally invariant problems will follow in the second paper in this series [49].

II. DIMENSIONAL ANALYSIS, RENORMALIZATION, AND DIMENSIONAL TRANSMUTATION

The description of a physical system in the context of a given theory, either in terms of a Lagrangian or of a Hamiltonian, includes a certain number of parameters. Usually, their values may all be fixed from the start by the laws of nature, in the form of "constants," but one always enjoys the mathematical freedom to make some of them become variable parameters as needed. Then, for the discussion that follows, these parameters will arbitrarily be classified into two groups: "constant" or fundamental and "variable" or dynamical. By varying the variable parameters one introduces a whole class $\mathscr{C}$ of physical systems, all characterized by the same values of the "constants" [50].

Fundamental parameters are those that are fixed constants for all the members of the given class of systems. Needless to say, they are dimensional because "fundamental" dimensionless parameters amount to plain numerical constants. Typical fundamental parameters of choice are the dimensional universal constants of nature, for example, $\hbar$ and c.

The second group is composed of the dynamical parameters that acquire different values for the different members of the class $\mathscr{C}$. As an example we could mention the masses of particles or the coupling constants of the interactions.

The reduction in the number of dimensionally independent quantities can be accomplished by arbitrarily assigning particular numerical values to a subset of the fundamental parameters. This procedure amounts to the selection of a generalized *natural system of units*, in which the number of fundamental dimensions is reduced. For example, in relativistic quantum field theory, it is customary to choose $c=1$ and $\hbar=1$, so that the theory is described in terms of a single fundamental dimension—usually taken as inverse length $\Lambda=L^{-1}$, which is equivalent to mass, momentum, and energy. Even though all the physical dimensions can be restored easily at any stage of the calculation, it is clear that great simplification is achieved in the dimensional analysis of various physical quantities.

Similarly, in one-particle nonrelativistic quantum mechanics, one has the freedom to use $\hbar$ and m (where m is the particle's mass) as fundamental parameters that define a particular generalized natural system of units; in this paper, we will choose $\hbar=1$ and $m=1/2$. Then, we will be left with only one fundamental dimension, which we will take again as inverse length $\Lambda=L^{-1}$ or momentum. Consequently, in what follows, we will define the inverse-length dimensionality $q=\dim[Q]$ of a physical quantity Q as the exponent that expresses its physical dimension Λ^q in terms of inverse length, that is,

$$q=\dim[Q]=\frac{\Lambda}{[Q]}\frac{\partial[Q]}{\partial\Lambda}. \tag{2.1}$$

For nonrelativistic quantum mechanics, Table I summarizes the dimensionalities of the most common physical quantities.

Let us now explore the consequences of the possible existence of dimensional parameters. For a given physical system, characterized by a Lagrangian or a Hamiltonian, oftentimes there exists at least one dimensional parameter, which can be used to define a system-specific or intrinsic scale. To illustrate how this is done, let us consider the nonrelativistic quantum-mechanical dynamics of a single particle in one dimension, such that the external-interaction potential contributes only one dimensional parameter λ; for example, for an attractive power-law potential, one may study the possible existence of bound states through the Schrödinger equation

$$\left[-\frac{d^2}{dx^2}+\mathrm{sgn}(\beta)\,\lambda\,|x|^{\beta}\right]\Psi(x)=E\Psi(x). \tag{2.2}$$

TABLE I

Physical Dimensions of Various Physical Quantities

Physical quantity	Ordinary dimensions	"Natural" dimensions	Dimensionality
Length	L	L	-1
Time	T	L^2	-2
Velocity	LT^{-1}	L^{-1}	1
Linear momentum	MLT^{-1}	L^{-1}	1
Angular momentum	ML^2T^{-1}	1	0
Energy	ML^2T^{-2}	L^{-2}	2
Cross section	L^{D-1}	L^{D-1}	$-(D-1)$
Wave function (normalized)	$L^{-D/2}$	$L^{-D/2}$	$D/2$

Note. "Natural" dimensions are defined by the choice $h=1$ and $2m=1$. The geometric dimension of position space is D.

Dimensional analysis shows that $\dim[\lambda]=\ell=2+\beta$; then $\lambda^{1/\ell}$ will define a basic unit of inverse length or momentum. Any dimensional quantity Q of dimensionality q will then be equal to $\lambda^{q/\ell}$, up to a numerical factor; similarly, a function $Q(x)$ of position (or $Q(p)$ of momentum), with dimensionality q, will then be equal to $\lambda^{q/\ell}$ times a dimensionless function of $\lambda^{1/\ell}x$ (or of $\lambda^{-1/\ell}p$). In particular, a characteristic ground-state energy may be estimated as $\lambda^{2/(2+\beta)}$. In other words, dimensional analysis gives nontrivial information about the system.

The obvious statements of the previous paragraph can be summarized in the Π theorem of dimensional analysis [44], which we state here without proof, with the intention of generalizing it later in this section. Consider a physical phenomenon described by M dimensional characteristic parameters $a_1, ..., a_M$, such that R of them are dimensionally independent. Then, given an equation

$$F(a_1, ..., a_M)=0 \tag{2.3}$$

involving these M parameters, there exist N independent dimensionless power products $\Pi_1, ..., \Pi_N$ of $a_1, ..., a_M$, such that Eq. (2.3) is equivalent to

$$\Phi(\Pi_1, ..., \Pi_N)=0, \tag{2.4}$$

with

$$N=M-R. \tag{2.5}$$

For example, working in a natural system of units with only one independent dimension, it follows that $N=M-1$, which describes the situation of the previous paragraph.

But what happens if the system exhibits no explicit dimensional dynamical parameter at the level of the Lagrangian or Hamiltonian? As we will see in

Section III, such a system is *scale-invariant*. An example is the power-law potential $-\lambda |x|^{-2} (\beta = -2)$ because its coupling λ is dimensionless. Then, naive dimensional analysis is at a loss to make any meaningful predictions. In this case, if a new scale arises (for example, a bound state under a scale-invariant potential), dimensional analysis implies that it has the following properties:

- It is *spontaneously generated*, in the sense that it characterizes the solution of a theory that is scale-invariant at the level of the classical Lagrangian or Hamiltonian. This amounts to an instance of quantum-mechanical breaking of classical scale symmetry—also called *scale anomaly* [11, 37, 51].
- It is totally *arbitrary* because no privileged value is defined a priori within the theory. If it were not arbitrary, it would violate the Π theorem in an irreconcilable way.

This manifestation of an arbitrary and spontaneously generated scale in a scale-invariant theory is known as dimensional transmutation [1].

In short, in the solution to a well-posed question within the scale-invariant theory, a dimensionally transmuted scale B may appear spontaneously. How does it come into existence in apparent violation of naive dimensional analysis? Our goal is to disentangle the mechanism that leads to this transmutation. This will be implemented by means of a regularization-renormalization procedure. The regularization technique introduces a dimensional parameter μ, in terms of which the scale B is expressed. Thus, a dimensional transfer takes place, whereby a dimensionless parameter λ is "transmuted" into or traded for a dimensional scale B. This simple process can be represented diagrammatically in the form

Initial problem	Technique	Solution
Lagrangian/Hamiltonian	*Regularization/renormalization*	*Physical quantity*
dimensionless	arbitrary	measurable
coupling λ	$\xrightarrow{\text{dimensional scale } \mu}$	dimensional scale B

We conclude this section by stating the modification of orthodox dimensional analysis needed to encompass this anomalous behavior. As discussed in Ref. [46], the usual assumption underlying the Π theorem is that the function $F(a_1, ..., a_M)$ of Eq. (2.3) is *uniquely* defined, an assumption that breaks down when the Lagrangian does not describe a single theory but a class of theories parametrized with renormalization parameters. This manifests itself in a theory that is ill-defined or exhibits singularities of some sort, in which case the Lagrangian or Hamiltonian cannot represent a complete description of the physics; thus, renormalization is needed. When the number of independent sliding scales or renormalization parameters is σ, the required modification is of the Π theorem is obviously

$$N = M + \sigma - R. \tag{2.6}$$

Equation (2.6) states that the number of "available" variables is $M' = M + \sigma$, rather than M; in particular, it provides the necessary freedom to permit the emergence of dimensional transmutation. The framework for deriving conclusions directly from Eq. (2.6) will be referred to as generalized dimensional analysis.

III. CHARACTERIZATION OF SCALE-INVARIANT POTENTIALS

A. *Scale Symmetry and Homogeneity*

In this section we set out to define and characterize mathematically the class of scale-invariant potentials $V(\mathbf{r})$ in one-particle nonrelativistic quantum physics. In a strict sense, we are referring to a physical system whose classical action

$$S = \int [\, \tfrac{1}{2} m \dot{\mathbf{r}}^2 - V(\mathbf{r}) \,]\, dt \tag{3.1}$$

is invariant under the scale transformations $\mathbf{r} \to \varrho \mathbf{r}$, $t \to \tau t$ (with $\varrho > 0$ and $\tau > 0$). This scale symmetry is satisfied if and only if each one of the two terms of the nonrelativistic action—the kinetic-energy term $\int dt\, m \dot{\mathbf{r}}^2/2$ and the potential-energy term $-\int dt\, V(\mathbf{r})$—are left unchanged. Due to the spatial and time dependence of the nonrelativistic kinetic-energy term, this invariance condition is satisfied only when $\varrho^2 = \tau$ (obviously consistent with the dimensional analysis of Table I), while the invariance of the potential-energy term requires that

$$V(\varrho \mathbf{r}) = \varrho^{-2} V(\mathbf{r}). \tag{3.2}$$

As Eq. (3.2) is valid for all $\varrho > 0$, the class of scale-invariant potentials is identical to that of *homogeneous potentials of degree* -2. As we will see next, this is the same condition to be satisfied when the potential does not exhibit any explicit dimensional scale.

B. *Dimensional Scaling in Nonrelativistic Quantum Mechanics*

One-particle nonrelativistic quantum mechanics in the presence of a stationary potential $V(\mathbf{r})$ is described in the D-dimensional position-space representation of the Schrödinger picture via the solutions of the time-independent Schrödinger equation, which in natural units reads

$$[\, -\nabla^2 + V(\mathbf{r}) \,]\, \Psi(\mathbf{r}) = E \Psi(\mathbf{r}). \tag{3.3}$$

The analysis and interpretation of the solutions to Eq. (3.3) will become more transparent when the transition to its dimensionless version is carried out. This can be accomplished by rescaling all quantities appearing in Eq. (3.3) by means of a dimensional parameter μ, which we will assume to represent an inverse-length standard; then, μ will satisfy the properties:

(i) inverse-length dimensionality,

$$\dim[\mu] = 1; \tag{3.4}$$

(ii) positivity,

$$\mu > 0. \tag{3.5}$$

In general, there are many possible characteristic scales that may serve as μ: they could either be intrinsic to the system or arbitrary scales introduced via regularization. In any case, we will not be concerned with the multi-scale case, because our ultimate goal is to analyze the extreme scenario where there is no intrinsic dimensional parameter, but an arbitrary sliding scale μ is introduced by the regularization procedure. Then any physical quantity Q of dimensionality q will be equal to a numerical coefficient times μ^q, whence its dimensionless counterpart will be defined as $\mu^{-q}Q$. If, in addition, the quantity is a function of either position or momentum, it will be of the form μ^q times a dimensionless function of the dimensionless position

$$\boldsymbol{\xi} = \mu \mathbf{r}, \tag{3.6}$$

or of the dimensionless momentum

$$\boldsymbol{\pi} = \mu^{-1} \mathbf{p}. \tag{3.7}$$

Correspondingly, dimensional analysis predicts that the potential energy function (whose dimensionality is 2) should have a dependence on the dimensional parameter μ given by

$$V(\mathbf{r}, \mu) = \mu^2 \mathscr{V}(\mu \mathbf{r}), \tag{3.8}$$

for arbitrary μ. In particular, one can define the reduced function $\mathscr{V}(\boldsymbol{\xi})$, via Eq. (3.8), in the form

$$\mathscr{V}(\boldsymbol{\xi}) = \mu^{-2} V(\mu^{-1}\boldsymbol{\xi}, \mu), \tag{3.9}$$

or straightforwardly by specializing to the unit value of the dimensional parameter, i.e.,

$$\mathscr{V}(\boldsymbol{\xi}) = V(\boldsymbol{\xi}, \mu = 1). \tag{3.10}$$

The right-hand side of Eq. (3.8) displays the two sources of possible scale dependence of the potential: the one associated with the dimensionality of V as a potential energy (i.e., μ^2) and the one associated with the functional form of the potential (as described by $\mathscr{V}(\mu \mathbf{r})$). Equation (3.8) implies that

$$\mu \frac{\partial |V(\mathbf{r}, \mu)|}{\partial \mu} = 2 |V(\mathbf{r}, \mu)| + \mathbf{r} \cdot \boldsymbol{\nabla} |V(\mathbf{r}, \mu)|, \tag{3.11}$$

TABLE II

Dimensionless Counterparts of Various Physical Quantities in Nonrelativistic Quantum Mechanics

Physical quantity	Symbol	Dimensionality	Dimensionless form
Position	$\mathbf{r}$	-1	$\xi = \mu \mathbf{r}$
Linear momentum	$\mathbf{p}$	1	$\boldsymbol{\pi} = \mu^{-1}\mathbf{p}$
Kinetic energy	$-\nabla_{\mathbf{r}}^2$	2	$-\nabla_{\xi}^2 = -\mu^{-2}\nabla_{\mathbf{r}}^2$
Potential energy	$V(\mathbf{r})$	2	$\mathscr{V}(\xi) = \mu^{-2} V(\mu^{-1}\xi, \mu)$
Energy	E	2	$\eta = \mu^{-2} E$
Wave function (normalized)	$\Psi(\mathbf{r})$	$D/2$	$\Phi(\xi) = \mu^{-D/2}\Psi(\mu^{-1}\xi)$

where the first term on the right-hand side is the dimensionality of the potential energy and the second term represents the degree of the functional dependence of the potential energy with respect to the given scale. For example, for a power-law potential $V(\mathbf{r}) \propto r^{\beta}$, the functional dependence amounts to $\mathscr{V}(\xi) \propto \xi^{\beta}$, whence $V(\mathbf{r}, \mu) \propto \mu^{2+\beta}$, which describes the *total* scale dependence of the potential energy function under arbitrary rescaling. Notice that, for $\beta = -2$, $V(\mathbf{r}, \mu)$ is independent of μ; i.e., it is scale-independent (see the next section). Table II gives a list of the various dimensionless quantities of interest.

Rescaling of Eq. (3.3) with the parameter μ yields

$$[-\nabla_{\xi}^2 + \mathscr{V}(\xi)]\, \Phi(\xi) = \eta \Phi(\xi), \tag{3.12}$$

which describes an eigenvalue problem for the dimensionless eigenfunctions $\Phi(\xi)$, with dimensionless eigenvalues

$$\eta = \mu^{-2} E, \tag{3.13}$$

in a space of arbitrary dimension D. In addition, it is convenient to normalize the wave function $\Phi(\xi)$ with respect to its dimensionless argument ξ, that is,

$$|\Phi(\xi)|^2\, d^D\xi = |\Psi(\mu^{-1}\xi)|^2\, d^D r, \tag{3.14}$$

a condition that yields the rescaling

$$\Phi(\xi) = \mu^{-D/2}\Psi(\mu^{-1}\xi). \tag{3.15}$$

As usual, the solutions to Eq. (3.12) should be separately obtained and interpreted for the bound-state and scattering sectors. Additional conclusions about these specific problems will be drawn in Subsections V.B, V.C, and V.D.

C. *Absence of Explicit Dimensional Scales and Homogeneity Revisited*

Let us now characterize the class of potentials that do not exhibit any dimensional scale, in a space of arbitrary dimension D. We will resort to the general framework developed in Subsection III.B, where we considered potentials that

depend upon only one dimensional parameter or none (once the choice $h=1$ and $2m=1$ has been made); from Eq. (3.8), the position and dimensional dependence of the potential energy are such that

$$V(\mu\mathbf{r}, \mu=1)=\mu^{-2}V(\mathbf{r}, \mu), \tag{3.16}$$

for arbitrary μ. In general, the function $V(\mathbf{r}, \mu)$ may have a nontrivial (i.e., not quadratic) dependence with respect to the parameter μ, as displayed in Eq. (3.11). However, those potentials devoid of explicit dimensional scales are independent of any dimensional parameter μ; then their functional dependence is of the form $V=V(\mathbf{r})$ rather than of the form $V=V(\mathbf{r}, \mu)$. We can also say that the second argument in $V=V(\mathbf{r}, \mu)$ is actually spurious. An explicit mathematical statement of this independence is

$$\frac{\partial}{\partial\mu}V(\mathbf{r}, \mu)=0. \tag{3.17}$$

Even though this may be taken as the primary definition, it is convenient to derive a more illuminating form by just eliminating the spurious μ dependence in Eq. (3.16), i.e.,

$$V(\mu\mathbf{r})=\mu^{-2}V(\mathbf{r}). \tag{3.18}$$

Equation (3.18), valid for $\mu>0$ (Eq. (3.5)), is identical to our earlier homogeneous property (3.2). An alternative derivation of this remarkably simple property can be obtained directly from Eq. (3.11) or by differentiation of Eq. (3.16) with respect to μ; then (after setting the arbitrary scale equal to unity),

$$\mathbf{r}\cdot\nabla V(\mathbf{r})=-2V(\mathbf{r}), \tag{3.19}$$

a relation that amounts to Euler's theorem for a homogeneous function of degree -2. In conclusion, the classes of scale-invariant potentials and potentials without any explicit dimensional scale are identical.

So far our discussion has only focused on the dimensions but not on the magnitude of the potentials. It is now due time to introduce a dynamical coupling parameter λ to characterize the "strength" of a given potential, according to

$$V(\mathbf{r})=-\lambda W(\mathbf{r}), \tag{3.20}$$

where $W(\mathbf{r})$ is a homogeneous function of degree -2.

The homogeneity displayed by Eq. (3.18) has a straightforward consequence on the position dependence of scale-invariant potentials. In effect, writing $\mathbf{r}=r\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit position vector, we conclude that

$$V(\mathbf{r})=r^{-2}V(\hat{\mathbf{r}}), \tag{3.21}$$

so that the general scale-invariant potential has either one of the following two forms:

(i) A generalized inverse square potential in any number D of dimensions,

$$V(\mathbf{r}) = -\lambda \frac{v(\Omega^{(D)})}{r^2}, \tag{3.22}$$

where the dimensionless function $v(\Omega^{(D)})$ explicitly depends upon the D-dimensional solid angle $\Omega^{(D)} \equiv \hat{\mathbf{r}}$. In particular, $v = 1$ corresponds to the ordinary inverse square potential [7–12],

$$V(\mathbf{r}) = -\frac{\lambda}{r^2}, \tag{3.23}$$

and $v = \cos\theta$ amounts to the dipole potential [19],

$$V(\mathbf{r}) = -\lambda \frac{\cos\theta}{r^2}, \tag{3.24}$$

where θ is the polar angle (measured from the orientation of the dipole moment) and λ is proportional to the magnitude of the dipole moment.

(ii) A homogeneous pseudopotential of degree -2, for which the most general form is

$$V(\mathbf{r}) = -\lambda r^{D-2} \delta^{(D)}(\mathbf{r}). \tag{3.25}$$

It should be noticed that Eq. (3.25) can be transformed into a number of alternative forms involving the radial delta function; in particular, it is equivalent to $\delta(r)/r$ and $\delta'(r)$ [52]. In this class, the two-dimensional delta-function potential

$$V(\mathbf{r}) = -\lambda \delta^{(2)}(\mathbf{r}) \tag{3.26}$$

is the best known example, which we will analyze in Section VI in order to illustrate our general theory.

Equation (3.21) provides the limiting form of the scale-invariant potential at infinity,

$$\lim_{r \to \infty} V(\mathbf{r}) = 0 \tag{3.27}$$

(which is identically true for pseudopotentials, for any $r \neq 0$). Equation (3.27) then implies that all states with $E \geqslant 0$ are scattering states for any scale-invariant potential. The sign of λ in Eqs. (3.20)–(3.26) has been chosen so that $\lambda > 0$ corresponds to an *attractive* potential, wherever this concept is applicable.

D. *Scale Invariance and Eigenvalue Spectrum*

The Hamiltonian H_r associated with a scale-invariant potential,

$$H_{\mathbf{r}} = -\nabla^2 - \lambda W(\mathbf{r}), \tag{3.28}$$

is homogeneous of degree -2 with respect to the position vector, that is,

$$H_{\zeta\mathbf{r}} = \zeta^{-2} H_{\mathbf{r}} \tag{3.29}$$

for arbitrary ζ, because both the kinetic and potential energies have the same property. From our analysis of the last section, this is another way of saying that

$$\mathscr{H}_{\xi} = \mu^{-2} H_{\xi/\mu} = -\nabla_{\xi}^2 - \lambda \mathscr{W}(\xi), \tag{3.30}$$

where

$$\mathscr{W}(\xi) = \mu^{-2} W(\xi/\mu) \tag{3.31}$$

(cf. Eq. (3.9)), is a dimensionless Hamiltonian totally independent of any explicit dimensional scale.

A straightforward consequence of the scale invariance implied by Eq. (3.29) is the breaking of the discrete character of the bound-state energy spectrum for attractive potentials. Let us now show this. First, the Hamiltonian (3.30) associated with a scale-invariant potential is an example of a local operator. As is well known, the representative $A_{\mathbf{r}}$ of an abstract local operator A is defined via

$$A_{\mathbf{r}} \delta^{(D)}(\mathbf{r} - \mathbf{r}') = \langle \mathbf{r} | \, A \, | \mathbf{r}' \rangle, \tag{3.32}$$

which implies that

$$A_{\mathbf{r}} \psi(\mathbf{r}) = \langle \mathbf{r} | \, A \, | \psi \rangle. \tag{3.33}$$

Equation (3.29) refers to a particular case of a local operator with scale dimension s, defined via

$$A_{\zeta\mathbf{r}} = \zeta^{-s} A_{\mathbf{r}}, \tag{3.34}$$

for some exponent s and for arbitrary scalar $\zeta > 0$. With arguments similar to those of Subsection III.C, it is easy to see that $s = \dim[A_{\mathbf{r}}]$. Thus, the Hamiltonian of scale-invariant potentials is a local operator with scale dimension $s = 2$.

The statement we wish to prove is that, for any local operator with scale dimension s, the spectrum can only be continuous. In effect, if $\psi_a(\mathbf{r})$ is an eigenfunction with eigenvalue a,

$$A_{\mathbf{r}} \psi_a(\mathbf{r}) = a \psi_a(\mathbf{r}), \tag{3.35}$$

then the rescaling

$$\mathbf{r}' = \zeta \mathbf{r} \tag{3.36}$$

implies that

$$A_{\mathbf{r}} \psi_a(\zeta \mathbf{r}) = \zeta^s a \psi_a(\zeta \mathbf{r}), \tag{3.37}$$

whence $\psi_a(\zeta \mathbf{r})$ is an eigenfunction of the same operator with eigenvalue $\zeta^s a$. As this should be so for arbitrary ζ, one concludes that:

(i) If a is a finite eigenvalue, then all real numbers of the same sign are eigenvalues.

(ii) The eigenvalue spectrum is continuous.

Of course, this is what should be expected on intuitive grounds, because if the spectrum were discrete then one would be able to identify preferential scales where none is defined a priori. For example, the rescaling $\mathbf{r}' = \zeta \mathbf{r}$ for the plane-wave eigenstates $e^{i\mathbf{k}\cdot\mathbf{r}}$ of the scale-invariant free-particle Hamiltonian amounts to a rescaling of the corresponding momentum $\mathbf{k}' = \zeta \mathbf{k}$, whence all positive energies $E = k'^2 = \zeta^2 k^2$ can be reached by continuously varying the parameter ζ.

Therefore, for the Hamiltonian of attractive scale-invariant potentials, the corresponding implications are

(i) The energy spectrum is either not bounded from below or, if it is, it can only start at $E = 0$ and be of the scattering type.

(ii) The bound-state energy spectrum, if it exists, is continuous.

Thus, it is clear that, for a given unregularized scale-invariant potential, in addition to the continuous scattering spectrum with energies $0 \leqslant E < \infty$ (as required by Eq. (3.27)), there are only three possibilities for the bound-state spectrum:

1. Spectrum devoid of bound states.

2. Continuous bound-state spectrum with energies between $E = -\infty$ and $E = 0$.

3. Singular bound-state spectrum with a unique energy level at $E = -\infty$.

The fact that no other cases are possible is just a consequence of the scale symmetry.

In other words, in the first category the potential is so "weak" that if fails to generate any bound states; in fact, this situation is familiar: it is characteristic of any repulsive potential. The subtlety lies in that a "weak" attractive potential behaves in all respects as a repulsive one: it only has a continuous scattering spectrum extending from $E = 0$ to $E = \infty$. As an alternative, the potential may be so "strong" that it produces an example of the second category, where it breaks down both the existence of a lower bound and the discrete nature of the bound-state spectrum. It is well known that both cases (categories 1 and 2) are realized by the

inverse square potential with weak and strong coupling, respectively [7–12]. Yet, the singular case of category 3 provides an alternative for a "strong" potential—a behavior exhibited by the two-dimensional delta-function potential [2, 5, 6].

With regard to scattering, one may use the fact that the theory is scale-invariant and that the poles of the scattering matrix on the imaginary energy axis correspond to the bound states. These facts imply that:

(i) When there are no bound states, the scattering matrix has no poles and is manifestly scale-invariant, i.e., independent of the energy.

(ii) When the spectrum is singular (either categories 2 or 3), the scattering matrix exhibits the corresponding singular behavior.

A remark about the need for renormalization is in order. A theory that produces no bound states and a scale-invariant S-matrix (category 1) needs no regularization. In effect, such theory displays no divergence whatsoever and, as we have seen, its spectrum is identical in every respect to that of repulsive potentials, so that scale-invariance is maintained even in the quantum-mechanical theory. Instead, regularization and renormalization are needed for cases 2 and 3 above, an issue to which we now turn our attention.

IV. DIMENSIONAL TRANSMUTATION VIA DIMENSIONAL REGULARIZATION

A. *Dimensional Regularization of Scale-Invariant Potentials*

Let us consider a scale-invariant potential $V(\mathbf{r})$ in D_0 dimensions. The corresponding D_0-dimensional Schrödinger equation is

$$[-\nabla^2_{\mathbf{r},D_0} + V(\mathbf{r})]\,\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{4.1}$$

where $\nabla^2_{\mathbf{r},D_0}$ is the D_0-dimensional Laplacian.

We have seen in Subsection III.D that if $V(\mathbf{r})$ is of the scale-invariant type, then its unregularized bound-state spectrum is either nonexistent or not bounded from below. Therefore, the difficulty here resides in that, in the initial dimension D_0, the problem is singular and has to be regularized.

In this paper we will use dimensional regularization, a technique originally developed for quantum field theory [47, 48] and which we now adapt to non-relativistic quantum mechanics, The D-dimensional generalization of Eq. (4.1) is of the form

$$[-\nabla^2_{\mathbf{r},D} + V^{(D)}(\mathbf{r})]\,\Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{4.2}$$

where

$$V^{(D)}(\mathbf{r}) = -\lambda_B W^{(D)}(\mathbf{r}), \tag{4.3}$$

with λ_B being the bare coupling constant (see Subsection IV.B), while $W^{(D)}(\mathbf{r})$ is an appropriate generalization to D dimensions of the original D_0-dimensional potential, with the only constraint

$$\lim_{D \to D_0} W^{(D)}(\mathbf{r}) = W(\mathbf{r}), \tag{4.4}$$

and with $W(\mathbf{r})$ defined by Eq. (3.20).

Of course, the whole purpose of this regularization is to produce a scenario where the generalized potential is no longer scale-invariant in a dimension close but not equal to D_0. In other words, we require that, for arbitrary $\varepsilon \neq 0$, $V^{(D_0-\varepsilon)}(\mathbf{r})$ not be a homogeneous function of degree -2.

Even though the requirement above allows infinitely many possible generalizations, a particularly simple prescription can be developed by the use of Fourier transforms, just like it is done in quantum field theory. It turns out that a simple property of Fourier analysis immediately suggests the generalization: if $f(\mathbf{r})$ is a D-dimensional homogeneous function of degree β, then its Fourier transform $\tilde{f}(\mathbf{k})$ is homogeneous of degree $-(D+\beta)$. With this property in mind, we define the Fourier transform of the original scale-invariant potential,

$$\tilde{W}(\mathbf{k}) = \mathscr{F}_{(D_0)}\{W(\mathbf{r})\} = \int d^{D_0}r\, e^{-i\mathbf{k}\cdot\mathbf{r}} W(\mathbf{r}), \tag{4.5}$$

which we analytically continue to D dimensions with the prescription that, in Fourier space, the D-dimensional functional form should be the same as the D_0-dimensional functional form, i.e,

$$\tilde{W}^{(D)}(\mathbf{k}) = \tilde{W}(\mathbf{k}). \tag{4.6}$$

Finally a D-dimensional inverse Fourier transform $\mathscr{F}^{-1}_{(D)}$ provides the desired generalization in the position representation, i.e.,

$$W^{(D)}(\mathbf{r}) = \mathscr{F}^{-1}_{(D)}\{\tilde{W}(\mathbf{k})\} = \int \frac{d^D k}{(2\pi)^D} e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{W}(\mathbf{k}), \tag{4.7}$$

where all the vectors are now D-dimensional.

The process represented by Eqs. (4.5) and (4.7) involves a dimensional continuation that can be summarized in the following succinct expression for the potential

$$W^{(D)}(\mathbf{r}_D) = \int \frac{d^D k_D}{(2\pi)^D} e^{i\mathbf{k}_D\cdot\mathbf{r}_D} \left[\int d^{D_0} r_{D_0} e^{-i\mathbf{k}_{D_0}\cdot\mathbf{r}_{D_0}} W(\mathbf{r}_{D_0}) \right]_{\mathbf{k}_{D_0} \to \mathbf{k}_D}, \tag{4.8}$$

where the subscripts D and D_0 explicitly indicate the dimension of the corresponding vector and the symbol $\mathbf{k}_{D_0} \to \mathbf{k}_D$ stands for the dimensional "jump" in momentum space that defines the dimensionally continued potential. The whole process can be represented by means of the commutative diagram

$$\begin{array}{lcc} & \text{real space} & \text{reciprocal space} \\ D_0 \text{ dimensions:} & W(\mathbf{r}) = W^{(D_0)}(\mathbf{r}) \xrightarrow{\mathscr{F}_{(D_0)}} & \tilde{W}(\mathbf{k}) = \tilde{W}^{(D_0)}(\mathbf{k}) \\ & \Big\downarrow \mathscr{D}_{D_0 \to D} & \Big\downarrow \mathscr{D}_{D_0 \to D} \\ D \text{ dimensions:} & W^{(D)}(\mathbf{r}) \xleftarrow{\mathscr{F}^{-1}_{(D)}} & \tilde{W}^{(D)}(\mathbf{k}) = \tilde{W}(\mathbf{k}) \end{array}, \tag{4.9}$$

where $\mathscr{D}_{D_0 \to D}$ is a shorthand for dimensional continuation from D_0 to D dimensions. Correspondingly, the degree of homogeneity of a scale-invariant potential is transformed according to

$$\begin{array}{ccc} W(\mathbf{r}) & \xrightarrow{\mathscr{F}_{(D_0)}} & \tilde{W}(\mathbf{k}) \\ \text{degree} = -2 & & \text{degree} = 2 - D_0 \\ \Big\downarrow \mathscr{D}_{D_0 \to D} & & \Big\downarrow \mathscr{D}_{D_0 \to D} \\ W^{(D)}(\mathbf{r}) & \xleftarrow{\mathscr{F}^{-1}_{(D)}} & \tilde{W}^{(D)}(\mathbf{k}) \\ \text{degree} = -2 + \varepsilon & & \text{degree} = 2 - D_0 \end{array}, \tag{4.10}$$

where

$$\varepsilon = D_0 - D. \tag{4.11}$$

Diagram (4.10) explicitly shows that the D-dimensional real-space continuation of the potential is *not* of the scale-invariant type, because its degree of homogeneity is $-2 + \varepsilon$ (with $\varepsilon \neq 0$), rather than -2.

For example, when the criteria above are applied to the two-dimensional delta-function and inverse square potentials, one obtains the dimensional continuations summarized in Table III [53].

TABLE III

Dimensional Continuations $W^{(D)}(\mathbf{r})$ for the Two-Dimensional Delta-Function and Inverse Square Potentials

Potential $W(\mathbf{r})$	Dimension D_0	Dimensional continuation $W^{(D)}(\mathbf{r})$
$\delta^{(2)}(\mathbf{r})$	2	$\delta^{(D)}(\mathbf{r})\|_{D=2-\varepsilon}$
r^{-2}	Arbitrary	$\pi^{\varepsilon/2}\Gamma(1-\varepsilon/2)\, r^{-(2-\varepsilon)}$

B. *Dimensional Transmutation of the Coupling Parameter*

In the analysis above we have defined the appropriate functional dependence of the dimensionally continued potential but we have not spelled out the dimensionality change experienced by the coupling parameter. This change occurs in Eq. (4.8), when the dimensional "jump" $\mathbf{k}_{D_0} \to \mathbf{k}_D$ is performed, according to

$$[\, W^{(D)}(\mathbf{r}_D)\,] = \Lambda^{D-D_0}[\, W^{(D_0)}(\mathbf{r}_{D_0})\,]. \tag{4.12}$$

Equation (4.12) amounts to the "creation" of a physical dimension $L^{-(D-D_0)} = L^{\varepsilon}$. If the physical dimensions of the potential energy are to remain unchanged, then the bare coupling constant λ_B should acquire the physical dimensions $L^{-\varepsilon}$, that is, its dimensionality should become $\dim[\,\lambda_B\,] = \varepsilon$. A convenient way of parametrizing this dimensionality change is by the introduction of an *arbitrary* inverse-length scale μ, i.e., by the replacement

$$\lambda \to \lambda_B = \lambda \mu^{\varepsilon}, \tag{4.13}$$

where λ is dimensionless. When this dimensionality change is made explicit in Eq. (4.8), we obtain a dimensionally continued potential

$$V^{(D)}(\mathbf{r}) = -\lambda \mu^{\varepsilon} W^{(D)}(\mathbf{r}), \tag{4.14}$$

$$= -\lambda \mu^{\varepsilon} \int \frac{d^D k_D}{(2\pi)^D}\, e^{i\mathbf{k}_D \cdot \mathbf{r}_D} \left[\int d^{D_0} r_{D_0}\, e^{-i\mathbf{k}_{D_0} \cdot \mathbf{r}_{D_0}} V(\mathbf{r}_0) \right]_{\mathbf{k}_{D_0} \to \mathbf{k}_D}, \tag{4.15}$$

i.e., the dimensional jump is performed simultaneously with the introduction of the factor μ^{ε}. Notice that Eqs. (4.8) and (4.15), as well as diagram (4.10) show that even though

$$\dim[\, V^{(D)}(\mathbf{r})\,] = 2, \tag{4.16}$$

in fact

$$\dim[\, W^{(D)}(\mathbf{r})\,] = 2 - \varepsilon. \tag{4.17}$$

Thus, one can write

$$W^{(D)}(\mathbf{r}) = \mu^{2-\varepsilon} \mathscr{W}^{(D)}(\mu \mathbf{r}), \tag{4.18}$$

where $\mathscr{W}^{(D)}(\boldsymbol{\xi})$ is the dimensionless counterpart of $W^{(D)}(\mathbf{r})$; then, from Eqs. (3.9), (4.14), and (4.18),

$$\mathscr{V}^{(D)}(\boldsymbol{\xi}) = -\lambda \mathscr{W}^{(D)}(\boldsymbol{\xi}). \tag{4.19}$$

The dimensionality change represented by Eq. (4.13) introduces a completely arbitrary dimensional scale μ. The replacement of a dimensionless coupling constant by an arbitrary dimensional scale is the phenomenon of dimensional transmutation seen from the dimensional-regularization viewpoint. In addition, Eq. (4.13) indicates that the space dimension D plays a pivotal role in the determination of the physical dimensions of the coupling constant.

Equations (4.2) and (4.14) imply that the dimensionally regularized Schrödinger equation has the explicit form

$$[-\nabla^2_{\mathbf{r},D} - \lambda\mu^{\varepsilon} W^{(D)}(\mathbf{r})]\ \Psi(\mathbf{r}) = E\Psi(\mathbf{r}), \tag{4.20}$$

where $W^{(D)}(\mathbf{r})$ is a homogeneous function of degree $-2+\varepsilon$. Ultimately, in order to make the potential less singular, it is necessary to choose $\varepsilon > 0$ (that is, $D < D_0$), whence proper regularization is achieved in the limit

$$\varepsilon = 0^+. \tag{4.21}$$

Alternatively, the dimensionally regularized Schrödinger equation (4.20) can be rewritten in the dimensionless form of Eq. (3.12), i.e.,

$$[-\nabla^2_{\boldsymbol{\xi},D} - \lambda \mathscr{W}^{(D)}(\boldsymbol{\xi})]\ \Phi(\boldsymbol{\xi}) = \eta\Phi(\boldsymbol{\xi}). \tag{4.22}$$

If the problem posed by Eq. (4.22) with the limit (4.21) is *regular*, then its solution provides regular eigenfunctions $\Phi(\boldsymbol{\xi})$ corresponding to the eigenvalues η. These eigenvalues depend upon the dimensionless parameters ε and λ. Unlike the dependence of η on ε, which is potential-dependent, the dependence of η on λ is the *same* for all scale-invariant potentials. This can be understood as follows:

1. The Schrödinger equation (4.22) is dimensionless and independent of μ.

2. Instead, the Schrödinger equation (4.20) is more explicit in that it displays the dimensional scales E and μ. In addition, λ and μ appear in it *only* in its second term and through the bare coupling constant λ_B, Eq. (4.13). This can be made more explicit by rewriting Eq. (4.20) in terms of λ_B, i.e.,

$$[-\nabla^2_{\mathbf{r},D} - \lambda_B W^{(D)}(\mathbf{r})]\ \Psi(\mathbf{r}) = E\Psi(\mathbf{r}). \tag{4.23}$$

3. Equation (4.13) is the basis for the introduction of an effective inverse length

$$\hat{\mu} = \lambda_B^{1/\varepsilon} = \lambda^{1/\varepsilon}\mu, \tag{4.24}$$

where $\lambda > 0$ is assumed because we only need to regularize the potential when it is attractive. Equation (4.24) allows for the rescaling of the energy

$$\hat{\eta} = \hat{\mu}^{-2} E = \lambda^{-2/\varepsilon}\eta, \tag{4.25}$$

of the potential energy

$$\hat{\mathscr{W}} = \hat{\mu}^{-(2-\varepsilon)} W = \lambda^{-2/\varepsilon+1} \mathscr{W}, \tag{4.26}$$

as well as of the dimensionless position

$$\hat{\boldsymbol{\xi}} = \hat{\mu}\,\mathbf{r} = \lambda^{1/\varepsilon} \boldsymbol{\xi}. \tag{4.27}$$

Then, the Schrödinger equation (4.23) takes the form

$$[\,-\nabla^2_{\hat{\boldsymbol{\xi}},\,D} - \hat{\mathscr{W}}^{(D)}(\hat{\boldsymbol{\xi}})\,]\;\hat{\Phi}(\hat{\boldsymbol{\xi}}) = \hat{\eta}\hat{\Phi}(\hat{\boldsymbol{\xi}}). \tag{4.28}$$

Equations (4.20), (4.22), and (4.28) are equivalent before taking the limit $\varepsilon = 0^+$. When actually solving the Schrödinger equation for a specific potential, it is more straightforward to use (4.20) because it makes all the variables explicit, while (4.22) is just a convenient way of relating the dimensional scales from scratch. On the other hand, at the conceptual level, Eq. (4.28) provides the "universal" connection between λ and η for all scale-invariant potentials. In other words, Eq. (4.25) defines the relationship between the mathematical eigenvalues $\hat{\eta}$ of Eq. (4.28) and the physical or dimensional eigenvalues E of Eq. (4.20), which are explicit functions of the parameters μ and λ. A convenient form of this "universal" condition satisfied by the energy eigenvalues of Eq. (4.28) reads (from Eq. (4.25))

$$\lambda \Xi(\varepsilon)\;|\eta(\varepsilon)|^{-\varepsilon/2} = 1, \tag{4.29}$$

which, in terms of dimensional variables, explicitly states that

$$\lambda \mu^{\varepsilon} \Xi(\varepsilon)\;|E(\varepsilon)|^{-\varepsilon/2} = 1. \tag{4.30}$$

Any of the Eqs. (4.29) and (4.30) would be referred to as the "master eigenvalue equation," which provides the required energies if the mathematical function

$$\Xi(\varepsilon) = |\hat{\eta}|^{\varepsilon/2} \geqslant 0 \tag{4.31}$$

is known. In fact, Eq. (4.31) shows that $\Xi(\varepsilon)$ is completely determined by the functional form of the potential $\hat{\mathscr{W}}^{(D)}(\hat{\boldsymbol{\xi}})$, through the solution of the differential equation (4.28). We shall refer to $\Xi(\varepsilon)$ as the "energy generating function."

In this paper, we will exemplify the regularization procedure and computation of the energy generating function for the two-dimensional delta-function potential. A more detailed discussion of this potential, as well as the corresponding treatment of the inverse square potential, can be found in the second paper of this series [49].

V. RENORMALIZATION OF SCALE-INVARIANT POTENTIALS

A. *Regularized Bound-State Sector*

The theory of Subsection IV.B applies equally well to both bound and scattering states. In both cases, however, we may assume an attractive potential $\lambda > 0$, which is the type that possibly requires renormalization.

Let us now consider the bound-state sector, for which Eq. (4.22) provides a discrete sequence of energy eigenvalues η_n, in the regularized version of the theory. These eigenvalues explicitly depend upon the discrete set of quantum numbers $n = (n_1, ..., n_D)$, ordered as an increasing sequence in such a way that, for sufficiently small ε, $E_n(\varepsilon) \leqslant E_{n'}(\varepsilon)$ if $n_j \leqslant n'_j$ for all $j = 1, ..., D$ (additional ordering rules are needed in the presence of degeneracies, but they are immaterial to our discussion). In particular, the ground state will be labeled with the lowest numbers of the sequence.

Our goal is to extract additional information from Eq. (4.29), which we now rewrite

$$\lambda \Xi_n(\varepsilon)\, |\eta_n(\varepsilon)|^{-\varepsilon/2} = 1, \tag{5.1}$$

with $\Xi_n(\varepsilon) = |\hat{\eta}_n|^{\varepsilon/2}$. This can be accomplished by defining the variables

$$\lambda_n^{(*)} = [\lim_{\varepsilon \to 0} \Xi_n(\varepsilon)]^{-1} = \lim_{\varepsilon \to 0} |\hat{\eta}_n(\varepsilon)|^{-\varepsilon/2}, \tag{5.2}$$

such that

$$\lim_{\varepsilon \to 0} |\eta_n(\varepsilon)| = \lim_{\varepsilon \to 0} \left[\frac{\lambda}{\lambda_n^{(*)}} \right]^{2/\varepsilon}. \tag{5.3}$$

In Eq. (5.3) one can see that the unregularized energy is critically dependent on the ratio $\lambda/\lambda_n^{(*)}$, in the limit $\varepsilon \to 0$. Thus, $\lambda_n^{(*)}$ acts as a critical coupling strength for the given energy level labeled by n. From now on, we will identify the following three regimes:

(i) Strong coupling, characterized by $\lambda > \lambda_n^{(*)}$, for which Eq. (5.3) gives a bound state at $-\infty$.

(ii) Weak coupling, characterized by $\lambda < \lambda_n^{(*)}$, for which Eq. (5.3) gets rid of the bound state by pushing it all the way up to 0.

(iii) Critical coupling, characterized by $\lambda = \lambda_n^{(*)}$, for which additional analysis is needed.

In fact, Eq. (5.3) implies the following behavior according to the values of the critical coupling:

(a) $\lambda_n^{(*)} = 0$ amounts to a strong coupling for all finite and positive λ, a condition that is manifested by the "collapse" of the given bound state, $\eta_n \to -\infty$.

(b) $\lambda_n^{(*)} = \infty$ amounts to a weak coupling for all finite $\lambda > 0$, a condition that is manifested by the loss of the regularized bound state, i.e., $\eta_n \to 0$.

(c) $0 < \lambda_n^{(*)} < \infty$ permits the existence of the three possible regimes.

A few results are implied by the above analysis. First, because of the assumed ordering of quantum numbers, from Eq. (5.2), it follows that $\lambda_n^{(*)} \leqslant \lambda_{n'}^{(*)}$, when $n_j \leqslant n'_j$ for all $j = 1, ..., D$. In particular, for the ground state, which we will subsequently label with (gs), we define the "principal" critical coupling $\lambda^{(*)}$, which satisfies the condition

$$\lambda^{(*)} = \lambda_{(\mathrm{gs})}^{(*)} \leqslant \lambda_n^{(*)}. \tag{5.4}$$

For example, if the coupling is weak for the ground state, it is also weak for all other states, so that the unregularized theory is completely deprived from bound states.

The analysis above assumes that λ is independent of ε and displays the singular behavior associated with dimensional transmutation as $\varepsilon \to 0$. Renormalization is called for in order to obtain meaningful results.

B. *Renormalized Bound-State Sector*

Equation (5.1) provides a regularization of the energy levels in terms of the parameter ε. In this section we introduce the general strategy for renormalization and use it to reach a few general conclusions about dimensional transmutation.

In order to obtain finite results, it is necessary to renormalize the energy levels by the following procedure:

(i) Letting the coupling constant λ be a function of the regularization parameter ε, i.e., $\lambda = \lambda(\varepsilon)$.

(ii) Adjusting $\lambda(\varepsilon)$ by comparison with a specific bound state, which is conveniently chosen to be the ground state of the theory; notice that if bound states exist at all, the ground state is the only one that is guaranteed to exist. We will refer to this procedure as ground-state renormalization.

Consequently, in the following analysis, it will prove useful to compare the values of the function $\Xi_n(\varepsilon)$ with its ground-state value $\Xi_{(\mathrm{gs})}(\varepsilon)$, by means of the replacement

$$\Xi_n(\varepsilon) = \Xi_{(\mathrm{gs})}(\varepsilon) \left[1 + \frac{\varepsilon}{2} \mathscr{R}_n(\varepsilon) \right], \tag{5.5}$$

which defines a new function $\mathscr{R}_n(\varepsilon)$, with the obvious implication that

$$\mathscr{R}_{(\mathrm{gs})}(\varepsilon) = 0. \tag{5.6}$$

Then, the regularized dependence of the energy levels with respect to ε can be derived from (5.1) in the limit (4.21), and using Eq. (5.5), which implies

$$|\eta_n(\varepsilon)| = [\,\lambda(\varepsilon)\,\Xi_{(\mathrm{gs})}(\varepsilon)\,]^{2/\varepsilon}\left[\,1+\frac{\varepsilon}{2}\,\mathscr{R}_n(\varepsilon)\right]^{2/\varepsilon} \tag{5.7}$$

$$\approx [\,\lambda(\varepsilon)\,\Xi_{(\mathrm{gs})}(\varepsilon)\,]^{2/\varepsilon}\exp[\,\mathscr{R}_n(\varepsilon)\,]. \tag{5.8}$$

Furthermore, the analysis of the previous section shows that finite results follow *only* if the coupling constant takes a critical value. For the ground state, this requires that

$$\lambda(\varepsilon) \overset{(\varepsilon\to 0)}{\sim} [\,\Xi_{(\mathrm{gs})}(\varepsilon)\,]^{-1}. \tag{5.9}$$

Even though Eq. (5.9) is sufficient for renormalization purposes, let us consider a more general expression

$$\lambda(\varepsilon) = [\,\Xi_{(\mathrm{gs})}(\varepsilon)\,]^{-1}\left[\,1+\frac{\varepsilon}{2}\,g(\varepsilon)\right]. \tag{5.10}$$

Equation (5.10) defines a residual coupling function $g(\varepsilon)$, which—according to the definition of critical coupling, Eq. (5.2)—should have the limiting behavior

$$\varepsilon g(\varepsilon) = o(1). \tag{5.11}$$

As a consequence, from Eqs. (5.8) and (5.10),

$$|\eta_n(\varepsilon)| = \exp[\,g(\varepsilon)+\mathscr{R}_n(\varepsilon)\,], \tag{5.12}$$

and

$$\left|\frac{\eta_n(\varepsilon)}{\eta_{(\mathrm{gs})}(\varepsilon)}\right| = \exp[\,\mathscr{R}_n(\varepsilon)\,]. \tag{5.13}$$

From the form of Eqs. (5.12) and (5.13), it proves convenient to resolve both $\mathscr{R}_n(\varepsilon)$ and $g(\varepsilon)$ into their various components, i.e.,

$$\mathscr{R}_n(\varepsilon) = \mathscr{R}_n^{(-)}(\varepsilon) + \mathscr{R}_n^{(0)} + \mathscr{R}_n^{(+)}(\varepsilon) \tag{5.14}$$

and

$$g(\varepsilon) = g^{(-)}(\varepsilon) + g^{(0)} + g^{(+)}(\varepsilon),$$

where $\mathscr{R}_n^{(-)}(\varepsilon)$ and $g^{(-)}(\varepsilon)$ are the divergent pieces; $\mathscr{R}_n^{(0)}$ and $g^{(0)}$ are the limits, for $\varepsilon = 0^+$, of the finite parts; and $\mathscr{R}_n^{(+)}(\varepsilon)$, $g^{(+)}(\varepsilon) = o(1)$. Then Eq. (5.12) will again give 0 or ∞, unless

$$[\, g^{(-)}(\varepsilon) + \mathscr{R}_n^{(-)}(\varepsilon)\,] + [\, g^{(0)} + \mathscr{R}_n^{(0)}\,] + [\, g^{(+)}(\varepsilon) + \mathscr{R}_n^{(+)}(\varepsilon)\,] = O(1). \tag{5.15}$$

From now on, the terms $g^{(+)}(\varepsilon)$ and $\mathscr{R}_n^{(+)}(\varepsilon)$ can and will be omitted, as they are clearly irrelevant at the level of the renormalized energies. In turn, in Eq. (5.15), the terms $[\, g^{(-)}(\varepsilon) + \mathscr{R}_n^{(-)}(\varepsilon)\,]$ would give a divergent contribution unless

$$g^{(-)}(\varepsilon) = -\mathscr{R}_n^{(-)}(\varepsilon). \tag{5.16}$$

Condition (5.16), in general, cannot be satisfied for all bound states, but it should be satisfied, in particular, for the ground state, so that (from Eqs. (5.6) and (5.10))

$$\lambda(\varepsilon) = [\, \Xi_{(\mathrm{gs})}(\varepsilon)\,]^{-1} \left[\, 1 + \frac{\varepsilon}{2}\, g^{(0)} + o(\varepsilon) \right] \tag{5.17}$$

and (from Eqs. (5.6) and (5.12))

$$|\eta_{(\mathrm{gs})}| = e^{g^{(0)}}. \tag{5.18}$$

Once the ground-state renormalization is established, one can analyze the existence and properties of the excited states. In this regard, Eq. (5.4) selects only a subset of the states of the regularized theory as bound states of the renormalized theory, once the limit $\varepsilon \to 0$ is taken. More precisely, for any set of quantum numbers n for which Eq. (5.4) is a strict inequality, the coupling becomes weak in the limit $\varepsilon = 0^+$, so that the given state is merged with the continuum. Thus, the equality

$$\lambda^{(*)} = \lambda_n^{(*)} \tag{5.19}$$

is a necessary condition for the state labeled by n to survive as a bound state. Then, for any state that satisfies Eq. (5.19), the function $\mathscr{R}_n(\varepsilon)$ defined in (5.5) is constrained by the limiting form

$$\varepsilon \mathscr{R}_n(\varepsilon) = o(1) \tag{5.20}$$

and (from Eqs. (4.31) and (5.4)) satisfies the inequality

$$\mathscr{R}_n(\varepsilon) \leqslant 0; \tag{5.21}$$

in particular,

$$\mathscr{R}_n^{(-)}(\varepsilon) \leqslant 0. \tag{5.22}$$

If the inequality (5.22) were strict, then Eq. (5.13) would annihilate the state labeled by n in the bound-state sector by exponential suppression; as a consequence, the only alternative option for the state to "survive," as allowed by the inequality (5.22), is

$$\mathscr{R}_n^{(-)}(\varepsilon) = 0, \tag{5.23}$$

in which case

$$\left| \frac{\eta_n(\varepsilon)}{\eta_{(\mathrm{gs})}(\varepsilon)} \right| = \exp[\mathscr{R}_n^{(0)}]. \tag{5.24}$$

It should be pointed out that, when Eqs. (5.19) and (5.23) are satisfied, the inequality $\mathscr{R}_n^{(0)} < 0$ (from Eq. (5.21)) guarantees that $E_n > E_{(\mathrm{gs})}$.

In summary, Eqs. (5.19), (5.23), and (5.24) give the following conditions for the existence of bound states. An excited state labeled with the index $n \neq (\mathrm{gs})$ exists if:

(i) The critical coupling $\lambda_n^{(*)}$ satisfies the equality (5.19).

(ii) The function $\mathscr{R}_n^{(-)}(\varepsilon)$ is identically zero (condition (5.23)) for the states that already satisfy Eq. (5.19).

(iii) $\mathscr{R}_n^{(0)} \neq \mathscr{R}_{(\mathrm{gs})}^{(0)} = 0$.

These are indeed very stringent conditions, so "ordinarily" dimensional transmutation will produce a single bound state, as is the case for the two-dimensional delta-function and inverse square potentials (Ref. [49]).

A final digression on strategy may provide a more direct path in a typical problem. If the energy generating function $\Xi_n(\varepsilon)$ admits the expansion

$$\Xi_n(\varepsilon) = [L_n(\varepsilon)]^{-1} \left[1 + \frac{\varepsilon}{2} \mathscr{G}_n(\varepsilon) \right], \tag{5.25}$$

with a power-law leading term

$$L_n(\varepsilon) = a_n \varepsilon^{\tau_n} \tag{5.26}$$

(where a_n and τ_n are constants), and

$$\varepsilon \mathscr{G}_n(\varepsilon) = o(1), \tag{5.27}$$

then the following results will directly apply. First, the critical coupling will be (from Eqs. (5.2), (5.25), and (5.27))

$$\lambda_n^{(*)} = \lim_{\varepsilon \to 0} L_n(\varepsilon), \tag{5.28}$$

with a regularized coupling (Eq. (5.17))

$$\lambda(\varepsilon) = L_{(\mathrm{gs})}(\varepsilon) \left\{ 1 + \frac{\varepsilon}{2} [\, g^{(0)} - \mathscr{G}_{(\mathrm{gs})}(\varepsilon) \,] \right\} + o(\varepsilon), \tag{5.29}$$

while the function $\mathscr{R}_n(\varepsilon)$ of Eq. (5.5) will become

$$\mathscr{R}_n(\varepsilon) \overset{(\varepsilon \to 0)}{\sim} \mathscr{G}_n(\varepsilon) - \mathscr{G}_{(\mathrm{gs})}(\varepsilon), \tag{5.30}$$

up to higher order corrections. Thus, the condition for the existence of the ground state will be $\tau_{(\mathrm{gs})} \geqslant 0$, while the conditions for the existence of excited states will amount to the existence of an index $n \neq (\mathrm{gs})$, such that: (i) $a_n = a_{(\mathrm{gs})}$ and $\tau_n = \tau_{(\mathrm{gs})}$; (ii) $\mathscr{G}_n^{(-)}(\varepsilon) = \mathscr{G}_{(\mathrm{gs})}^{(-)}(\varepsilon)$; and (iii) $\mathscr{G}_n^{(0)} \neq \mathscr{G}_{(\mathrm{gs})}^{(0)}$.

We now turn to the scattering problem.

C. *Renormalized Scattering Sector*

For the scattering sector, the scattering amplitude $f_k^{(D)}(\Omega^{(D)})$ and the differential scattering cross section $d\sigma^{(D)}(k, \Omega^{(D)})/d\Omega_D = |f_k^{(D)}(\Omega^{(D)})|^2$ are functions of the energy $E = k^2$ associated with the incident momentum k, as well as of the hyperspherical angles $\Omega^{(D)}$ (with $d\Omega_D$ being the corresponding element of the D-dimensional solid angle; see Appendix A).

As discussed in Subsection III.D and using the language developed in Subsections V.A and V.B, there are two distinct regimes for scattering.

In the weak-coupling regime, $\lambda < \lambda^{(*)}$, the scattering is well defined for all values of the coupling constant (consistent with the inequality defining the weak-coupling regime). In particular, this scattering is either scale-invariant (energy-independent) or trivial and needs no regularization whatsoever. For example, the inverse square potential gives scale-invariant scattering [7, 11], when $\lambda < (l + D/2 - 1)^2$ (where l is the angular momentum quantum number), while the two-dimensional delta potential yields no scattering for $\lambda < 0$. These examples and issues will be analyzed in greater detail in the second paper in this series.

Instead, in the strong-coupling regime, $\lambda > \lambda^{(*)}$, the coupling constant gets renormalized according to the theory developed in Subsection V.B. In particular, this implies that the coupling parameter of the regularized theory is ε-dependent, as displayed in Eq. (5.17), with a limiting critical value $\lambda = \lambda^{(*)} + 0^+$. Moreover, Eq. (4.29) is still applicable, as it was derived solely using scaling arguments—but the function $\Xi(\varepsilon)$ will now have a different specific form, one that is no longer discrete. Then, the scattering matrix and amplitude are determined from the asymptotic form of the scattering wave function, which is an appropriate linear combination of scattering solutions with arguments $kr = (\hat{\eta})^{1/2} |\hat{\xi}| = [\, \Xi(\varepsilon) \,]^{1/\varepsilon} |\hat{\xi}|$. Equation (4.29) then implies that the scattering depends upon

$$|\eta(\varepsilon)|^{-\varepsilon/2} = 1 - \frac{\varepsilon}{2} \ln(k^2/\mu^2), \tag{5.31}$$

which, after taking the limit $\varepsilon = 0^+$ and using Eq. (5.17), should provide a breakdown of the scale symmetry through the logarithmic dependence $\ln(k^2/\mu^2)$. This suggests that the scattering amplitude should be of the form

$$f_k^{(D)}(\Omega^{(D)}) = F(k, \ln(k^2/\mu^2), \Omega^{(D)}), \tag{5.32}$$

where F is a dimensional quantity. This procedure will be illustrated for the two-dimensional delta-function potential in Section VI.

In Eq. (5.32), the variable k^2/μ^2 explicitly appears in a characteristic logarithmic form. However, the function F is not dimensionless so that its form can be simplified by the Π theorem. We now turn to such dimensional considerations.

D. *Dimensional Analysis Revisited*

Let us now rephrase some of the results of Subsections V.B and V.C in terms of dimensional variables.

The dimensional bound-state energies are arranged in a spectrum

$$E_n = \mu^2 \eta_n, \tag{5.33}$$

a conclusion that can be directly drawn from dimensional analysis. In particular, the ground state defines a conventional characteristic scale

$$E_{(\mathrm{gs})} = \mu^2 \eta_{(\mathrm{gs})} \rightsquigarrow -\mu^2, \tag{5.34}$$

where the symbol $\rightsquigarrow$ refers to the freedom to make a convenient choice of $g^{(0)}$, due to its arbitrariness; in this case, we selected $g^{(0)} = 0$. This point will be further discussed and illustrated in Subsection VI.B for the particular case of the two-dimensional delta-function potential. Equation (5.34) displays in its purest form the emergence of an energy scale from the renormalization procedure; in addition, it shows that naive generalized dimensional analysis (including renormalization parameters according to Eq. (2.6)) gives straightforwardly the correct result.

Of course, Eq. (5.33) also refers to the excited states, if they exist. Again, by the generalized Π theorem, the only remaining information about the spectrum is conveyed by the ratios (cf. Eqs. (5.24) and (5.30))

$$\rho_n = \frac{\eta_n}{\eta_{(\mathrm{gs})}} = \exp[\mathscr{R}_n(\varepsilon)] \overset{(\varepsilon \to 0)}{\sim} \exp[\mathscr{G}_n(\varepsilon) - \mathscr{G}_{(\mathrm{gs})}(\varepsilon)], \tag{5.35}$$

which give its characteristic "structure," with the restrictions discussed in Subsection V.B.

For the scattering sector, as the dimensionality of the cross section is $-(D-1)$ ("area" of hypersurface), it follows that Eq. (5.32) can be rewritten in the form

$$f_k^{(D)}(\Omega^{(D)}) = k^{-(D-1)/2} \Pi(\ln(k^2/\mu^2), \Omega^{(D)}), \tag{5.36}$$

where $\Pi(u, \Omega^{(D)})$ is a dimensionless function of the dimensionless ratio $u = (k/\mu)^2$. Equation (5.36) will be valid, whether the system is capable of producing bound states or not; in the weak-coupling regime, the function Π is identically constant. On the other hand, if there is at least one bound state, the existence of a characteristic energy scale $E_{(\mathrm{gs})}$, Eq. (5.34), yields an alternative form of (5.36),

$$f_k^{(D)}(\Omega^{(D)}) = k^{-(D-1)/2} \check{\Pi}(\ln(k^2/E_{(\mathrm{gs})}), \Omega^{(D)}), \tag{5.37}$$

where $\check{\Pi}(\ln(E/E_{(\mathrm{gs})}), \Omega^{(D)}) = \Pi(\ln(E/\mu^2), \Omega^{(D)})$ is another dimensionless function. In fact, when the assignment $g^{(0)} = 0$ is made (Eq. (5.34)), the simple identity $\check{\Pi} = \Pi$ takes place.

Equations (5.33), (5.36), and (5.37) are just a consequence of generalized dimensional analysis.

VI. TWO-DIMENSIONAL DELTA-FUNCTION POTENTIAL

One of the most basic properties of a quantum field theory is locality, which leads to a nonrelativistic limit with highly singular potentials of zero range, also known as pseudopotentials [14]. The simplest pseudopotentials are plain delta functions, which display a large number of unusual features; however, in this section we will only explore those properties that relate to the dimensional transmutation produced by the two-dimensional representative of this class. As we will see, this potential displays all the characteristic fingerprints of dimensional transmutation that we discussed in previous sections. In fact, the two-dimensional delta-function potential has been extensively studied in the literature, mainly using cutoff regularization [2, 6] and square-well regularization [54]. In our approach, we will exclusively use dimensional regularization within the framework defined in Sections IV and V.

Our strategy will be to compare the calculations with the predictions and requirements of the general theory of Sections IV and V. However, we will use the dimensional form (4.20) of the Schrödinger equation from scratch, rather than any of the dimensionless equations (4.22) or (4.28). In effect, the dimensionless counterparts are most useful at a theoretical level, in establishing the relationships between all relevant parameters for our problem; yet, in practice, it is more direct to set up the "ordinary" (dimensional) dimensionally regularized Schrödinger equation.

A. *Dimensional Regularization of the Two-Dimensional Delta-Function Potential*

The two-dimensional delta-function potential is a particular zero-range or contact interaction of the form

$$V(\mathbf{r}) = -\lambda \delta^{(2)}(\mathbf{r}). \tag{6.1}$$

We have already seen that this potential is scale-invariant. Using the techniques of Section IV, we now apply the dimensional-continuation prescription of Eq. (4.8) to $W^{(2)}(\mathbf{r}) = \delta^{(2)}(\mathbf{r})$, with $D_0 = 2$, i.e.,

$$\begin{aligned} W^{(D)}(\mathbf{r}_D) &= \int \frac{d^D k_D}{(2\pi)^D} e^{i\mathbf{k}_D \cdot \mathbf{r}_D} \left[\int d^2 r_2 \, e^{-i\mathbf{k}_2 \cdot \mathbf{r}_2} \delta^{(2)}(\mathbf{r}_2) \right]_{\mathbf{k}_2 \to \mathbf{k}_D}, \\ &= \int \frac{d^D k_D}{(2\pi)^D} e^{i\mathbf{k}_D \cdot \mathbf{r}_D} [\,1\,]_{\mathbf{k}_2 \to \mathbf{k}_D} \\ &= \int \frac{d^D k_D}{(2\pi)^D} e^{i\mathbf{k}_D \cdot \mathbf{r}_D} = \delta^{(D)}(\mathbf{r}), \end{aligned} \tag{6.2}$$

which is the obvious dimensional extension of the original delta-function potential. Thus, in what follows, we will consider the dimensionally regularized problem

$$[\, -\nabla^2_{\mathbf{r},D} - \lambda \mu^\varepsilon \delta^{(D)}(\mathbf{r}) \,]\, \Psi(\mathbf{r}) = E \Psi(\mathbf{r}). \tag{6.3}$$

Straightforward solution of Eq. (6.3) in this context should *not* be interpreted as a way of drawing conclusions about the D-dimensional delta-function potential. Instead, it is just the means to regularize the $D_0 = 2$ case. Of course, one could adjust the regularization to be applied around a value $D_0 \neq 2$; however, we will not attempt such modification in this paper, as it is not directly relevant to dimensional transmutation.

B. *Bound-State Sector for a Two-Dimensional Delta-Function Potential*

Equation (6.3) can be easily solved in momentum space; for the bound-state sector,

$$\tilde{\Psi}(\mathbf{q}) = \lambda \mu^\varepsilon \frac{\Psi(\mathbf{0})}{q^2 - E}, \tag{6.4}$$

which, via the inverse Fourier transform, yields the position-space eigenfunctions. However, if we are only interested in the eigenvalue equation, it suffices to consider the value of the wave function at the origin,

$$\Psi(\mathbf{0}) = \int \frac{d^D q}{(2\pi)^D} \tilde{\Psi}(\mathbf{q}), \tag{6.5}$$

so that Eq. (6.4) gives

$$\frac{\lambda \mu^\varepsilon}{(2\pi)^D} \int \frac{d^D q}{q^2 + |E|} = 1, \tag{6.6}$$

where $E = -|E|$ ($E < 0$ for the possible bound states). Equation (6.6) can be straightforwardly integrated using Eq. (A13), which implies that

$$\int \frac{d^D q}{q^2 + |E|} = \pi^{D/2} |E|^{D/2-1} \Gamma\left(1 - \frac{D}{2}\right), \tag{6.7}$$

and the regularized eigenvalue equation takes the form

$$\frac{\lambda \mu^\varepsilon}{4\pi} \left(\frac{|E|}{4\pi}\right)^{D/2-1} \Gamma\left(1 - \frac{D}{2}\right) = 1. \tag{6.8}$$

It is a simple exercise to verify that Eq. (6.8) reduces to the familiar textbook result $\kappa = \sqrt{|E|} = \lambda/2$ for $D = 1$ [55]. On the other hand, the left-hand side is divergent for $D = 2, 4, 6, \ldots$. However, the restriction on the spatial dimension D of regular potentials is even stronger because more stringent conditions are dictated by the eigenfunctions, as we will see next. In our subsequent analysis, both for two-dimensional delta-function and inverse square potentials, the dimension D will usually appear in terms of the variable

$$\nu = D/2 - 1, \tag{6.9}$$

which will thereby simplify the form of most formulas; for example, the eigenvalue Eq. (6.8) reads (with $\varepsilon = -2\nu$)

$$\frac{\lambda \mu^{-2\nu}}{4\pi} \left(\frac{|E|}{4\pi}\right)^{\nu} \Gamma(-\nu) = 1. \tag{6.10}$$

Then the inverse Fourier transform,

$$\Psi(\mathbf{r}) = \lambda \mu^\varepsilon \Psi(\mathbf{0}) \int \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2 + |E|} \frac{d^D q}{(2\pi)^D}, \tag{6.11}$$

is recognized to be proportional to the Green's function $\mathscr{D}_D(\mathbf{r}; \kappa)$ for the modified Helmholtz equation (see Appendix B, in particular Eqs. (B1)–(B3)),

$$\Psi(\mathbf{r}) = -\lambda \mu^\varepsilon \Psi(\mathbf{0}) \mathscr{K}_D(\mathbf{r}; \kappa) \tag{6.12}$$

$$= \frac{\lambda \mu^\varepsilon \Psi(\mathbf{0})}{2\pi} \left(\frac{\kappa}{2\pi r}\right)^{\nu} K_\nu(\kappa r), \tag{6.13}$$

where $\kappa = \sqrt{|E|}$, $r = |\mathbf{r}|$, and $K_\nu(z)$ is the modified Bessel function of the second kind of order ν. The asymptotic behavior of the wave function $\Psi(\mathbf{r})$ (and of the Green's function $\mathscr{K}_D(\mathbf{r}; \kappa)$) is governed by that of $K_\nu(z)$ [56],

$$K_\nu(z) \overset{(z \to \infty)}{\sim} \sqrt{\frac{\pi}{2z}}\, e^{-z} [1 + O(1/z)], \tag{6.14}$$

whence

$$\Psi(\mathbf{r}) \overset{(r\to\infty)}{\sim} \frac{\lambda\mu^{\varepsilon}\Psi(\mathbf{0})}{4\pi}\left(\frac{\kappa}{2\pi}\right)^{(D-3)/2}\frac{e^{-\kappa r}}{r^{(D-1)/2}}\,[1+O(1/r)], \tag{6.15}$$

which displays the correct behavior for a bound-state wave function at infinity. However, near the origin, the modified Bessel function has a singular behavior [57] of the form

$$K_p(z) \overset{(z\to 0)}{\sim} \frac{1}{2}\left[\Gamma(p)\left(\frac{z}{2}\right)^{-p}+\Gamma(-p)\left(\frac{z}{2}\right)^{p}\right][1+O(z^2)], \tag{6.16}$$

$$\overset{(z\to 0)}{\sim} \begin{cases} \frac{1}{2}\Gamma(|p|)(2/z)^{|p|}\,[1+O(z^2)] & \text{for} \quad p\neq 0 \\ -[\ln(z/2)+\gamma][1+O(p,z^2)] & \text{for} \quad p\approx 0, \end{cases} \tag{6.17}$$

where γ is the Euler–Mascheroni constant; then, the explicit form of the wave function is

$$\Psi(\mathbf{r}) \overset{(r\to 0)}{\sim} \frac{\lambda\mu^{\varepsilon}\Psi(\mathbf{0})}{4\pi}\left(\frac{\kappa^2}{4\pi}\right)^{\nu}\times\begin{cases}\Gamma(\nu)(\kappa r/2)^{-2\nu} & \text{for} \quad D>2 \\ -2[\ln(\kappa r/2)+\gamma] & \text{for} \quad D=2 \\ \Gamma(-\nu) & \text{for} \quad 0<D<2,\end{cases} \tag{6.18}$$

which shows that the nature of the solution changes around $\nu=0$, i.e., for $D=2$. This confirms the *critical* character of the dimension $D=2$ for the delta-function potential. Notice that the wave functions are singular at the origin for any dimension $D\geqslant 2$. Parenthetically, this is an example of an ultraviolet divergence: the wave function is singular at small distances or due to large momenta (cf. Eq. (6.6)). For $D<2$, we can regularize the two-dimensional delta-function potential and take the limit $\nu\to 0$ in Eq. (6.18), thus recovering self-consistently the eigenvalue equation (6.10), which, with $D=2-\varepsilon$, i.e., $\nu=-\varepsilon/2$, reads

$$\frac{\lambda\mu^{\varepsilon}}{4\pi}\left(\frac{|E|}{4\pi}\right)^{-\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}\right)=1. \tag{6.19}$$

Alternatively, in the language of Eq. (6.13), this eigenvalue equation can be simply enforced by the condition

$$-\lambda\mu^{\varepsilon}\mathscr{K}_D(\mathbf{0};\kappa)=1. \tag{6.20}$$

Having completed the exploratory analysis of the bound-state sector and found the eigenvalue equation ((6.8), (6.10), (6.19), or (6.20)) we are ready to compare

these expressions with the general eigenvalue equation (5.1), which will now include an energy generating function

$$\Xi(\varepsilon) = \frac{1}{4\pi}(4\pi)^{\varepsilon/2}\,\Gamma\left(\frac{\varepsilon}{2}\right). \tag{6.21}$$

It is immediately apparent from Eq. (6.21) that the theory has only one bound state, so that there is no need for a quantum number. In order to determine whether this ground state survives the renormalization process, we should look at the $\varepsilon = 0^+$ limit of Eq. (6.21), which can be conveniently examined through the expansion

$$\Xi(\varepsilon) = \frac{1}{2\pi\varepsilon}\left[\,1 + \frac{\varepsilon}{2}(\ln 4\pi - \gamma) + O(\varepsilon^2)\right]. \tag{6.22}$$

From Eqs. (5.2) and (6.22), the critical coupling is found to be

$$\lambda^{(*)} = 0, \tag{6.23}$$

so that the theory looks asymptotically free but still engenders a unique bound state. It should be noticed that this is achieved by the regularization of the coupling constant through the strategy of Eq. (5.17) (or Eq. (5.29)), so that

$$\lambda(\varepsilon) = 2\pi\varepsilon\left\{1 + \frac{\varepsilon}{2}[\,g^{(0)} - (\ln 4\pi - \gamma)\,]\right\}, \tag{6.24}$$

leading to a ground state

$$E_{(\mathrm{gs})} = -\mu^2 e^{g^{(0)}}. \tag{6.25}$$

A final remark about renormalization shows additional parallels with the corresponding field-theory problems. As usual, the arbitrariness in the choice of the finite part $g^{(0)}$ can be used to simplify the expressions above in such a way that $|E_{(\mathrm{gs})}| = \mu^2$, as displayed in Eq. (5.34). On the other hand, the singular nature of the ground state has been tamed by subtracting the divergent part of Eq. (6.21), which amounts to the subtraction of the pole $1/2\pi\varepsilon$. However, due to the arbitrariness in the choice of $g^{(0)}$, at the level of the ground-state energy, we have also subtracted—along with the pole—the term $\ln 4\pi - \gamma$ (which is an artifact of the dimensional-regularization technique). This is recognized to be the usual modified minimal subtraction ($\overline{\mathrm{MS}}$) scheme [58].

In conclusion, the unregularized problem has a singular spectrum with a unique energy level at $-\infty$ and vanishing critical coupling. The regularization process brings this level to a finite value, which, upon renormalization, becomes the unique ground state of the two-dimensional delta-function potential.

C. *Scattering Sector for a Two-Dimensional Delta-Function Potential*

The scattering sector of the Schrödinger equation is described by its equivalent Lippmann–Schwinger equation (C3), which, for a two-dimensional delta-function potential (6.3), takes the simple form

$$\Psi^{(+)}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \lambda\mu^{\varepsilon}\mathscr{G}_D^{(+)}(\mathbf{r}; k)\ \Psi^{(+)}(\mathbf{0}), \tag{6.26}$$

where $\mathscr{G}_D^{(+)}(\mathbf{r}; k)$ is the Green's function for the Helmholtz equation, with outgoing boundary conditions (See Appendix B, in particular Eqs. (B4)–(B8)). In particular, Eq. (6.26) implies that

$$\Psi^{(+)}(\mathbf{0}) = [\,1 + \lambda\mu^{\varepsilon}\mathscr{G}_D^{(+)}(\mathbf{0}; k)\,]^{-1}. \tag{6.27}$$

The asymptotic form of Eq. (6.26) is obtained as described in Appendix C, according to which (e.g., Eqs. (C5) and (C8)) the scattering amplitude becomes

$$f_k^{(D)}(\Omega^{(D)}) = -\Gamma_D(k)\ \lambda\mu^{\varepsilon}\Psi^{(+)}(\mathbf{0}), \tag{6.28}$$

where

$$\Gamma_D(k) = -\frac{1}{4\pi}\left(\frac{k}{2\pi}\right)^{(D-3)/2}. \tag{6.29}$$

Finally, Eqs. (6.27) and (6.28) provide the desired expression,

$$f_k^{(D)}(\Omega^{(D)}) = -\Gamma_D(k)[\,(\lambda\mu^{\varepsilon})^{-1} + \mathscr{G}_D^{(+)}(\mathbf{0}; k)\,]^{-1}. \tag{6.30}$$

Equation (6.30) is singular for $D \geqslant 2$, as can be seen from the divergent small-argument limit of Eq. (B8). However, one can use the renormalization of the bound-state sector to eliminate this divergence through the regularized coupling, Eq. (6.24). More precisely, for an attractive potential, we found that the coupling constant can be traded in favor of dimensional parameters, e.g., using the Green's function $\mathscr{K}_D(\mathbf{r}; \kappa)$ for the bound-state sector in Eq. (6.20). In other words, using the renormalization for the bound-state sector, we will now obtain directly the renormalized scattering amplitude, which is explicitly given by the limit

$$f_k(\Omega^{(D)}) = \Gamma_D(k) \lim_{r\to 0}[\,\mathscr{K}_D(\mathbf{r}; \sqrt{E_{(\mathrm{gs})}}) - \mathscr{G}_D^{(+)}(\mathbf{r}; k)\,]^{-1}, \tag{6.31}$$

where the replacement

$$\kappa = \sqrt{|E_{(\mathrm{gs})}|} \tag{6.32}$$

was made. Equation (6.31) already displays a remarkable property of the scattering by a delta-function potential: it is *isotropic*, i.e., it only scatters s-waves, as it corresponds intuitively to a contact interaction.

Let us now evaluate the limit in Eq. (6.31). First, from Eqs. (6.7) and (B2),

$$\lim_{r\to 0} \mathscr{K}_D(\mathbf{r}; \sqrt{|E_{(\mathrm{gs})}|}) = -\frac{1}{(2\pi)^D} \int \frac{d^D q}{q^2 + |E_{(\mathrm{gs})}|}$$

$$= -\frac{1}{(4\pi)^{D/2}} |E_{(\mathrm{gs})}|^{D/2-1} \Gamma\left(1 - \frac{D}{2}\right). \tag{6.33}$$

Next, $\lim_{r\to 0} \mathscr{G}_D^{(+)}(\mathbf{r}; k)$ can be obtained by analytic continuation

$$\mathscr{G}_D^{(+)}(\mathbf{r}; k) = \mathscr{K}_D(\mathbf{r}; \kappa)|_{\kappa^2 \to -(k^2 + i\delta)}, \tag{6.34}$$

where $\delta = 0^+$, which implies that

$$\lim_{r\to 0} \mathscr{G}_D^{(+)}(\mathbf{r}; k) = \frac{1}{(2\pi)^D} \int \frac{d^D q}{k^2 - q^2 + i\delta}$$

$$= -\frac{1}{(4\pi)^{D/2}} (-k^2 - i\delta)^{D/2-1} \Gamma\left(1 - \frac{D}{2}\right). \tag{6.35}$$

Thus,

$$\lim_{r\to 0} [\mathscr{G}_D^{(+)}(\mathbf{r}; k) - \mathscr{K}_D(\mathbf{r}; \kappa)]$$

$$= \frac{1}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) [|E_{(\mathrm{gs})}|^{D/2-1} - (-k^2 - i\delta)^{D/2-1}], \tag{6.36}$$

which can be evaluated in the limit $\varepsilon \to 0^+$, with $D = 2 - \varepsilon$,

$$\lim_{\varepsilon\to 0} \lim_{r\to 0} [\mathscr{G}_D^{(+)}(\mathbf{r}; k) - \mathscr{K}_D(\mathbf{r}; \kappa)] = -\frac{1}{4\pi} (\ln |E_{(\mathrm{gs})}| - \ln k^2 + i\pi), \tag{6.37}$$

where the identity $\ln[-(k^2 + i\delta)] = \ln k^2 - i\pi$ was used. Finally, the scattering amplitude is obtained by replacing Eqs. (6.29) (with $D = 2$) and (6.37) in (6.31), i.e.,

$$f_k^{(2)}(\Omega^{(2)}) = \sqrt{\frac{2\pi}{k}} \left[\ln\left(\frac{k^2}{E_{(\mathrm{gs})}}\right) - i\pi \right]^{-1}. \tag{6.38}$$

Equation (6.38) is seen to agree with the prediction of generalized dimensional analysis, Eqs. (5.36)–(5.37), with a dimensionless variable $\Pi(u) = \sqrt{2\pi}[\ln u - i\pi]^{-1}$, and $u = k^2/E_{(\mathrm{gs})}$.

Finally, the differential scattering cross section $d\sigma^{(2)}(k, \Omega^{(2)})/d\Omega_2$, from Eq. (C9), again agrees with the prediction of generalized dimensional analysis, Eqs. (5.36)–(5.37), providing a dimensionless variable $\Pi(u) = 2\pi[(\ln u)^2 + \pi^2]^{-1}$, for the energy ratio $u = k^2/E_{(gs)}$.

VII. CONCLUSIONS

Until recently, it had been generally assumed that generic field-theoretic tools and concepts are useful only for systems with infinitely many degrees of freedom. While this perception is essentially correct for "regular" systems, it is now recognized, as discussed in our series of papers, that such techniques can be generalized and used to extract meaningful physical results for certain "singular" systems with a finite number of degrees of freedom.

In this paper, we developed systematic uses of the techniques of dimensional regularization and renormalization, and of the concept of dimensional transmutation, with the purpose of gathering information about the class of scale-invariant potentials. Our discussion relied on dimensional regularization, which we argued provides a generic tool for the treatment of all members of that class, by establishing a simple link between the two meanings of the word dimension. From our fairly general investigation, we have learned that all scale-invariant potentials are homogeneous of degree -2 and share a number of remarkable properties; here, without attempting to give an exhaustive list, we summarize a few of the most outstanding:

(i) There exists a critical coupling $\lambda^{(*)}$ such that, for $\lambda < \lambda^{(*)}$ (weak coupling) the Hamiltonian is self-adjoint but produces no bound states, while for $\lambda > \lambda^{(*)}$ (strong coupling) it loses its self-adjoint character, generating a continuum of bound states not bounded from below and requiring renormalization.

(ii) Solution of the regularized theory for strong coupling yields a master eigenvalue equation (4.29), which condenses all the information about the given scale-invariant potential and requires proper renormalization.

(iii) The ground state of a given "strong" scale-invariant potential exists provided that $\lambda^{(*)} = [\lim_{\varepsilon \to 0} \Xi_n(\varepsilon)]^{-1}$ (Eq. (5.2)) be finite.

(iv) Excited states exist under the demanding conditions listed after Eq. (5.24). Thus, "strong" scale-invariant potentials have a tendency to suppress excited states.

(v) The scattering sector remains scale-invariant or trivial in the weak-coupling regime, while it displays a logarithmic dependence $\ln(k^2/\mu^2)$, with respect to the energy k^2, in the strong-coupling regime.

(vi) In short, in the strong-coupling regime, a given scale-invariant potential leads to dimensional transmutation, which manifests itself on the existence of at least one bound state and a scale-dependent scattering matrix. The dimensional transmutation exhibited for strong coupling amounts to the emergence of a scale anomaly, i.e., quantum-mechanical breaking of the classical scale symmetry.

Additional progress in understanding these singular quantum-mechanical systems can best be achieved by studying specific cases. A first attempt was made in this paper by considering some aspects of the two-dimensional delta-function potential. In that regard, the second paper in this series [49] will present a more thorough analysis of the two-dimensional delta-function potential, as well as a novel treatment of the anomalous transmuting behavior of the inverse square potential.

APPENDIX A

Dimensional Regularization in D-Dimensional Euclidean Spaces

Just like for the corresponding case in quantum field theory [58], our approach is based on the dimensional extension of mathematical expressions for a system that is assumed to be embedded in a D-dimensional Euclidean space. Then, starting from the Cartesian coordinates $(x_1, ..., x_D)$, one can introduce an alternative set of D-dimensional hyperspherical polar coordinates $(q_0 = r,\ q_1 = \theta_1, ..., q_{D-1} = \theta_{D-1})$ via the transformation equations

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ &\vdots \\ x_j &= r\left(\prod_{k=1}^{j-1}\sin\theta_k\right)\cos\theta_j \\ &\vdots \\ x_D &= r\prod_{k=1}^{D-1}\sin\theta_k, \end{aligned} \tag{A.1}$$

where the ranges for the hyperspherical polar variables are $0 \leqslant \theta_j \leqslant \pi$ for $j = 1, ..., D-2$; $0 \leqslant \phi \equiv \theta_{D-1} \leqslant 2\pi$; and $0 \leqslant r < \infty$. All the basic geometric quantities associated with hyperspherical coordinates can be constructed through the scale coefficients h_j for the diagonal metric $(g_{ij}) = \mathrm{diag}(h_j^2)_{0 \leqslant j \leqslant D-1}$ [59]; these coefficients are straightforwardly given by $h_0 = 1$, $h_1 = r$, and $h_j = r\prod_{k=1}^{j-1}\sin\theta_k$ (for $2 \leqslant j \leqslant D-1$), while their product is

$$h^{(D)} = \prod_{j=0}^{D-1} h_j = r^{D-1}\prod_{j=1}^{D-1}\sin^{D-j-1}\theta_j. \tag{A2}$$

In our series of papers, both the Laplacian operator and the element of volume are needed. The Laplacian can be computed from [60]

$$\nabla_D^2 = \frac{1}{h^{(D)}} \sum_{j=0}^{D-1} \frac{\partial}{\partial q_j}\left(\frac{h^{(D)}}{h_j^2}\frac{\partial}{\partial q_j}\right)$$

$$= \Delta_r^{(D)} + \frac{1}{r^2}\Delta_{\Omega^{(D)}}, \tag{A3}$$

where its radial part is, explicitly,

$$\Delta_r^{(D)} = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right), \tag{A4}$$

while its angular part is

$$\Delta_{\Omega^{(D)}} = \sum_{j=1}^{D-1}\left[\left(\prod_{k=1}^{j-1}\sin^2\theta_k\right)\sin^{D-j-1}\theta_j\right]^{-1}\frac{\partial}{\partial\theta_j}\left(\sin^{D-j-1}\theta_j\frac{\partial}{\partial\theta_j}\right), \tag{A5}$$

in which the notation $\Omega^{(D)} \equiv (\theta_1, ..., \theta_{D-1})$ has been introduced and it is implied that $\prod_{k=k_1}^{k_2} \equiv 1$ when $k_1 > k_2$ (i.e., for $j = 1$).

Similarly, the element of the D-dimensional solid angle becomes

$$d\Omega_D = \prod_{j=1}^{D-1} h_j\, d\theta_j = h^{(D)} \prod_{j=1}^{D-1} d\theta_j = \prod_{j=1}^{D-1}\sin^{D-j-1}\theta_j\, d\theta_j, \tag{A6}$$

in terms of which the D-dimensional volume element is given by

$$d^D r = r^{D-1}\, d\Omega_D\, dr. \tag{A7}$$

Equation (A6) can be integrated to a total D-dimensional solid angle

$$\Omega_D = \int d\Omega_D = \left(\prod_{j=1}^{D-2}\int_0^\pi d\theta_j \sin^{D-j-1}\theta_j\right)\int_0^{2\pi} d\theta_{D-1}, \tag{A8}$$

where the angular integrals can be evaluated using the beta-function identity [56],

$$\int_0^{\pi/2}\sin^m\theta\, d\theta = \frac{1}{2}B((m+1)/2, 1/2) = \frac{\sqrt{\pi}\,\Gamma((m+1)/2)}{2\Gamma((m+2)/2)}, \tag{A9}$$

whence

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{A10}$$

With the given element of volume, Eq. (A7), one can compute the integral of any function $f(r)$ that exhibits D-dimensional central symmetry, that is,

$$\int f(r)\, d^D r = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty r^{D-1} f(r)\, dr. \tag{A11}$$

In particular, Eqs. (A6)–(A11) are essential for the evaluation of expressions in dimensional regularization, in conjunction with another beta-function identity [56],

$$\int_0^\infty \frac{x^{2\alpha-1}}{(x^2+1)^{\alpha+\beta}}\, dx = \frac{1}{2} B(\alpha, \beta), \tag{A12}$$

whence

$$\int \frac{(q^2)^n}{(q^2+a^2)^m}\, d^D q = \pi^{D/2} a^{D+2n-2m} \frac{\Gamma(n+D/2)\, \Gamma(m-n-D/2)}{\Gamma(D/2)\, \Gamma(m)}. \tag{A13}$$

Finally, let us consider the general D-dimensional Fourier transform $\tilde{f}(\mathbf{s})$ of a function $f(\mathbf{u})$, defined by

$$\tilde{f}(\mathbf{s}) = \frac{n_D}{(2\pi)^{D/2}} \int d^D u\, e^{\mp i \mathbf{s}\cdot\mathbf{u}} f(\mathbf{u}), \tag{A14}$$

with an arbitrary normalization constant n_D (usually $n_D = 1$ or $n_D = (2\pi)^{\pm D/2}$). Its computation can be simplified considerably for the particular case when the function displays central symmetry, i.e., $f(\mathbf{u}) = f(u)$. In effect, for the integration of $f(u)$, selecting coordinates according to $\mathbf{s}\cdot\mathbf{u} = su\cos\theta_1$, it follows that

$$\begin{aligned}
\tilde{f}(\mathbf{s}) &= \frac{n_D}{(2\pi)^{D/2}} \left(\prod_{j=2}^{D-2} \int_0^\pi d\theta_j \sin^{D-j-1}\theta_j \right) \int_0^{2\pi} d\theta_{D-1} \\
&\quad \times \int_0^\infty du\, u^{D-1} f(u) \int_0^\pi d\theta_1 \sin^{D-2}\theta_1\, e^{\mp isu\cos\theta_1} \\
&= n_D (2\pi)^{-D/2}\, \Omega_{D-1} \int_0^\infty du\, u^{D-1} f(u)\, \mathscr{I}(D/2-1, su),
\end{aligned} \tag{A15}$$

where [56]

$$\mathscr{I}(v, z) = \int_0^\pi e^{\mp iz\cos\theta} \sin^{2v}\theta\, d\theta = \frac{\Gamma(v+1/2)\, \Gamma(1/2)}{(z/2)^v} J_v(z), \tag{A16}$$

which implies that ($\nu = D/2 - 1$)

$$\tilde{f}(\mathbf{s}) = \frac{n_D}{(2\pi)^{D/2}} \int d^D u \, e^{-i\mathbf{s}\cdot\mathbf{u}} f(\mathbf{u})$$

$$= \frac{n_D}{s^{D/2-1}} \int_0^\infty f(u) \, J_{D/2-1}(su) \, u^{D/2} \, du. \tag{A17}$$

Equation (A17), which is a Hankel transform, is sometimes referred to as Bochner's theorem.

APPENDIX B

D-Dimensional Green's Functions

As an example of Bochner's theorem, we will now compute the infinite-space Green's function for the D-dimensional Helmholtz equation. We will start with the modified Helmholtz equation,

$$[\nabla^2_{\mathbf{r},D} - \kappa^2] \, K_D(\mathbf{r}, \mathbf{r}'; \kappa) = \delta^{(D)}(\mathbf{r} - \mathbf{r}'), \tag{B1}$$

whose Green's function $K_D(\mathbf{r}, \mathbf{r}'; \kappa)$ can be computed by applying translational invariance, i.e., $K_D(\mathbf{r}, \mathbf{r}'; \kappa) = \mathscr{K}_D(\mathbf{R}; \kappa)$, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Its Fourier transform $\tilde{\mathscr{K}}_D(\mathbf{q}; \kappa) = -(q^2 + \kappa^2)^{-1}$ leads to an integral expression

$$\mathscr{K}_D(\mathbf{R}; \kappa) = -\int \frac{d^D q}{(2\pi)^D} \frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{q^2 + \kappa^2}$$

$$= -(2\pi)^{-D/2} \, R^{-(D/2-1)} \int_0^\infty \frac{q^{D/2} J_{D/2-1}(qR)}{q^2 + \kappa^2} \, dq, \tag{B2}$$

in which Eq. (A17) was used. Equation (B2) can be explicitly evaluated in terms of the modified Bessel function of the second kind $K_\nu(\kappa R)$, of order $\nu = D/2 - 1$, i.e. [56],

$$\mathscr{K}_D(\mathbf{R}; \kappa) = -\frac{1}{2\pi} \left(\frac{\kappa}{2\pi R} \right)^\nu K_\nu(\kappa R), \tag{B3}$$

where the dimensional variable $\nu = D/2 - 1$ (cf. Eq. (6.9)) has been explicitly introduced.

Likewise, for the ordinary Helmholtz equation,

$$[\nabla^2_{\mathbf{r},D} + k^2] \, G_D(\mathbf{r}, \mathbf{r}'; k) = \delta^{(D)}(\mathbf{r} - \mathbf{r}') \tag{B4}$$

in infinite space, translational invariance implies that $G_D(\mathbf{r}, \mathbf{r}'; k) = \mathscr{G}_D(\mathbf{R}; k)$, with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. However, its Fourier transform $\tilde{\mathscr{G}}_D(\mathbf{q}; k) = (k^2 - q^2)^{-1}$ leads to an ill-defined integral expression that needs to be evaluated by a prescription defining the boundary conditions at infinity; for outgoing $(+)$ and incoming $(-)$ boundary conditions,

$$\mathscr{G}_D^{(\pm)}(\mathbf{R}; k) = \int \frac{d^D q}{(2\pi)^D} \frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{k^2 - q^2 \pm i\delta}$$

$$= (2\pi)^{-D/2} R^{-(D/2-1)} \int_0^\infty \frac{q^{D/2} J_{D/2-1}(qR)}{k^2 - q^2 \pm i\delta} dq, \tag{B5}$$

where Eq. (A17) was used and $\delta = 0^+$. Equation (B5) can be explicitly evaluated in terms of Hankel functions of order $\nu = D/2 - 1$; in fact, it is easy to see that Eq. (B4) can be obtained from (B1) with the replacement $\kappa = \mp ik$, so that

$$\mathscr{G}_D^{(\pm)}(\mathbf{R}; k) = \mathscr{K}_D(\mathbf{R}; \kappa = \mp ik), \tag{B6}$$

and the choice of signs amounts to the choice of boundary conditions at infinity or the $i\delta$ prescription. From the identity [57]

$$K_\nu(\mp iz) = \pm \frac{\pi i}{2} e^{\pm i\pi\nu/2} H_\nu^{(1,2)}(z), \tag{B7}$$

Eq. (B6) is converted into

$$\mathscr{G}_D^{(\pm)}(\mathbf{R}; k) = \mp \frac{i}{4} \left(\frac{k}{2\pi R}\right)^\nu H_\nu^{(1,2)}(kR). \tag{B8}$$

Equations (B3) and (B8) are well known [61] and reduce to the familiar results in one, two, and three dimensions [62].

APPENDIX C

Scattering in D Dimensions

Just as in the standard 3-D scattering formalism, the D-dimensional time-independent operator Schrödinger equation

$$(H_0 - E)\,|\Psi\rangle = -V\,|\Psi\rangle \tag{C2}$$

is equivalent to a Lippmann–Schwinger equation [63]

$$|\Psi^{(+)}\rangle = |\chi\rangle + (E - H_0 + i\delta)^{-1}\, V\, |\Psi^{(+)}\rangle \tag{C2}$$

in which the state vector $|\Psi^{(+)}\rangle$ is explicitly resolved into an incident wave $|\chi\rangle$ (solution of the free-particle case) and a second term that represents the outgoing scattered wave (with an appropriate boundary condition summarized by the $i\delta = i0^+$ prescription). In what follows, we will assume that $|\chi\rangle = |\chi_{\mathbf{k}}\rangle$ represents a D-dimensional plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$. Equation (C2) can be converted into the integral form

$$\Psi^{(+)}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \int d^D r' G_D^{(+)}(\mathbf{r}, \mathbf{r}'; k)\; V(\mathbf{r}')\; \Psi^{(+)}(\mathbf{r}'), \tag{C3}$$

by the introduction of one of the Green's functions computed in Appendix A, namely, $G_D^{(+)}(\mathbf{r}, \mathbf{r}'; k) = \langle \mathbf{r}|\, (E - H_0 + i\delta)^{-1}\, |\mathbf{r}'\rangle$, which is a solution to Eq. (B4), and explicitly given by Eq. (B8).

For the scattering problem, one is interested in the asymptotic form of the Green's function, which follows from [56]

$$H_\nu^{(1)}(z) \overset{(z\to\infty)}{\sim} \sqrt{\frac{2}{\pi z}}\, e^{i(z - \nu\pi/2 - \pi/4)}, \tag{C4}$$

whence

$$\begin{aligned} G_D^{(+)}(\mathbf{r}, \mathbf{r}'; k) \quad &= \quad \mathscr{G}_D^{(+)}(\mathbf{R}; k) \\ &\overset{(r\to\infty)}{\sim} -\frac{1}{4\pi}\left(\frac{k}{2\pi}\right)^{(D-3)/2} e^{i\gamma_D}\, \frac{e^{ikr}}{r^{(D-1)/2}}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \end{aligned} \tag{C5}$$

where $\mathbf{k}' = k\mathbf{r}/r$ and

$$\gamma_D = (3 - D)\,\frac{\pi}{4}. \tag{C6}$$

From Eqs. (C3) and (C5), in the position representation,

$$\Psi^{(+)}(\mathbf{r}) \overset{(r\to\infty)}{\sim} e^{i\mathbf{k}\cdot\mathbf{r}} + f_k^{(D)}(\Omega^{(D)})\, e^{i\gamma_D}\, \frac{e^{ikr}}{r^{(D-1)/2}}, \tag{C7}$$

where the scattering amplitude

$$f_k^{(D)}(\Omega^{(D)}) = -\frac{1}{4\pi}\left(\frac{k}{2\pi}\right)^{(D-3)/2} \int d^D r' e^{-i\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}')\; \Psi^{(+)}(\mathbf{r}') \tag{C8}$$

leads to the usual expression for the differential scattering cross section,

$$\frac{d\sigma^{(D)}(k, \Omega^{(D)})}{d\Omega_D} = |f_k^{(D)}(\Omega^{(D)})|^2. \tag{C9}$$

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