On the global version of Euler–Lagrange equations

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Abstract

The introduction of a covariant derivative on the velocity phase space is needed for a global expression of Euler–Lagrange equations. The aim of this paper is to show how its torsion tensor turns out to be involved in such a version.

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An increasing attention has been recently paid to coordinate-free formulations of motion equations in classical mechanics (see for instance [1, 2] and references therein). In this work we write down intrinsic Euler–Lagrange equations and show the appearance of a torsion term. Furthermore, we shall see that this term should also be present in the horizontal Lagrange–Poincaré equations considered in [1, 2], if the torsion of the chosen derivative does not vanish.

It is worth noting that covariant derivatives with non-vanishing torsion naturally arise in several branches of physics; namely dynamics with nonholonomic constraints [3, 4], E Cartan’s theory of gravity (see for instance [5]) and modern string theories (see for example [6]), among others.

Let us consider a physical system with configuration manifold Q and Lagrangian $L(q, \dot{q}) : TQ \rightarrow \mathbb{R}$ (for this geometrical setting, see for instance [7]).

If a coordinate-free characterization of the Euler–Lagrange equations associated with the system is required a covariant derivative D must be introduced to TQ , for $\frac{\partial L}{\partial \dot{q}}$ is involved (see for instance [8]). Once such D is chosen, $\frac{DL}{Dq}$ is defined in the standard way

$$\frac{DL}{Dq}(q_0, \dot{q}_0) = \left. \frac{\partial}{\partial \lambda} \right|_{\lambda=0} L \circ \gamma(\lambda) \quad (1)$$

with $\gamma(\lambda) = (q(\lambda), \dot{q}_{0\parallel}(\lambda))$, $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$ and $\dot{q}_{0\parallel}(\lambda)$ the parallel transport of $\dot{q}_0$ along $q(\lambda)$.

Moreover, an associated covariant derivative on T^*Q , that we will also denote by D , is naturally defined through the Leibnitz rule: for any curves $\alpha(t)$ and $v(t)$ in T^*Q and TQ , respectively

$$\frac{d}{dt} \langle \alpha(t), v(t) \rangle = \left\langle \frac{D\alpha(t)}{Dt}, v(t) \right\rangle + \left\langle \alpha(t), \frac{Dv(t)}{Dt} \right\rangle \quad (2)$$

where $\langle, \rangle$ denotes the pairing between T^*Q and TQ .

It is worth noting that $\frac{\partial}{\partial \dot{q}}$ has a coordinate-free sense: it is the derivative along the fibre.

Proposition 1. *Let D be an arbitrary covariant derivative on TQ . Then the coordinate-free expression of the Euler–Lagrange equations is*

$$\frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{DL}{Dq} = \frac{\partial L}{\partial \dot{q}} T(\dot{q}(t), \cdot) \quad (3)$$

where $T(\cdot, \cdot)$ is the torsion tensor of D .

Proof. The curve $q(t)$ is a solution of the Euler–Lagrange equations if and only if it is a critical point for the action

$$S = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt \quad (4)$$

for variations of the curves such that q_0 and q_1 remain fixed. That is, for each $q(t, \lambda) : [t_0, t_1] \times (-\varepsilon, \varepsilon) \rightarrow Q$ such that $q(t, 0) = q(t)$, $q(t_0, \lambda) = q(t_0)$ and $q(t_1, \lambda) = q(t_1)$,

$$\frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} \int_{t_0}^{t_1} L(q(t, \lambda), \dot{q}(t, \lambda)) dt = \int_{t_0}^{t_1} \delta L(q(t), \dot{q}(t)) dt = 0 \quad (5)$$

where $\delta L(q(t), \dot{q}(t)) = \frac{\partial}{\partial \lambda} \big|_{\lambda=0} L(q(t, \lambda), \dot{q}(t, \lambda))$.

But

$$\begin{aligned} \delta L(q(t), \dot{q}(t)) &= \lim_{\lambda \rightarrow 0} \frac{L(q(t, \lambda), \dot{q}(t, \lambda)) - L(q(t, 0), \dot{q}(t, 0))}{\lambda} \\ &= \lim_{\lambda \rightarrow 0} \frac{L(q(t, \lambda), \dot{q}(t, \lambda)) - L(q(t, \lambda), \dot{q}_{\parallel}(t, \lambda))}{\lambda} \\ &\quad + \lim_{\lambda \rightarrow 0} \frac{L(q(t, \lambda), \dot{q}_{\parallel}(t, \lambda)) - L(q(t, 0), \dot{q}(t, 0))}{\lambda} \end{aligned} \quad (6)$$

where $\dot{q}_{\parallel}(t, \lambda)$ is the parallel translate of the vector $\dot{q}(t, 0)$ along the curve $q(t, \lambda)|_{\text{fixed } t}$ (see figure 1).

Then

$$\delta L(q(t), \dot{q}(t)) = \frac{\partial L}{\partial \dot{q}} D_{\delta q(t)} \dot{q}(t) + \frac{DL}{Dq} \delta q(t) \quad (7)$$

where we have denoted

$$\delta q(t) = \frac{\partial q(t, \lambda)}{\partial \lambda} \bigg|_{\lambda=0} \quad \text{and} \quad \dot{q}(t) = \frac{\partial q(t, \lambda)}{\partial t} \bigg|_{\lambda=0}. \quad (8)$$

By definition of the torsion tensor $T(\cdot, \cdot)$, we have

$$T(\dot{q}(t), \delta q(t)) = D_{\dot{q}(t)} \delta q(t) - D_{\delta q(t)} \dot{q}(t) - [\dot{q}(t), \delta q(t)]. \quad (9)$$

Thus, by using (2) and taking into account that $[\dot{q}(t), \delta q(t)]$ vanishes, we have

$$\begin{aligned} \delta L(q(t), \dot{q}(t)) &= \frac{\partial L}{\partial \dot{q}} D_{\dot{q}(t)} \delta q(t) + \frac{DL}{Dq} \delta q(t) - \frac{\partial L}{\partial \dot{q}} T(\dot{q}(t), \delta q(t)) \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q(t) \right) - \frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q(t) + \frac{DL}{Dq} \delta q(t) - \frac{\partial L}{\partial \dot{q}} T(\dot{q}(t), \delta q(t)). \end{aligned} \quad (10)$$

Figure 1. The vector $\dot{q}_{\parallel}(t, \lambda)$ is the parallel transport of the vector $\dot{q}(t, 0)$ along the dashed curve.

Now integrating along the curve $q(t)$ we finally get

$$\frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q(t) - \frac{DL}{Dq} \delta q(t) = \frac{\partial L}{\partial \dot{q}} T(\dot{q}(t), \delta q(t)). \quad (11)$$

□

Remark 1. Of course, regardless of the covariant derivative D we introduced, in *any* coordinate patch the Euler–Lagrange equations always read

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} = 0. \quad (12)$$

Remark 2. As well known (e.g., [8]), a torsion-free D can always be chosen. It is obvious that for such a connection, global Euler–Lagrange equations read

$$\frac{D}{Dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{DL}{Dq} = 0. \quad (13)$$

So in this case, the global expression can be obtained merely by replacing the usual derivatives by D in (12).

Remark 3. A similar result holds for the horizontal Lagrange–Poincaré equations considered in [1, 2]. One of the goals of these references is to analyse the intrinsic meaning of motion equations for constrained systems with symmetries. Let us recall that, in such a system, the Lagrangian L and the constraints remain invariant under the lifting to TQ of a suitable action of a Lie group G on Q . A connection A , related to the constraints, is introduced on the principal bundle $Q \xrightarrow{\pi} Q/G$. If $\tilde{\mathfrak{g}}$ is the adjoint bundle to the principal bundle Q , the connection A yields an isomorphism α between TQ/G and the Whitney sum $T(Q/G) \oplus \tilde{\mathfrak{g}}$ in the following way

$$\alpha_A[q, \dot{q}]_G = (x, \dot{x}, \tilde{v}) = \pi_*(q, \dot{q}) \oplus [q, A(q, \dot{q})]_G. \quad (14)$$

Now, one can define the reduced Lagrangian $\ell : T(Q/G) \oplus \tilde{\mathfrak{g}} \rightarrow \mathbb{R}$ as

$$\ell(x, \dot{x}, \tilde{v}) = L(q, \dot{q}). \quad (15)$$

A variation δq of a curve in Q is said to be *horizontal* if $A(\delta q) = 0$. In this case, the corresponding variation $\alpha(\delta q(t))$ of the curve $\alpha(q(t))$ in $T(Q/G) \oplus \tilde{\mathfrak{g}}$ is [1]

$$\alpha(\delta q(t)) = \delta x \oplus \tilde{B}(\delta x, \dot{x}) \quad (16)$$

where $\tilde{B}$ is the $\tilde{\mathfrak{g}}$ -valued two-form on Q/G defined by

$$\tilde{B}([q]_G)(X, Y) = [q, B(X^h(q), Y^h(q))]_G \quad (17)$$

with X^h, Y^h the horizontal lifts to Q of X and Y , and B the curvature of the connection A .

The horizontal Lagrange–Poincaré equations for L are defined as the Euler–Lagrange ones for ℓ restricted to horizontal variations δq . A coordinate-free version of them can be written down by introducing an arbitrary covariant derivative D on $T(Q/G)$ and using the covariant derivative $\tilde{D}$ induced by A on $\tilde{\mathfrak{g}}$.

Under the implicit assumption that the torsion of D vanishes, it is shown in [1, 2] that, for horizontal variations δq ,

$$\delta \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt = 0 \quad (18)$$

if and only if the following horizontal Lagrange–Poincaré equations hold

$$\frac{D}{Dt} \left(\frac{\partial \ell}{\partial \dot{x}} \right) (x, \dot{x}, \tilde{v}) - \frac{D\ell}{Dx} (x, \dot{x}, \tilde{v}) = - \left\langle \frac{\partial \ell}{\partial \tilde{v}}, \tilde{B}(x)(\dot{x}, \cdot) \right\rangle. \quad (19)$$

It is easy to see that, for an arbitrary covariant derivative D on $T(Q/G)$, its torsion tensor T must be taken into account in the previous equations. Arguing as above one gets

$$\frac{D}{Dt} \left(\frac{\partial \ell}{\partial \dot{x}} \right) (x, \dot{x}, \tilde{v}) - \frac{D\ell}{Dx} (x, \dot{x}, \tilde{v}) = - \left\langle \frac{\partial \ell}{\partial \tilde{v}}, \tilde{B}(x)(\dot{x}, \cdot) \right\rangle + T(\dot{q}, \cdot). \quad (20)$$

Assuming as in [1, 2] the torsion-free requirement for D , the last term clearly vanishes and we recover the horizontal Lagrange–Poincaré equations found in those references.

Again, in *any* coordinate patch, the expression of horizontal Lagrange–Poincaré equations is independent of the choice of the covariant derivative D .

Remark 4. When considered as a map of the second-order tangent bundle $T(TQ)$ to T^*Q , the Euler–Lagrange operator turns out to be intrinsic without any choice of connection (e.g., [9]). The need for connections appears if one prefers to stay in the framework of tangent bundles, as it is usually done, and not to deal with second-order ones.

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References

- [1] Cendra H, Marsden J E and Ratiu T S 2001 Lagrangian reduction by stages *Mem. Am. Math. Soc.* **722** 152
- [2] Cendra H, Marsden J E and Ratiu T S 2001 Geometric mechanics. Lagrangian reduction, and nonholonomic system *Mathematics Unlimited-2001 and Beyond* ed B Engquist and W Schmid (Berlin: Springer) pp 221–73
- [3] Vershik A M and Faddeev L D 1981 *Sel. Math. Sov.* **1** 339

- [4] Vershik A M and Ya Gershkovich V 1994 Non-holonomic Riemannian manifolds *Dynamical Systems* 7 (*Encyclopaedia of Mathematics*, vol 16) (Berlin: Springer)
- [5] Hehl F W, von der Heyde P and Kerlick G D 1976 *Rev. Mod. Phys.* **48** 393
- [6] Polchinski J 1998 *String Theory* vol 2 (Cambridge: Cambridge University Press)
- [7] Abraham R and Marsden J E 1967 *Foundations of Mechanics* (New York: Benjamin)
- [8] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 (New York: Wiley)
- [9] Marsden J E and Ratiu T S 1994 *Introduction to Mechanics and Symmetry* (New York: Springer)