On the connection between Complementarity and Uncertainty Principles in the Mach–Zehnder interferometric setting

G.M. Bosyk, M. Portesi, F. Holik, A. Plastino
Instituto de Física La Plata, CONICET, and Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, 115 y 49, C.C. 67, 1900 La Plata, Argentina
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We revisit, in the framework of Mach–Zehnder interferometry, the connection between the complementarity and uncertainty principles of quantum mechanics. Specifically, we show that, for a pair of suitably chosen observables, the trade-off relation between the complementary path information and fringe visibility is equivalent to the uncertainty relation given by Schrödinger and Robertson, and to the one provided by Landau and Pollak as well. We also employ entropic uncertainty relations (based on Rényi entropic measures) and study their meaning for different values of the entropic parameter. We show that these different values define regimes which yield qualitatively different information concerning the system, in agreement with findings of [A. Luis, Phys. Rev. A **84**, 034101 (2011)]. We find that there exists a regime for which the entropic uncertinty relations can be used as criteria to pinpoint non trivial states of minimum uncertainty.

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I. INTRODUCTION

The Complementarity Principle (CP) [1] lies at the heart of Quantum Mechanics (QM). Many years have passed since its original formulation but, still today, there is an important debate regarding its adequate interpretation and its precise definition in several contexts [2–5].

The Complementarity Principle has been both theoreticallyand experimentallystudied in the framework of Mach–Zehnder (MZ) interferometrics. The MZ framework is particularly suitable for discussions regarding wave–particle duality, and there is a debate concerning the complementarity of the fringe-visibility observable (wave aspect) and the which-way-has-passed question (particle aspect). In this regard, the wave and particle properties are represented by measurable quantities P and V, respectively, which satisfy the duality relation [2, 3]

$$P^2 + V^2 \leq 1. \tag{1}$$

Despite the fact that this quantitative formulation of the complementarity principle is expressed in a way that resembles inequalities typical of the uncertainty principle, the derivation of Eq. (1) does not involve any mention of inherent fluctuations in the measured quantities. Inspired in this quantitative similarity, we are interested in looking deeper at the connection between these two important principles of Quantum Mechanics. Specifically, we address the question: is Eq. (1) the expression of an uncertainty relation? This basic issue has been the subject of intense debate in the literature. Answers in both the affirmative and the negative have been provided by various authors (see, for instance, Refs. [4, 6–8]). Our goal is, with regards to Eq. (1), to shed some new light on the issue by considering several ways of quantifying uncertainty, concentrating attention on variance-based and entropy-based inequalities.

The article's outline is as follows: in Sect. II we review details of the discussion concerning the duality relation (1), introducing relevant operators that account for the path information and fringe visibility in double-slit-like experiments. Sect. III is devoted to summarize various formulations of the uncertainty principle that were applied to our problem, i.e. for a pair of two-level discrete operators, by employing variances as well as entropic and other measures. In Sect. IV we provide an affirmative answer to the question posed in the case of the uncertainty inequalities prescribed by Shrödinger–Robertson and by Landau–Pollak, demonstrating the full equivalence between them. Additionally, our analysis of a class of entropic uncertainty inequalities shows that they are not on the same footing as the above ones but that they yield nonetheless nontrivial information about the system. This is achieved by studying states that saturate the entropic inequality. We find that, according to the value of the Rényi parameter, different regimes can be discerned, a fact that can be interpreted as giving support to previous investigations [9]. Finally, some conclusions are drawn in Sect. V.

II. MACH–ZEHNDER INTERFEROMETER SCHEME AND COMPLEMENTARITY RELATION

The Mach–Zehnder interferometer (Fig. 1) is a device that has been used in several branches of physics, in particular, for the study of the Complementarity Principle. An important quantity is the "*which way*" information, that is quantified by the *predictability* P defined as $P = 2L - 1$, where $L = \max\{w_+, w_-\}$, and w_+ and w_- are the probabilities of the particle taking path "+" or path "-", respectively. On the other hand, the *fringe visibility* is quantified via a natural extension of the usual measure for intensity of light, that is $V = \frac{p_{\max} - p_{\min}}{p_{\max} + p_{\min}}$ where p stands for the probability that the particle be detected in some position in space, with $p_{\max}$ and $p_{\min}$ denoting, respectively, the maximum and minimum of this probability.

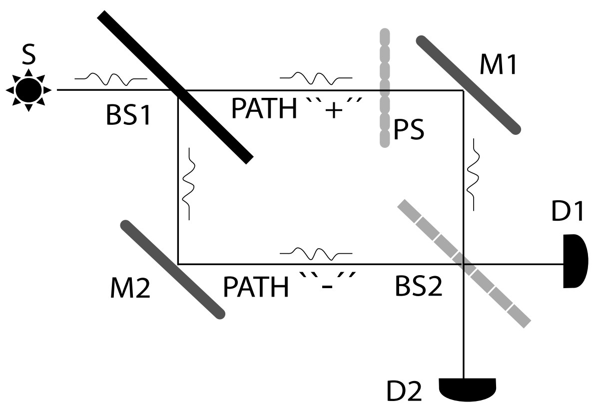

FIG. 1: A source S emits a photon which splits, after passing through the beam splitter BS1, into paths "+" and "-". It reflects in mirrors M1 and M2, and is finally observed using detectors D1 and D2. A phase shifter PS and another beam splitter BS2 may be inserted into the set-up, in order to produce interference.

The quantitative formulation of CP in the MZ-interferometer scheme is the celebrated duality relation [2, 3] given by Eq. (1), where the equal sign holds (only) for pure states. This relation was also implicitly alluded to in the pioneering works of Refs. [10, 11].

The MZ interferometer, having two relevant spatial modes, can be represented by a two-dimensional Hilbert space spanned for instance by $\{|0\rangle, |1\rangle\}$, which is the so-called computational basis. The states $|0\rangle$ and $|1\rangle$ are eigenstates of the Pauli spin operator σ_z, representing the two paths. We use the Bloch representation to describe quantum density operators as

$$\rho = \frac{I + \vec{s} \cdot \vec{\sigma}}{2} \tag{2}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denote the Pauli matrices, I is the 2×2 identity matrix, and $\vec{s} = (s_x, s_y, s_z)$ is the Bloch vector (with $\|\vec{s}\| \leq 1$) that characterizes the state. The action of the 50:50 beam splitter can be described by the unitary transformation $U_{\text{BS}} = e^{-i\pi\sigma_y/4}$, which implies a rotation of $\pi/2$ of the Bloch vector around the y-axis. The phase shifter $U_\phi = e^{-i\phi\sigma_z/2}$ introduces a phase difference equal to ϕ between the paths.

Following Ref. [6], a sharp observable $\hat{P}$ can be associated to the predictability, while two families of sharp observables $\hat{V}_\phi$ and $\hat{V}_\phi^\perp$ can be associated to the visibility. It is possible to express these operators in terms of the Pauli spin ones as

$$\hat{P} = \sigma_z \tag{3}$$

$$\hat{V}_\phi = (\cos\phi)\ \sigma_x + (\sin\phi)\ \sigma_y \tag{4}$$

$$\hat{V}_\phi^\perp = -(\sin\phi)\ \sigma_x + (\cos\phi)\ \sigma_y \tag{5}$$

with ϕ ranging, in principle, between 0 and 2π. Note that $\hat{P}$, $\hat{V}_\phi$, and $\hat{V}_\phi^\perp$ are a set of mutually complementary observables, that is, if one is certain about the value of one observable, then maximum ignorance reigns concerning the value of any of the other two.

For a system in state ρ (with Bloch vector $\vec{s}$) the predictability P is obtained by taking the modulus of the expectation value of observable $\hat{P}$: $P = |\langle\hat{P}\rangle| = |s_z|$. The visibility V can be derived either from the observable $\hat{V}_\phi$ or from $\hat{V}_\phi^\perp$ by properly choosing the parameter ϕ. Defining ρ as $\begin{pmatrix} \omega_+ & re^{-i\theta} \\ re^{i\theta} & \omega_- \end{pmatrix}$, the auxiliary state variables $r \equiv \frac{1}{2}\sqrt{s_x^2 + s_y^2}$ and $\tan\theta \equiv \frac{s_y}{s_x}$ allow us to write $\langle\hat{V}_\phi\rangle = 2r\cos(\theta - \phi)$ and $\langle\hat{V}_\phi^\perp\rangle = 2r\sin(\theta - \phi)$. Thus the visibility, which is given by the maximum absolute expectation value of these observables, is equal to $2r$ and can be obtained using $\hat{V}_\phi$ if one sets $\phi = \theta$, or $\hat{V}_\phi^\perp$ setting $\phi = \theta - \pi/2$ (arranging the apparatus with a phase difference of π with respect to these angles gives also the same value of visibility). Finally, due to the positivity of the density matrix, the complementarity relation (1) is directly obtained:

$$P^2 + V^2 = s_x^2 + s_y^2 + s_z^2 \leq 1, \tag{6}$$

and it is seen to be saturated whenever $\|\vec{s}\| = 1$, i.e. for any pure state.

We note that the measurements of the two observables (3) and (4), or (3) and (5), can only be carried out in two *incompatible* experimental set-ups and that joint measurement is not involved. Therefore the trade off relation (1) expresses the *preparation complementarity* [4], that is the impossibility to prepare the system in a state where the two observables have simultaneously sharp values.

III. UNCERTAINTY RELATIONS

The Uncertainty Principle (UP) states that the probability distributions associated to the outcomes of two incompatible observables cannot be simultaneously sharp. Quantitative formulations of the UP are known as *uncertainty relations* (UR), and there is now a collection of inequalities that express this principle (see, for instance, the recent review articles [12, 13]). Before summarizing the UP formulations to be employed, let us introduce the relevant quantities and fix the notation.

In general, the state of an N-level system is described by a density operator ρ, with $\mathrm{Tr}\rho = 1$ and $\rho \geq 0$. Physical observables like A and B are represented by Hermitian operators which in their spectral decomposition can be written as $A = \sum_{i=1}^{N} a_i |a_i\rangle\langle a_i|$ and $B = \sum_{i=1}^{N} b_i |b_i\rangle\langle b_i|$, where a_i and b_i are real numbers, and $\{|a_i\rangle\}_{i=1}^{N}$ and $\{|b_i\rangle\}_{i=1}^{N}$ are the corresponding eigenbases. The probability to obtain a certain value a_i of observable A is given by Born rule: $p(A = a_i) = \mathrm{Tr}(\rho |a_i\rangle\langle a_i|)$. The so-called overlap between operators A and B is defined by $c = \max_{i,j} |\langle a_i | b_j\rangle|$ and lies between $\frac{1}{\sqrt{N}}$ and 1.

In the particular case of two-level or qubit systems ($N = 2$), one can use the Bloch representation. Hence, the density operator is given by Eq. (2). Similarly, we can write operators A and B as

$$A = \alpha_1 I + \alpha_2 \vec{a} \cdot \vec{\sigma} \quad (7)$$

$$B = \beta_1 I + \beta_2 \vec{b} \cdot \vec{\sigma} \quad (8)$$

where α_i and β_i are real numbers, and $\vec{a}$ and $\vec{b}$ are unit vectors on the Bloch sphere. Therefore, in this representation the probability distributions associated to both observables take the simple form

$$\{p(A)\} = \left\{ \frac{1 + \vec{a} \cdot \vec{s}}{2}, \frac{1 - \vec{a} \cdot \vec{s}}{2} \right\} \quad (9)$$

$$\{p(B)\} = \left\{ \frac{1 + \vec{b} \cdot \vec{s}}{2}, \frac{1 - \vec{b} \cdot \vec{s}}{2} \right\} \quad (10)$$

for a qubit characterized by the Bloch vector $\vec{s}$. Meanwhile, the overlap is

$$c = \frac{1 + |\vec{a} \cdot \vec{b}|}{2} \in \left[\frac{1}{\sqrt{2}}, 1 \right] \quad (11)$$

where the case $c = 1/\sqrt{2}$ corresponds to A and B being *complementary observables*.

A. Variance-based uncertainty relations

Heisenberg, in his famous 1927 paper [14], was the first to propose an uncertainty relation for position and momentum observables in terms of their variances. The generalization of Heisenberg inequality for any arbitrary pair of Hermitian operators A and B is due to Robertson [15] and contains the commutator $[A, B]$. A further tighter relation was derived by Schrödinger [16] and includes also the anticommutator $\{A, B\}$, namely

$$(\Delta A)^2 (\Delta B)^2 \geq \left(\frac{1}{2} \langle \{A, B\} \rangle - \langle A \rangle \langle B \rangle \right)^2 + \left(\frac{1}{2i} \langle [A, B] \rangle \right)^2 \quad (12)$$

with $(\Delta O)^2 = \langle O^2 \rangle - \langle O \rangle^2$ being the variance of observable O. If one does not consider the first squared term in the right-hand side (rhs) of (12), one deals with the usual Heisenberg–Robertson (HR) uncertainty relation.

In the particular case of observables (7) and (8), the Schrödinger–Robertson (SR) uncertainty relation (12) reads

$$\left[1 - (\vec{a} \cdot \vec{s})^2\right] \left[1 - (\vec{b} \cdot \vec{s})^2\right] \geq \left[\vec{a} \cdot \vec{b} - (\vec{a} \cdot \vec{s})(\vec{b} \cdot \vec{s})\right]^2 + \left[(\vec{a} \times \vec{b}) \cdot \vec{s}\right]^2 \quad (13)$$

where we used that $\langle A \rangle = \alpha_1 + \alpha_2 \vec{a} \cdot \vec{s}$, $(\Delta A)^2 = \alpha_2^2 \left[1 - (\vec{a} \cdot \vec{s})^2\right]$, and analogously for B, while $\{A, B\} = 2\left[\left(\alpha_1 \beta_1 + \alpha_2 \beta_2 \vec{a} \cdot \vec{b}\right) I + \alpha_2 \beta_1 \vec{a} \cdot \vec{\sigma} + \alpha_1 \beta_2 \vec{b} \cdot \vec{\sigma}\right]$ and $[A, B] = 2i \alpha_2 \beta_2 (\vec{a} \times \vec{b}) \cdot \vec{\sigma}$.

Variance-based UP formulations have been doubly criticized. On the one hand, the lower bound to the product of variances depends, in general, on the state of the system via the expectation values and thus lacks a universal character [17, 18]. Moreover, it can be easily seen [19] that for discrete, bounded operators the lower bound is trivially zero, yielding no valuable information. On the other hand, the use of the variance as measure of uncertainty (spreading) of a given probability distribution exhibits some limitations [13, 20]. It might also be the case that the variance is not well-defined.

B. Landau-Pollak uncertainty relation

An alternative UP formulation was introduced by Landau and Pollak (LP) in the context of time-frequency analysis [21], and adapted to the quantum framework by Maassen and Uffink [18]. Using the notation $M_\infty(A; \rho) = \max_i p_i(A)$ for the maximum probability of the outcomes of observable A, then the LP uncertainty relation reads

$$\arccos \sqrt{M_\infty(A; |\Psi\rangle\langle\Psi|)} + \arccos \sqrt{M_\infty(B; |\Psi\rangle\langle\Psi|)} \geq \arccos c \quad (14)$$

The LP relation captures the essence of the uncertainty principle for quantum pure state, indeed the rhs is state-independent. A generalization of a weak version of the LP inequality for positive operator valued measures was recently given in Ref. [22], and an application of this inequality to separability problems was developed in Ref. [23].

For N-dimensional systems, the extension of (14) to general (mixed) states is not obvious, due to the lack of definite concavity of $\arccos \sqrt{M_\infty(A; \rho)}$. However, for two-dimensional systems it can been shown that the LP relation remains valid for mixed states.

For our purposes, we express the LP inequality (14) as

$$\sqrt{M_\infty(A) M_\infty(B)} - \sqrt{[1 - M_\infty(A)][1 - M_\infty(B)]} \leq c \quad (15)$$

where we have used the trigonometric identity $\arccos x + \arccos y = \arccos\left(xy - \sqrt{(1-x^2)(1-y^2)}\right)$ for $x+y \geq 0$ [24], and that $\arccos(x)$ is a decreasing function. In our 2D case, using (9)–(10), we have

$$\sqrt{(1+|\vec{a}\cdot\vec{s}|)(1+|\vec{b}\cdot\vec{s}|)} - \sqrt{(1-|\vec{a}\cdot\vec{s}|)(1-|\vec{b}\cdot\vec{s}|)} \leq 1 + |\vec{a}\cdot\vec{b}| \quad (16)$$

C. Entropy-based uncertainty relations

Information-theory tools have shown their usefulness in the study of uncertainty relations [25, 26]. Consider now, as a measure of uncertainty (ignorance), the one parameter generalization of Shannon entropy given by Rényi [27], that in the case of an N-dimensional, discrete probability distribution reads

$$H_q(\{p_i\}) = \frac{1}{1-q}\ln\left(\sum_{i=1}^{N} {p_i}^q\right) \quad (17)$$

where $0 \leq p_i \leq 1$, $\sum_{i=1}^{N} p_i = 1$, and the real parameter $q > 0$ with $q \neq 1$. If we let $q \to 1$, then this definition includes by continuity the Shannon case: $H_1(\{p_i\}) = -\sum_{i=1}^{N} p_i \ln p_i$. Other special q values of interest, for instance in quantum information process and quantum cryptography, are $q = 2$ and $q \to \infty$. In the former case, $H_2(\{p_i\}) = -\sum p_i^2$ is known as collision entropy. The latter is known as min-entropy, due to the property $H_{q'} < H_q$ if $q' > q$ for fixed $\{p_i\}$, and its value is $H_\infty(\{p_i\}) = -\ln(\max_i\{p_i\})$.

An *entropic uncertainty relation* (EUR) has the form

$$H(A;\rho) + H(B;\rho) \geq \mathcal{B}(A,B) \quad (18)$$

where H is an entropic measure like the ones in Eq. (17) with the probability distributions of the observables calculated via Born rule, while $\mathcal{B}$ is a function of them. More precisely, it depends on the overlap between both eigenbases, being state-independent (i.e. it is not a function of the state ρ) and a positive quantity. The search of tight bounds $\mathcal{B}$ for different pairs of observables with discrete or continuous spectra, using diverse entropic forms, has been subject of intense interest, for instance in Refs. [12, 13, 28, 29].

In the particular case of spin-1/2 observables, using Eq. (9), the Rényi entropy reads

$$H_q(A;\rho) = \frac{1}{1-q}\ln\left[\left(\frac{1+\vec{a}\cdot\vec{s}}{2}\right)^q + \left(\frac{1-\vec{a}\cdot\vec{s}}{2}\right)^q\right] \quad (19)$$

Note that this is a measure of the degree of uncertainty associated to the observable A, in the following sense: when one is certain about the observable's value, i.e. $\{p(A)\} = \{1,0\}$ or $\{0,1\}$, then the entropy takes its minimum value $H_q = 0$. Contrariwise, for total ignorance concerning the value of A, i.e. $\{p(A)\} = \{\frac{1}{2},\frac{1}{2}\}$, the entropy is maximal and equal to $H_q = \ln 2$ (irrespective of q). Rényi entropy is a concave function in ρ for q lying in the interval $(0,2]$, that is, if $\rho = \sum_n \lambda_n |\Psi_n\rangle\langle\Psi_n|$, with $0 \leq \lambda_n \leq 1$ and $\sum_n \lambda_n = 1$, then $H_q(A;\rho) \geq \sum_n \lambda_n H_q(A;|\Psi_n\rangle\langle\Psi_n|)$ [30, 31]. In the following we restrict the value of q to the above interval.

Optimal entropic uncertainty relations for two arbitrary quantum observables in the 2-dimensional case were obtained for the Shannon [19] and collision [32] entropies. Specifically, when A and B are spin-1/2 complementary observables ($c = 1/\sqrt{2}$) the optimal lower bounds are $\ln 2$ and $2\ln 4/3$, respectively.

IV. CONNECTIONS BETWEEN COMPLEMENTARITY AND UNCERTAINTY RELATIONS

A. Equivalence with variance-based uncertainty relations

The relationship between the predictability–visibility inequality (1) and the uncertainty relations based on variances (12) are readily analyzed using the Bloch representation of the pertinent operators and the density matrix. First of all, the variances of the operators defined in Eqs. (3)–(5) are given, in terms of the predictability P and visibility V, by

$$(\Delta\hat{P})^2 = 1 - P^2 \quad (20)$$

$$(\Delta\hat{V}_\phi)^2 = 1 - V^2\cos^2(\theta-\phi) \quad (21)$$

$$(\Delta\hat{V}_\phi^\perp)^2 = 1 - V^2\sin^2(\theta-\phi) \quad (22)$$

The connection between the CP relation and variance-based URs has been analyzed in Refs. [7], [6], and [4]. In [7] the authors highlight the *equivalence* between both principles. Indeed, they compute the HR-UR for the pair of observables $\hat{P}$ and $\hat{V}_\theta^\perp$, and also for $\hat{V}_\theta$ and $\hat{V}_\theta^\perp$ (setting the phase shifter to an angle $\phi = \theta$). By doing so, they obtain the following uncertainty inequalities

$$(\Delta\hat{P})^2(\Delta\hat{V}_\theta^\perp)^2 = 1 - P^2 \geq V^2 \quad (23)$$

$$(\Delta\hat{V}_\theta)^2(\Delta\hat{V}_\theta^\perp)^2 = 1 - V^2 \geq P^2 \quad (24)$$

and notice that both are equivalent to (1). The main drawback that they note in their derivation is the use of $\hat{V}_\theta^\perp$, which has no direct interpretation in terms of neither predictability nor visibility in connection with the MZ interferometry experiment, since $\langle\hat{V}_\theta^\perp\rangle = 0$ and $\Delta\hat{V}_\theta^\perp = 1$. Moreover, when dealing with $\hat{P}$ and $\hat{V}_\theta$, the corresponding HR-UR becomes trivial: $(\Delta\hat{P})^2(\Delta\hat{V}_\theta)^2 \geq 0$.

Independently, Björk *et al.* also dealt with the problem of connecting CP with UP. Although the authors of Ref. [6] mention the SR-UR, they do not actually calculate the lower bound of the product of variances as prescribed by the rhs of (12). Their analysis is, in this respect, limited to obtain expressions (20) and (21) followed by an appeal to $(\Delta\hat{V}_\phi)^2 \geq (\Delta\hat{V}_\theta)^2$ (basic trigonometry) with the purpose of linking the two fundamental principles of quantum mechanics.

A complete proof of the alluded to equivalence dealing with the *appropriate* observables $\hat{P}$ and $\hat{V}_\theta$ and the *full* SR-UR, is given in Ref. [4]. We reproduce it here –although in a slightly different way– for the sake of completeness. For arbitrary ϕ, the UR prescribed by Schrödinger and Robertson reads

$$(1-P^2)[1-V^2\cos^2(\theta-\phi)] \geq P^2V^2\cos^2(\theta-\phi)+V^2\sin^2(\theta-\phi) \quad (25)$$

where equality holds for any pure state. It is straightforward to show that this inequality is *equivalent* to the duality relation (1). We stress that (25) is valid for *any* phase ϕ introduced by the phase shifter in the MZ interferometer. We then conclude that the appropriate choice $\phi = \theta$ implies *equivalence* with the trade-off relation between predictability and visibility. This circumvents the drawback pointed out by Dürr and Rempe. With this simple result, a rather sharp conclusion is drawn from the discussion about complementarity between P and V, including the status of (1) as an uncertainty relation. Finally, we mention that in Ref. [5] a relation between wave–particle duality and quantum uncertainty has been investigated, both theoretically and experimentally, by recourse to variances of the operators $\hat{P}$ and $\hat{V}_\theta$, although without appealing to Heisenberg-like inequalities.

B. Equivalence with Landau–Pollak uncertainty relation

Let us now see just how inequality (1) becomes equivalent to Landau–Pollak uncertainty relation. The maximum probabilities associated to observables $\hat{P}$ and $\hat{V}_\theta$, in terms of the predictability and visibility, are

$$M_\infty(\hat{P}) = \frac{1+P}{2} \quad (26)$$

$$M_\infty(\hat{V}_\theta) = \frac{1+V}{2} \quad (27)$$

Replacing these probabilities in (15), and setting $c = 1/\sqrt{2}$ as corresponds to the case of complementary operators, we obtain

$$\sqrt{\left(\frac{1+P}{2}\right)\left(\frac{1+V}{2}\right)} - \sqrt{\left(\frac{1-P}{2}\right)\left(\frac{1-V}{2}\right)} \leq \frac{1}{\sqrt{2}} \quad (28)$$

Squaring both sides of this inequality and grouping terms conveniently, we immediately arrive at the relation

$$(1-P^2)(1-V^2) \geq (PV)^2 \quad (29)$$

which coincides with (25) for $\phi = \theta$, and, as mentioned before, can be easily recast in the fashion $P^2 + V^2 \leq 1$. This implies that the duality relation (1) can be deduced from the LP inequality, and viceversa.

C. Relationship with entropic uncertainty relations

Having clarified the above equivalences, we now consider the problem of elucidating the connection between EURs and the duality relation (1). Using Eq. (19) and the Bloch representation of $\hat{P}$ and $\hat{V}_\theta$ we obtain

$$H_q(P) = \frac{1}{1-q}\ln\left[\left(\frac{1+P}{2}\right)^q + \left(\frac{1-P}{2}\right)^q\right] \quad (30)$$

$$H_q(V) = \frac{1}{1-q}\ln\left[\left(\frac{1+V}{2}\right)^q + \left(\frac{1-V}{2}\right)^q\right] \quad (31)$$

where to simplify notation we have renamed $H_q(\hat{P};\rho) \equiv H_q(P)$ and $H_q(\hat{V}_\theta;\rho) \equiv H_q(V)$. Our goal is to find the minimum of the sum of these Rényi entropies over all available states, that is, $\min_\rho\{H_q(\hat{P};\rho) + H_q(\hat{V}_\theta;\rho)\}$. Appealing to the concavity of Rényi entropy for $q \in (0,2]$, we can restrict our calculations to pure states and then the conditioned minimization problem can be recast in the fashion

$$\min_{P^2+V^2=1}\{H_q(P) + H_q(V)\} \quad (32)$$

For arbitrary values of q, this problem can be solved numerically. It is seen that three qualitatively different regimes appear: (i) for $0 < q < q^*$ with $q^* \approx 1.4316$, the minimum is $\ln 2$ and it is attained at $V = 0$ and $P = 1$, or $V = 1$ and $P = 0$; (ii) at $q = q^*$ the minimum value is also $\ln 2$ but it corresponds to the cases $V = 0$ and $P = 1$, $V = 1$ and $P = 0$, or also $V = P = 1/\sqrt{2}$; and (iii) for $q^* < q \leq 2$, the minimum is the q-dependent function $\frac{2}{1-q}\ln\left[\left(\frac{1+1/\sqrt{2}}{2}\right)^q + \left(\frac{1-1/\sqrt{2}}{2}\right)^q\right]$, attained at $V = P = 1/\sqrt{2}$. The value of q^* is obtained solving (numerically) the equation $2H_{q^*}(1/\sqrt{2}) = \ln 2$.

In Fig. 2 we display, in the V-P plane, the constraint $P^2 + V^2 = 1$ together with several contour lines of the sum of q-Rényi entropies for two representative values of the entropic parameter in the regimes (i) and (iii) mentioned above. In both cases the contour lines correspond to decreasing values towards the origin. In case (i) $\ln 2$ is the minimum-value contour line that intersects (tangencially) the constraint, at the points $(V,P) = (0,1)$ or $(V,P) = (1,0)$. In case (iii) the curve $P^2 + V^2 = 1$ is intersected by the minimum-value contour line $H_q(P) + H_q(V) = 2H_q(1/\sqrt{2})$, precisely at $(V,P) = (1/\sqrt{2}, 1/\sqrt{2})$.

The existence of these three regimes sheds light on the meaning of the sum of Rényi entropies. The fact that there appear three qualitatively different regimes agrees with a result suggested in previous work [9], where different values of the parameter q yield different entropic measures which, in turn, give rise to qualitatively different information about the system. As stated before, information-theoretic entropy gives a measure of the uncertainty related to the outcome of a variable in terms of the corresponding probability distribution. Thus, solving the problem raised in (32) we get the minimum of

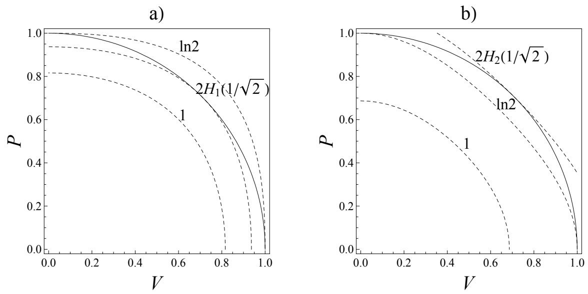

FIG. 2: Constraint $P^2+V^2=1$ (solid line) and contour plots (dashed lines) of the sum of q-Rényi entropies with entropic indices: a) $q=1$, b) $q=2$. The value of the entropy sum is indicated near each contour line: the values shown are 1, $2H_1(1/\sqrt{2}) \approx 0.833$, $\ln 2 \approx 0.693$, and $2H_2(1/\sqrt{2}) \approx 0.576$.

the sum of uncertainties (notice that in this spirit we are using the same q parameter for both entropies). We are also able to pinpoint the optimum (minimizing) states of this problem.

States which saturate an uncertainty relation are used in several contexts. An important example has to do with coherent states, which saturate the position–momentum Heisenberg UR. Because of having this property, coherent states are usually interpreted as the most classical ones. In the present case, in which we consider the predictability–visibility relation in the context of the MZ interferometer, we find that regime (i) is a trivial one, being the eigenstates of $\hat{V}_\theta$ or $\hat{P}$ those states of minimum uncertainty sum. However, an interesting non-trivial situation appears in regimes (ii) and (iii), where the extremum is attained at the symmetric case. Which are the characteristic features of states which make $|\langle \hat{V}_\theta \rangle| = |\langle \hat{P} \rangle| = \frac{1}{\sqrt{2}}$? They are the pure states of the form (2) with the four different unit Bloch vectors: $\pm\left(\frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\sin\theta, \pm\frac{1}{\sqrt{2}}\right)$. These are precisely the states which saturate the concomitant EURs in the most unbiased way (in the sense of simultaneously having the maximum visibility *and* maximum predictability that is possible).

Let us consider in more detail the question of getting the states with maximum value for both predictability and visibility, a situation which one expects to correspond to the best description of the system. From simple geometric arguments in the V-P plane (taking into account that $P^2+V^2 \leq 1$ has to be fulfilled), it is straightforward how to compute those states. However, one may deal as well with a situation in which one does not have at hand the whole set of states available, but only a fraction of it. In such circumstances, it is convenient to delve further into the usefulness of the minimization of the measure $H_q(P)+H_q(V)$ (when $q > q^*$). For example, this situation may appear if the source in Fig. 1 has limitations for producing certain states, and one is thus restricted to deal with a given region of the convex set of quantum states. Another interesting situation has to do with the case in which the second beam splitter is a Schrödinger cat (as is the case in Refs. [33] and [34]) or if there is a noisy environment. In both situations, the states of the system which pass trough the interferometer are limited by the state of the environment (and cannot be controlled, in the second case), being mixed states the more general case. Thus, not all states are available and (32) gives a way to solve the problem posed by conditions mentioned above in this non-symmetrical setting.

V. CONCLUSIONS

We studied here connections between the Complementarity and Uncertainty Principles in the Mach–Zehnder interferometer scheme. Following Ref. [6] and related work, we employed quantum-mechanical operators $\hat{P}$ and $\hat{V}_\theta$ to represent the particle and wave aspects of a quantum system, respectively.

We have thoroughly analyzed some drawbacks concerning the approaches of Björk *et al.*, and of Dürr and Rempe, who considered the link between the inequality (1) and variance-based uncertainty relations of the form (12). We showed the *equivalence* between the Schrödinger–Robertson UR and the duality relation in the relevant case, i.e. for observables which adequately represent predictability and visibility according to [4]. An alternative quantification of the Uncertainty Principle is given by the Landau–Pollak inequality (14). We proved the equivalence between (1) and the LP-UR (as specified for the observables of interest).

It is worth stressing then that in the present context the three inequalities (1), (25) and (28), are on an equal footing (which may well not be the case for other pairs of observables). We remark that our results give a precise (and quantitative) meaning to the assertion *P and V are complementary quantities* and, at the same time, settle pending question regarding the status of (1) as an uncertainty relation.

Moreover, we have studied the connection between (1) and entropic uncertainty relations (18) based on the q-Rényi entropy (17). We found that these EURs, for the pair $\hat{P}$-$\hat{V}_\theta$, are not equivalent to the duality relation. Nevertheless, we see that, when these uncertainty measures are applied to the MZ scheme, different regimes emerge, depending on the value of the entropic parameter q. We also noticed that this agrees with a previous investigation by Luis [9], in which the value chosen for q affects the qualitative behavior of the uncertainty relations. The fact that these different regimes are also found in the canonical example of MZ interferometry seems to provide support to the assertion *there is no preferred value for q*. Indeed, different q-values render the concomitant entropic measures useful for different purposes. In addition, looking at the states which correspond to an equality in the entropic uncertainty relation, we find regimes with nontrivial saturating states. We have in this vein established a procedure for solving the problem of finding a state having minimum uncertainty for the observables $\hat{P}$ and $\hat{V}_\theta$ in the most unbiased fashion. Finally, we also discussed the usefulness of such procedure for infomation-theoretical purposes, depending on the nature of the source and the beam splitters.

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