IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 205401 (12pp) doi:10.1088/1751-8113/42/20/205401

Local and semi-local vortices in the Yang–Mills–Chern–Simons model

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Received 5 March 2009
Published 28 April 2009
Online at stacks.iop.org/JPhysA/42/205401

Abstract

We study BPS vortex configurations in three-dimensional $U(N)$ Yang–Mills theories with Chern–Simons interaction coupled to scalar fields carrying flavor. We consider two kinds of configurations: local vortices (when the number of flavors $N_f = N$) and semi-local vortices (when $N_f > N$). In both cases, we carefully analyze the electric and magnetic properties and present explicit numerical solutions.

PACS numbers: 11.15.Kc, 11.27.+d, 11.10.Kk

1. Introduction

Non-Abelian vortices may find their application in a variety of problems ranging from particle physics and cosmology (e.g. confinement, supersymmetric and supergravity models, hot or dense QCD, cosmic strings) to condensed matter physics (e.g. quantum Hall effect). Of particular interest are those vortices solving first-order Bogomolny–Prasad–Sommerfield (BPS) equations, which correspond to the saturation of the Bogomolny bound for the mass and naturally arise in supersymmetric theories (see [1–3] for reviews with complete lists of references).

Non-Abelian BPS equations for vortices have been analyzed both for Yang–Mills–Higgs [4–8] and Chern–Simons–Higgs [9–15] models. In the former case, the gauge and the (fourth-order) Higgs potential coupling constants have to be related in order to pass from the second-order equations of motion to a first-order BPS system. When the Chern–Simons term, originally introduced in the context of topologically massive gauge theories [16], dictates the dynamics of the gauge field, one is forced to choose a sixth-order Higgs potential in order to find a Bogomolny bound for the vortex mass. Again, coupling constants should be related [11, 17, 18].

[4] Associated with CICBA.

The origin of these requirements can also be understood in the framework of supersymmetry: they are necessary conditions for the existence of an $\mathcal{N} = 2$ supersymmetry extension of the bosonic models. In this context, the first-order BPS equations arise studying the supersymmetry algebra and looking for supersymmetric states. The resulting selfdual and anti-selfdual solutions break 1/2 of the original supersymmetry [19–21].

The mixed case of Yang–Mills–Chern–Simons (YMCS) vortices was also recently discussed [22, 23]. As in the Abelian case [24], in order to have a Bogomolny bound and the first-order BPS equations the coexistence of the two terms giving dynamics to the gauge field requires a careful choice of the number and type of scalars. Moreover, the Gauss law, through which the Chern–Simons term enters into the energy, is no longer an algebraic equation for A_0, as in the case when the Yang–Mills term is absent, but a second-order differential equation that should be taken into account together with the first-order BPS system.

It is the purpose of this work to construct explicit BPS vortex solutions for the YMCS model thus completing the analysis presented in [23] where the low-energy vortex dynamics was the main aim of study. We shall take $U(N)$ as the gauge group, and include N_f scalars in the fundamental representation and one real scalar in the adjoint representation of $U(N)$. We shall consider two cases: when the number of flavors N_f is equal to the number of colors N and also when $N_f > N$, in which case vortices become semi-local [14, 25–27].

2. The model and the BPS equations

We shall consider the $d = 2 + 1$ dimensional $U(N)$ YMCS model discussed in [23] with dynamics governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{2e^2}\operatorname{Tr} F_{\mu\nu}F^{\mu\nu} - \frac{\kappa}{4\pi}\operatorname{Tr}\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho - \frac{2\mathrm{i}}{3}A_\mu A_\nu A_\rho\right) + \frac{1}{e^2}\operatorname{Tr}(\mathcal{D}_\mu\phi)^2$$
$$+ |\mathcal{D}_\mu q_i|^2 - q_i^\dagger(\phi - m_i)^2 q_i - \frac{e^2}{4}\operatorname{Tr}\left(q_i q_i^\dagger - \frac{\kappa\phi}{2\pi} - v^2\right)^2. \qquad (1)$$

Here q_i are N_f scalars with i being the flavor index ($i = 1, 2, \ldots, N_f$). Each q_i transforms in the fundamental representation of the gauge group $U(N)$, and ϕ is a real scalar in the adjoint. We shall first discuss the $N_f = N$ case (local vortices) and then extend the analysis to $N_f > N$ (semi-local vortices). Whenever it does not lead to confusion summation over flavors is implicit. The gauge field A_μ takes values in the Lie algebra of $U(N)$, $A_\mu = A_\mu^A t^A$, where $t^A = (t^A)^{ab}$ are the $U(N)$ generators ($A = 1, \ldots, N^2 - 1; a, b = 1, \ldots, N$) with normalization $\operatorname{Tr} t^A t^B = \delta^{AB}/2$. The curvature and covariant derivatives are defined as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu] \\ \mathcal{D}_\mu\phi &= \partial_\mu\phi - \mathrm{i}[A_\mu, \phi] \\ \mathcal{D}_\mu q_i &= \partial_\mu q_i - \mathrm{i}A_\mu q_i. \end{aligned} \qquad (2)$$

The Chern–Simons coefficient κ must be an integer for $N > 1$.

The masses m_i of the fundamental scalars break the flavor symmetry to $U(1)_f^{N-1}$. There exists a fully broken Higgs phase in which the scalars take the following expectation values:

$$q_{i\,\mathrm{vac}}^a = \delta_i^a\sqrt{v^2 + \frac{\kappa m_i}{2\pi}}, \qquad \phi_{\mathrm{vac}}^{ab} = \delta^{ab} m_b, \qquad A_{\mu\,\mathrm{vac}}^{ab} = 0. \qquad (3)$$

In this vacuum, the $U(N)$ gauge symmetry and the $U(1)_f^{N-1}$ flavor symmetry are spontaneously broken. There remains only a diagonal symmetry, $U(N) \times U(1)_f^{N-1} \to U(1)_{\mathrm{diag}}^{N-1}$. This corresponds to the combined action of a gauge group element $U^{ab} \in U(N)$ and a flavor transformation $V_{ij} = \delta_{ij}\,\mathrm{e}^{i\alpha_j} \in U(1)_f^{N-1}$ (with $N-1$ independent parameters α_j):

$$q^a_{i\,\mathrm{vac}} \to U^{ab} q^b_{j\,\mathrm{vac}} V_{ji}, \qquad \phi^{ab}_{\mathrm{vac}} \to U^{ac}\phi^{cd}_{\mathrm{vac}}(U^{-1})^{db}, \tag{4}$$

where $U = V^{-1}$.

Let us note that in the $e^2 \to \infty$ limit where the Yang–Mills term and the kinetic energy term for the ϕ field can be discarded, the adjoint field ϕ may be eliminated from the Lagrangian, and for $m_i = 0$ one ends up with the sixth-order potential which allows one to find BPS equations for the pure Chern–Simons–Higgs system both in the Abelian [17, 18] and non-Abelian [11] cases:

$$\lim_{e^2\to\infty} V[q,\phi,m_i=0] = \frac{(4\pi)^2}{\kappa^2}(|q_i|^2 - v^2)^2|q_i|^2. \tag{5}$$

The energy associated with Lagrangian (1) can be constructed from T_{00}, the time–time component of the energy–momentum tensor $T_{\mu\nu}$. It takes the form

$$E = \int \mathrm{d}^2x\, T_{00} = \int \mathrm{d}^2x \left[\frac{1}{e^2}\,\mathrm{Tr}\left(E_\alpha^2 + B^2\right) + \frac{1}{e^2}\,\mathrm{Tr}((\mathcal{D}_0\phi)^2 + (\mathcal{D}_\alpha\phi)^2) + |\mathcal{D}_0 q_i|^2 + |\mathcal{D}_\alpha q_i|^2 \right.$$
$$\left. + q_i^\dagger(\phi - m_i)^2 q_i + \frac{e^2}{4}\,\mathrm{Tr}\left(q_i q_i^\dagger - \frac{\kappa\phi}{2\pi} - v^2\right)^2\right], \tag{6}$$

where $E_\alpha = F_{0\alpha}$, $B = F_{12}$.

Following the Bogomolny procedure of square completion a lower energy bound can be obtained [23]:

$$E \geqslant \left|2\pi n v^2 + \sum_i Q_i m_i\right|. \tag{7}$$

Here $n \in \mathbb{Z}$ is the topological charge of the configuration, which corresponds to the topological degree associated with the q_i component that carries the winding,

$$n = \frac{1}{2\pi}\,\mathrm{Tr}\int \mathrm{d}^2x\, B, \tag{8}$$

and Q_i are the conserved Noether charges associated with the residual $U(1)^{N-1}$ flavor symmetry,

$$Q_i = \mathrm{i}\int \mathrm{d}^2x \left(q_i^\dagger \mathcal{D}_0 q_i - (\mathcal{D}_0 q_i)^\dagger\, q_i\right). \tag{9}$$

Using the Gauss law one can find the typical Chern–Simons term connection between charge and flux:

$$\sum_i Q_i = \frac{e\kappa}{2\pi}\Phi, \tag{10}$$

with Φ being the magnetic flux:

$$\Phi = \frac{1}{e}\,\mathrm{Tr}\int \mathrm{d}^2x\, B = \frac{2\pi}{e}n. \tag{11}$$

The bound (7) is saturated whenever the following BPS first-order equations hold:

$$\mathcal{D}_1 q_i \pm \mathrm{i}\mathcal{D}_2 q_i = 0 \tag{12}$$

$$B \pm \frac{e^2}{2}\left(q_i q_i^\dagger - \frac{\kappa\phi}{2\pi} - v^2\right) = 0 \tag{13}$$

$$\mathcal{D}_0\phi = 0 \tag{14}$$

$$E_\alpha \mp \mathcal{D}_\alpha \phi = 0 \tag{15}$$

$$\mathcal{D}_0 q_i \mp \mathrm{i}(\phi - m_i) q_i = 0. \tag{16}$$

It should be signaled that in obtaining the bound, it is necessary to use the Gauss law [23]:

$$-\frac{\kappa}{4\pi} B + \frac{\mathrm{i}}{2}\left[(\mathcal{D}_0 q_i) q_i^\dagger - q_i (\mathcal{D}_0 q_i)^\dagger\right] + \frac{1}{e^2}\mathcal{D}_\alpha E_\alpha + \frac{\mathrm{i}}{e^2}\left[\mathcal{D}_0\phi, \phi\right] = 0, \tag{17}$$

which should then be considered together with equations (12)–(16) when looking for explicit vortex solutions.

At the bound, the energy of configurations can be identified with the BPS soliton mass,

$$M = \left| 2\pi v^2 n + \sum_i Q_i m_i \right|. \tag{18}$$

In what follows we choose the upper sign in equations (12)–(16), which corresponds to a non-negative winding. Of course, the opposite choice is equally treatable.

3. The vortex ansatz

Starting from the trivial vacuum, a winding can be introduced through a singular gauge transformation generated by

$$\Omega(\varphi) = \mathrm{diag}[1, 1, \ldots, \mathrm{e}^{\mathrm{i}n\varphi}] = \mathrm{e}^{\frac{\mathrm{i}n\varphi}{N}}\, \mathrm{diag}\left[\mathrm{e}^{-\mathrm{i}\frac{n}{N}\varphi}, \mathrm{e}^{-\mathrm{i}\frac{n}{N}\varphi}, \ldots, \mathrm{e}^{\mathrm{i}\frac{n(N-1)}{N}\varphi}\right]. \tag{19}$$

We have written the formula above, so as to emphasize that Ω combines a $U(1)$ element with an element of $\mathbb{Z}_N$, the center of $SU(N)$. Then, a configuration of the form

$$q_{\mathrm{sing}} = \Omega(\varphi) q_{\mathrm{vac}} \tag{20}$$

with q_{vac} being the trivial vacuum (3) will lead to a topologically nontrivial but singular (at the origin) string configuration. To avoid the singularity the natural ansatz for a regular vortex should be

$$q = \mathrm{diag}[\eta_1, \eta_2, \ldots, \eta_N\, \mathrm{e}^{\mathrm{i}n\varphi} q_N(\rho)] \tag{21}$$

with $\eta_i^2 \equiv v^2 + \kappa m_i/2\pi$ and $q_N(\rho)$ vanishing at $\rho = 0$. Then, (21) should be supplemented with consistent ansätze for the remaining fields:

$$\phi = \mathrm{diag}[m_1, m_2, \ldots, m_N + h_N(\rho)] \tag{22}$$

$$A_\varphi = \mathrm{diag}[0, 0, \ldots, n - a_N(\rho)] \tag{23}$$

$$A_\rho = 0. \tag{24}$$

The complete set of appropriate boundary conditions ensuring finite energy is

$$a_N(0) = n, \qquad a_N(\infty) = 0, \qquad q_N(0) = 0 \tag{25}$$

$$q_N(\infty) = 1, \qquad h_N(\infty) = 0. \tag{26}$$

Concerning A_0, equations (14) and (15) require

$$[A_0, \phi] = 0, \qquad \partial_\rho(A_0 + \phi) = 0. \tag{27}$$

This suggests

$$A_0 = -\phi + C \qquad \text{with} \qquad [C, \phi] = 0, \tag{28}$$

with C determined by (16):

$$(\phi + A_0)^{ab} q_i^b = m_i q_i^a \quad \to \quad C^{ab} = \delta^{ab} m_b. \tag{29}$$

Unless all masses vanish, one cannot set $A_0 = -\phi$. The above equation fixes A_0 in our ansatz:

$$A_0 = \mathrm{diag}[0, 0, \ldots, -h_N(\rho)]. \tag{30}$$

4. The BPS vortex solution

Plugging ansatz (21)–(24) into equations (12), (13) and (17) gives

$$\rho\partial_\rho q_N - a_N q_N = 0 \tag{31}$$

$$\frac{1}{\rho}\partial_\rho a_N - \frac{e^2}{2}\left(\eta_N^2 q_N^2 - \frac{\kappa}{2\pi}h_N - \eta_N^2\right) = 0 \tag{32}$$

$$\frac{\kappa}{4\pi\rho}\partial_\rho a_N - h_N\eta_N^2 q_N^2 + \frac{1}{e^2}\left(\partial_\rho^2 h_N + \frac{1}{\rho}\partial_\rho h_N\right) = 0. \tag{33}$$

Note that the Gauss Law constraint is at the origin of the second-order derivative in h_N in (33). It will be convenient to define

$$\beta = \frac{e^2\eta_N^2}{2}, \qquad \gamma = \frac{\kappa}{2\pi\eta_N^2}, \tag{34}$$

and

$$a(\tau) = a_N(\rho), \qquad q(\tau) = q_N(\rho), \qquad h(\tau) = \gamma h_N(\rho), \tag{35}$$

where $\tau = \sqrt{\beta}\rho$. One can use (32) to eliminate $\partial_\rho a_N$ in (33) so that equations can be recast in the form

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = \tau(q^2 - h - 1) \tag{36}$$

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = \frac{1}{\tau}qa \tag{37}$$

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = u \tag{38}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = 2hq^2 - \alpha(q^2 - h - 1) - \frac{1}{\tau}u, \tag{39}$$

where $\alpha = \beta\gamma^2$. The boundary conditions imposed by finite energy read now

$$\begin{aligned} &a(0) = n, && a(\infty) = 0,\\ &q(0) = 0, && q(\infty) = 1,\\ &h(\infty) = 0. \end{aligned} \tag{40}$$

Concerning the behavior of h at the origin, it should go to a finite constant.

Using (18), the vortex mass for our ansatz takes the form

$$M = 2\pi v^2 n + Q_N m_N = 2\pi\eta_N^2 n. \tag{41}$$

The BPS vortex mass is solely determined by the topological charge, the role of η_N^2 being just that of a scale.

Our result is of course consistent with the re-parametrized form of the energy:

$$E = \frac{2\pi}{e^2}\int \tau\,\mathrm{d}\tau\left(\frac{2}{\gamma^2}(h')^2 + \frac{\beta}{\tau^2}(a')^2 + \frac{4}{\gamma^2}q^2h^2 + 2\frac{\beta}{\tau^2}a^2q^2 + 2\beta(q')^2 + \beta(q^2 - h - 1)^2\right), \tag{42}$$

which after some algebra and integration by parts can be written as follows:

$$\begin{aligned} E = \frac{2\pi}{e^2}\int \tau\,\mathrm{d}\tau\Bigg(&2\beta\left[q' \mp \frac{aq}{\tau}\right]^2 + \beta\left[\frac{a'}{\tau} \mp (q^2 - h - 1)\right]^2\\ &- \frac{2}{\gamma^2}h\left[\frac{1}{\tau}(\tau h')' - 2hq^2 \pm \alpha\frac{a'}{\tau}\right] \mp 2\beta\tau^{-1}a'\Bigg). \end{aligned} \tag{43}$$

The upper sign corresponds to our non-negative winding ansatz. The first three terms in the integral vanish as they are readily identified with the Bogomolny equations, while the last term gives the expected contribution to the vortex mass $M = 2\pi\eta_N^2 n$.

Let us note that for $m_N = 0$, equations (36)–(39) reduce to the Abelian case discussed in [24]. It is, indeed, typically observed that the non-Abelian $\mathbb{Z}_N$-vortex equations of a model reduce to the Abelian equations when the coupling constants of the $U(1)$- and the $SU(N)$-gauge groups are set equal (this choice has been made implicitly here by working with the gauge group $U(N)$). In the case $m_N \neq 0$, the only modification with respect to the $m_N = 0$ case arises through the parameter $\alpha = \frac{e^2}{4\pi}\kappa^2(2\pi v^2 + \kappa m_N)^{-1}$. Hence we can obtain the profile functions of any general $\{m_N \neq 0, \kappa\}$ vortex from a $\{m_N = 0, \kappa'\}$ solution by setting $\kappa' = \kappa/\sqrt{1 + \kappa m_N/2\pi v^2}$. However, the behavior of the physical observables depends upon the value of m_N. If the Chern–Simons term is absent ($\kappa = 0$), the Gauss law is satisfied by $h = 0$ and our ansatz reduces to the well-honored Abrikosov–Nielsen–Olesen vortex [28, 29].

To obtain numerical solutions to the BPS equations (36)–(39) we have used a relaxation method, selecting the following four boundary conditions:

$$a(0) = 1, \qquad q(\infty) = 1, \qquad h(\infty) = u(\infty) = 0. \tag{44}$$

Given our ansatz, the magnetic field $\mathcal{B}$ and the electric field $\mathcal{E}$ can be defined as

$$\mathcal{B} = \mathrm{Tr}\, F_{12}, \qquad \mathcal{E} = \mathrm{Tr}\, F_{0\rho}. \tag{45}$$

They are depicted in figure 1 where we have set $e^2/v^2 = 1$. It is interesting to observe the behavior of the magnetic field in the case $m_N \neq 0$. As κ is increased, the magnitude of the magnetic field at the origin initially increases. At large enough κ, however, the B-field starts to decrease at the origin and begins to show a characteristic *bump*, as encountered in the Abelian case [24]. Any further increase in κ amplifies the size of the bump. The electric field is considerably smaller than the magnetic field for small κ, but it also increases as the Chern–Simons coupling becomes important. As in the $m_N = 0$ case, the ratio $\mathcal{E}_{\max}(\kappa)/\mathcal{B}_{\max}(\kappa)$ increases linearly with κ for small κ, and eventually tends to a constant.

We may explain this behavior by observing that we have $\alpha(\kappa) \propto \kappa^2 \forall\, \kappa$ when $m_N = 0$ and $\alpha(\kappa) \propto \kappa$ for $\kappa \gg 2\pi v^2/m_N$ when $m_N \neq 0$. The limit $\kappa \gg 2\pi v^2/m_N$ is where the effects of non-zero m_N become noticeable. The profile functions a, h and q thus depend more sensitively on κ when the mass is zero. Furthermore, when computing the electric and magnetic fields from the functions a and h, their behavior is not explicitly dependent on κ for $m_N = 0$, whereas we have explicit dependence of order $\mathcal{O}(\kappa)$ in $\mathcal{B}$ and of order $\mathcal{O}(\sqrt{\kappa})$ in $\mathcal{E}$ for $m_N \neq 0$ in the same limit. In the zero-mass case [24], the magnetic field starts to exhibit the typical doughnut shape as κ is increased, at the same time as its overall strength decreases. The electric field develops the characteristic bump and also becomes weaker. As a consequence, the ratio $\mathcal{E}_{\max}/\mathcal{B}_{\max}$, which vanishes as $\kappa \to 0$, approaches a finite constant for $\kappa \gg 2\pi v^2/m_N$. For $m_N \neq 0$, on the other hand, the explicit κ-dependence of the electric and magnetic fields counteracts the trend of a and h becoming smaller for larger κ. The maxima of the two fields now increase for a larger range of κ and do not go to zero as $\kappa \to \infty$. As a result, the ratio $\mathcal{E}_{\max}(\kappa)/\mathcal{B}_{\max}(\kappa)$ still tends to a finite constant asymptotically, but it reaches it more slowly as m_N becomes larger. It is also worth noting that the radius of the bump in the electric field linearly increases with κ when $m_N = 0$, whereas it approaches a constant when $m_N = 0$ (see figure 1).

Let us end this section by computing the angular momentum for the vortex solution using the formula

$$J = \int \mathrm{d}^2x\, \varepsilon_{ij} x_i T_{0j} = \int \mathrm{d}^2x\, T_{0\varphi} \tag{46}$$

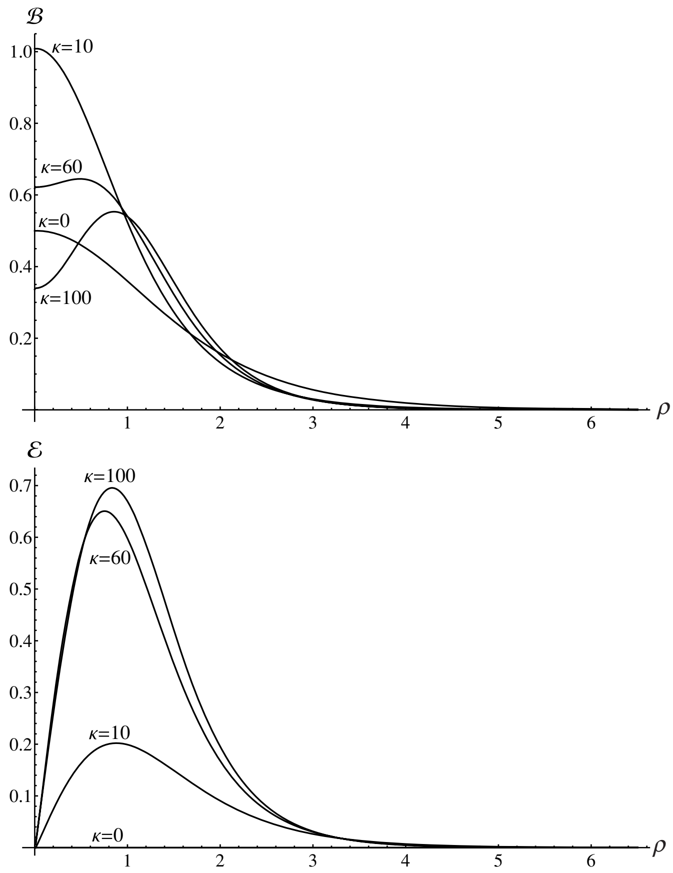

Figure 1. The magnetic field $\mathcal{B}$ and the electric field $\mathcal{E}$ of the $\{n = 1, m_N/v^2 = 1, \kappa\}$ vortex are shown for $\kappa = 0, 10, 60, 100$. The $\kappa = 0$ line corresponds to the Abelian–Nielsen–Olesen vortex, which exhibits no electric field.

(which is actually independent of the ansatz). Given Lagrangian (1), one has

$$T_{0\varphi} = \frac{2}{e^2}\rho\,\mathrm{Tr}(E_\rho B) + \frac{2}{e^2}\,\mathrm{Tr}(D_0\phi D_\varphi\phi) + ((D_0 q_a)^\dagger(D_\varphi q_a) + (D_\varphi q_a)^\dagger(D_0 q_a))$$
$$= \frac{2}{e^2}h'_N a'_N + 2\eta_N^2 h_N q_N^2 a_N. \tag{47}$$

Using the Gauss law, which for our ansatz reads

$$\eta_N^2 h_N q_n^2 = \frac{1}{e^2}\left(h''_N + \frac{1}{\rho}h'_N\right) + \frac{\kappa}{4\pi}(a'_N/\rho), \tag{48}$$

we can write

$$T_{0\varphi} = \frac{2}{e^2\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}(\rho a_N h'_N) + \frac{\kappa}{4\pi}\frac{1}{\rho}\frac{\mathrm{d}\left(a_N^2\right)}{\mathrm{d}\rho}. \tag{49}$$

So, we have for J

$$J = 2\pi\left(\frac{2}{e^2}\rho a_N h'_N + \frac{\kappa}{4\pi}a_N^2\right)\Big|_0^\infty \tag{50}$$

or finally

$$J = -\frac{\kappa}{2}n^2. \tag{51}$$

Since κ takes integer values, the angular momentum is quantized also at the classical level. Note that in view of equation (10), which in the present case reduces to $Q_N = \kappa n$, we see that the angular momentum can be written in terms of the square of the charge (in contrast to the pure Yang–Mills case in which it is proportional to the charge [9, 10]).

5. Semi-local vortices

Unlike Abrikosov–Nielsen–Olesen vortices and their non-Abelian extensions, the radius of a semi-local vortex is not fixed but it becomes a parameter. This kind of vortices emerges when $N_f > N$, that is, when there are $N_e = N_f - N$ additional fundamental scalars $\{p_e = q_{N+e}\}$, $e = 1, \ldots, N_f - N$ in comparison with the local vortices arising in the $N = N_f$ case.

We then start from Lagrangian (1) now with $N_f > N$ and consider minimization of the potential in the general case $m_i \neq m_j \forall i \neq j$, with $i, j \leqslant N$. The mass term in the $N_f > N$ Lagrangian,

$$L_m = q_i^{\dagger a}(\phi^{ab} - \delta^{ab}m_i)^2 q_i^b, \tag{52}$$

can only be made to vanish for N non-zero fundamental scalars, since we may only pick N of the diagonal entries in ϕ to cancel the $\delta^{ab}m_i$ terms. Then, in order to minimize the potential the remaining q_i need to vanish. Without loss of generality, we may choose $\{p_e = q_{N+e}\}$ to vanish. These fields then lie in the unbroken vacuum. Spontaneous symmetry breaking of $U(N)_g \times U(1)_f^{N_f-1}$ occurs as before for the original N-sector while the additional fields exhibit invariance under the more general transformation,

$$p^a_{e\text{vac}} \longrightarrow \exp(\mathrm{i}\alpha_e)U^{ab}p^b_{e\text{vac}}, \tag{53}$$

with unconstrained $\{\alpha_e\}$. These fields must be topologically trivial. If we adopt the previous ansatz for the original fields, equation (13), which as we explain below is still valid for $N_f > N$, requires all components of p_e^a to vanish identically except for $a = N$. This suggests the following ansatz for the p_e:

$$p_e^a = \eta_N \delta^{aN}\xi_e(\tau). \tag{54}$$

Furthermore, it is required by (16) that the mass of any additional (non-trivial) scalar p_e be equal to the mass of the field that carries the winding,

$$m_e = m_N \qquad \text{if} \quad \xi_e(\tau) \neq 0. \tag{55}$$

Equation (6) for the energy in the $N_f = N$ case is still valid for $N_f > N$ and so is the bound (7) and the BPS equations (12)–(16). Concerning the axially symmetric ansatz, it consists of that proposed in the local case, equations (21)–(24), augmented with (54) for the extra scalars. Inserting the ansatz in the BPS equations one now obtains

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = \tau(q^2 + |\xi_e|^2 - h - 1) \tag{56}$$

$$\frac{\mathrm{d}q}{\mathrm{d}\tau} = \frac{1}{\tau}qa \tag{57}$$

$$\frac{\mathrm{d}\xi_e}{\mathrm{d}\tau} = \frac{1}{\tau}(a - n)\xi_e, \qquad e = 1, \ldots, N_f - N_c \tag{58}$$

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = u \tag{59}$$

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = 2h(q^2 + |\xi_e|^2) - \alpha(q^2 + |\xi_e|^2 - h - 1) - \frac{1}{\tau}u, \tag{60}$$

with $|\xi_e|^2 = \sum_e \xi_e^\dagger \xi_e$. Equations (57) and (58) can be used to solve for the profile functions $\{\xi_e\}$ [25]:

$$\xi_e(\tau) = \chi_e \frac{q(\tau)}{\tau^n} \tag{61}$$

with $\chi_e \in \mathbb{C}$ arbitrary complex constants that parametrize the solutions. Of course, if we set all the χ_e parameters to zero, the system (56)–(60) coincides with (36)–(39) and the semi-local vortices become ordinary local ones.

The energy of the semi-local vortex is

$$E_s = \frac{2\pi}{e^2}\int \tau\,\mathrm{d}\tau \left(\frac{2}{\gamma^2}(h')^2 + \frac{\beta}{\tau^2}(a')^2 + \frac{4}{\gamma^2}h^2(q^2+|\xi_e|^2) + 2\frac{\beta}{\tau^2}a^2q^2 + 2\frac{\beta}{\tau^2}(a-n)^2|\xi_e|^2 \right.$$
$$\left. + 2\beta((q')^2 + |\xi_e'|^2) + \beta(q^2 + |\xi_e|^2 - h - 1)^2\right). \tag{62}$$

As in the local case, it is straightforward to show that this can be written as

$$E_s = \frac{2\pi}{e^2}\int \tau\,\mathrm{d}\tau \left(2\beta\left[q' \mp \frac{aq}{\tau}\right]^2 + 2\beta\left|\xi_e' \mp \frac{(a-n)\xi_e}{\tau}\right|^2 + \beta\left[\frac{a'}{\tau} \mp (q^2 + |\xi_e|^2 - h - 1)\right]^2 \right.$$
$$\left. - \frac{2}{\gamma^2}h\left[\frac{1}{\tau}(\tau h')' - 2h[q^2 + |\xi_e|^2] \pm \alpha\frac{a'}{\tau}\right] \mp 2\beta\tau^{-1}a'\right), \tag{63}$$

the upper sign corresponding to the non-negative winding vortex. Energy (63) reduces to the lower bound in (7) when the Bogomolny equations are satisfied:

$$M_s = \left|2\pi n v^2 + \sum_i Q_i m_i\right|. \tag{64}$$

In the case of semi-local vortices, one can define a parameter χ, the complexified size of the vortex, through the formula

$$|\chi|^2 = \sum_e |\chi_e|^2. \tag{65}$$

We thus see that, as expected, the vortex mass is χ-independent but the behavior of the fields at infinity drastically changes with respect to the local case: the fields have a long-range power falloff instead of an exponential one. This can be seen in figure 2, which shows the numerical solutions to (56)–(60). Indeed, at large distances ($\rho \gg 1/e\sqrt{\beta}$) and for very large transverse size of the vortex ($\chi \gg e\sqrt{\beta}$) one can see how the asymptotic behavior is no more that of an exponential falloff but a power one. The analytical asymptotic behavior of the fields a_N and q_N is

$$a_n \approx n\frac{|\chi|^2}{\rho^{2|n|}}, \qquad q_N \approx 1 - \frac{1}{2}\frac{|\chi|^2}{\rho^{2|n|}}, \tag{66}$$

whereas h_N has the same exponential falloff behavior as in the local vortex case:

$$h_N \approx \frac{\mathrm{e}^{-\eta_N\rho}}{\sqrt{\rho}}. \tag{67}$$

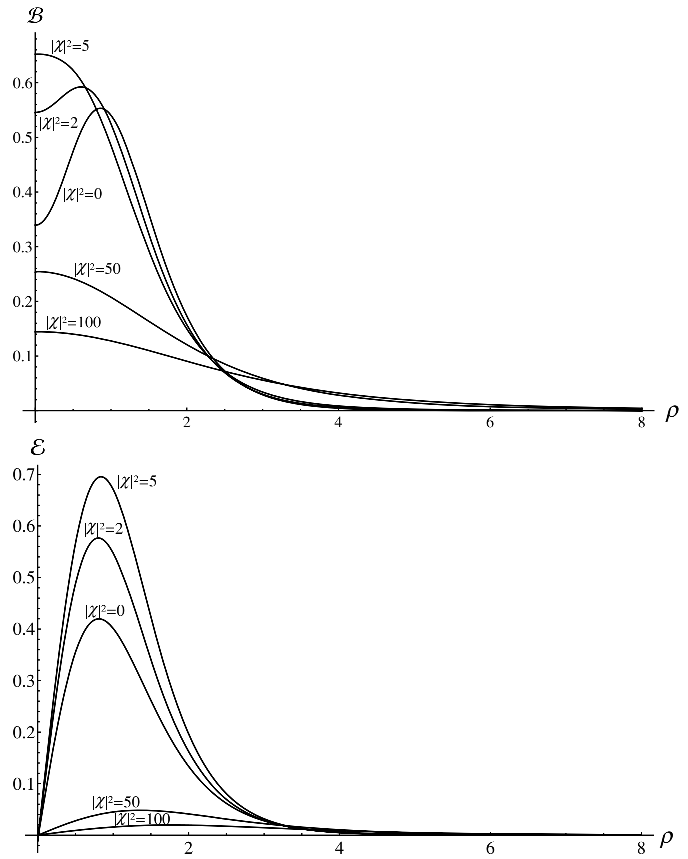

Figure 2. The magnetic and electric fields of an $\{n = 1, m_N/v^2 = 1, \kappa = 100\}$ vortex with $|\chi|^2 = 0, 2, 5, 50, 100$.

Let us finally note that the presence of the radius χ also reduces the Chern–Simons characteristic bump at the origin.

Concerning the angular momentum for the semi-local vortices, the component $T_{0\varphi}$ of the relevant energy–momentum tensor component obtains just an extra term $\Delta T_{0\varphi}$ with respect to the local case:

$$\begin{aligned}\Delta T_{0\varphi} &= \frac{\eta_N^2}{e^2}\left(\sum_e (D_0\xi_e)^*(D_\varphi\xi_e) + \text{h.c.}\right)\\ &= \mp 2\frac{\eta_N}{e^2}h_n(n-a_N)\eta_N^2|\xi_e|^2\end{aligned} \tag{68}$$

and then

$$T_{0\varphi} = \pm\frac{2}{e^2}h_N' a_N' \pm 2\eta_N^2 h_N\left(q_N^2 + |\xi_e|^2\right)a_N \mp 2n\eta_N^2 h_N|\xi_e|^2. \tag{69}$$

Using the Gauss law,

$$\rho^{-1}\frac{\mathrm{d}}{\mathrm{d}\rho}h_N(\rho)\rho) = \eta_N^2 h_n\left(q_N^2 + |\xi_e|^2\right) \mp \frac{\kappa}{4\pi}\rho^{-1}a_N', \tag{70}$$

we can write

$$T_{0\varphi} = \pm\frac{2}{e^2}\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}(h'_N a_N \rho) + \frac{\kappa}{4\pi}\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(a_N^2\right) \mp 2n\eta_N^2 h_N |\xi_e|^2, \tag{71}$$

where the first two terms coincide with the local vortex value for $T_{0\varphi}$. Taking into account the boundary conditions one now has

$$J = 2\pi \int T_{0\varphi}\rho\,\mathrm{d}\rho = -\frac{\kappa}{2}n^2 \pm 2n\eta_N^2 \int h_N |\xi_e|^2 \rho\,\mathrm{d}\rho. \tag{72}$$

Thus, unlike the case of local vortex, the angular momentum is not simply quantized in terms of κn^2 but it depends explicitly on the size of the semi-local vortex.

6. Discussion

A rich spectrum of non-Abelian vortices in theories where the dynamics of the gauge field is governed by both a Yang–Mills and a Chern–Simons action was shown to exist in [22, 23], where the low-energy vortex dynamics was described in terms of a gauged sigma model on the vortex worldline. Although the BPS equations were obtained, explicit solutions were not presented and this was precisely the main objective of our work. To this end, we proposed an axially symmetric ansatz leading to BPS vortex solutions for a YMCS $U(N)$ gauge theory coupled to scalars when the number of flavors $N_f \geqslant N$, analyzing the electric and magnetic properties of the local ($N_f = N$) and semi-local ($N_f > N$) vortices.

A first interesting feature of the local vortex solutions concerns the localization of the magnetic and electric fields. As expected, as the Chern–Simons coefficient κ grows, the magnetic maximum moves away from the vortex center and the electric field, also with an annulus shape, starts to develop. Semi-local vortices exhibit a similar behavior except that $\mathcal{B}$ and $\mathcal{E}$ have a long-range power falloff instead of an exponential one (only the real scalar field keeps its exponential falloff behavior). Another difference between local and semi-local vortices concerns the angular momentum which is a purely topological object in the former case while it depends on the vortex size in the latter semi-local case.

Our investigation started from the Lagrangian proposed in [23] with the scalar potential and constants chosen so as to guarantee the possibility of an $\mathcal{N} = 2$ supersymmetric extension, thus ensuring the existence of BPS equations [1–3]. Actually, it would be of interest to investigate the properties of the supersymmetric model and to construct the low-energy effective action describing moduli dynamics and analyze its properties both at the classical and quantum levels, following the approach presented in [13] for the pure Chern–Simons case. We hope to discuss these issues in a future work.

Acknowledgments

This work was partially supported by UNLP, UBA, CICBA, CONICET and ANPCYT and MinCyT. One of us (FAS) wishes to thank the ECM Department of the Universitat de Barcelona for hospitality during completion of this work.

References

[1] Tong D 2005 arXiv:hep-th/0509216
[2] Schaposnik F A 2006 arXiv:hep-th/0611028
[3] Shifman M and Yung A 2007 *Rev. Mod. Phys.* **79** 1139 (arXiv:hep-th/0703267)
[4] Hanany A and Tong D 2003 *J. High Energy Phys.* JHEP07(2003)037 (arXiv:hep-th/0306150)

[5] Auzzi R, Bolognesi S, Evslin J, Konishi K and Yung A 2003 *Nucl. Phys.* B **673** 187 (arXiv:hep-th/0307287)
[6] Shifman M and Yung A 2004 *Phys. Rev.* D **70** 045004 (arXiv:hep-th/0403149)
[7] Hanany A and Tong D 2004 *J. High Energy Phys.* JHEP04(2004)066 (arXiv:hep-th/0403158)
[8] Gorsky A, Shifman M and Yung A 2005 *Phys. Rev.* D **71** 045010 (arXiv:hep-th/0412082)
[9] de Vega H J and Schaposnik F A 1986 *Phys. Rev. Lett.* **56** 2564
[10] de Vega H J and Schaposnik F A 1986 *Phys. Rev.* D **34** 3206
[11] Cugliandolo L F, Lozano G, Manias M V and Schaposnik F A 1991 *Mod. Phys. Lett.* A **6** 479
[12] Cugliandolo L F, Lozano G and Schaposnik F A 1989 *Phys. Rev.* D **40** 3440
[13] Aldrovandi L G and Schaposnik F A 2007 *Phys. Rev.* D **76** 045010 (arXiv:hep-th/0702209)
[14] Lozano G S, Marques D, Moreno E F and Schaposnik F A 2007 *Phys. Lett.* B **654** 27 (arXiv:0704.2224 [hep-th])
[15] Navarro-Lerida F, Radu E and Tchrakian D H 2008 arXiv:0811.3524 [hep-th]
[16] Deser S, Jackiw R and Templeton S 1982 *Phys. Rev. Lett.* **48** 975
Deser S, Jackiw R and Templeton S 1982 *Ann. Phys.* **140** 372
[17] Hong J, Kim Y and Pac P Y 1990 *Phys. Rev. Lett.* **64** 2230
[18] Jackiw R and Weinberg E J 1990 *Phys. Rev. Lett.* **64** 2234
[19] Witten E and Olive D I 1978 *Phys. Lett.* B **78** 97
[20] Lee C k, Lee K M and Weinberg E J 1990 *Phys. Lett.* B **243** 105
[21] Edelstein J D, Nunez C and Schaposnik F 1994 *Phys. Lett.* B **329** 39 (arXiv:hep-th/9311055)
[22] Collie B and Tong D 2008 arXiv:0805.0602 [hep-th]
[23] Collie B 2008 arXiv:0809.0394 [hep-th]
[24] Lee C K, Lee K M and Min H 1990 *Phys. Lett.* B **252** 79
[25] Shifman M and Yung A 2006 *Phys. Rev.* D **73** 125012 (arXiv:hep-th/0603134)
[26] Eto M *et al* 2007 *Phys. Rev.* D **76** 105002 (arXiv:0704.2218 [hep-th])
[27] Auzzi R, Eto M, Gudnason S B, Konishi K and Vinci W 2008 arXiv:0810.5679 [hep-th]
[28] Abrikosov A 1957 *Sov. Phys.—JETP* **32** 1442
[29] Nielsen H B and Olesen P 1973 *Nucl. Phys.* B **61** 45