Ferromagnetic and Antiferromagnetic Properties of the Blume-Capel Model in $(1+1)$ and $(2+1)$ Dimensions*

C.M. Arizmendi

Facultad de Ingeniería, Universidad Nacional de Mar del Plata,
Mar del Plata, Argentina

L.N. Epele, H. Fanchiotti, and C.A. García Canal

Laboratorio de Física Teórica, Departamento de Física,
Universidad Nacional de La Plata, La Plata, Argentina

Received December 10, 1985; revised version April 1, 1986

A renormalization group realization, allowing the simultaneous and analytic study of both the ferromagnetic and antiferromagnetic phases of the Blume-Capel model, is applied in $(1+1)$ and $(2+1)$ dimensions. The model is also analyzed by using a variational technique. Critical and tricritical parameters are determined and the phase diagram is exhibited.

I. Introduction

The Blume-Capel (BC) spin model [1, 2] is a particular case of the Blume-Emery-Griffiths (BEG) one [3]. This BEG model was introduced by Griffith [4] in order to describe the tricritical behaviour of $^3\text{He}-^4\text{He}$ mixtures. It has also been useful in studying structural and metamagnetic transitions. The BC model shares some important characteristics of the BEG one, being at the same time much simpler and can be used, for example, to describe $^3\text{He}-^4\text{He}$ mixtures near a wall [5]. More recently there has been an increasing interest on this BC spin-one proposal because it is one of the simplest Ising-like models allowing for first and second order transitions and the corresponding tricritical point. For that reason it has been extensively studied particularly in a bidimensional lattice.

The quantum version of this model in d -dimensions, belonging to the same universality class than the $(d+1)$ classical one, has been attacked using finite size scaling techniques [6], real space renormalization group methods [7, 8] and mean field approximations [7]. Nevertheless in all these treatments, the antiferromagnetic behaviour of the model

was not fully taken into account. For that reason we decided to revisit the BC model, by means of an alternative renormalization group realization that allows us to study simultaneously both the ferromagnetic and the antiferromagnetic phases [9]. It is worth mentioning that the BC model is closed under our renormalization group realization in spite of the assertion in [8] that the minimal closed spin one quantum model is the BEG one. Notice that besides our realization, there exists another approach [10] allowing the same conclusion as ours. Another possibility of analysis of the different phases of the BC model is offered by a variational method [11] that we have also successfully applied.

The real space renormalization group realization used [9] allows a simple analytical study of the ferromagnetic and antiferromagnetic properties of the system. The central idea here is to consider spin blocks with a central spin in such a way that it participates in all the nearest-neighbour interactions present in the block so that the block hamiltonian can be easily decoupled in the variables of the peripheral sites. The block election and the symmetry link-site maintained all the way, distinguish two different sets of lattice sites and for that reason the antiferromagnetic properties appear without introducing higher order couplings. The variational calculation, on the other hand, is based on an iter-

* Partially supported by CONICET, CIC Pcia. de Buenos Aires and SUBCYT, Argentina

ative block hamiltonian procedure [11]. In every step a truncation is performed in order to retain only the lowest eigenstates to be used as the basis for the next step and in this way an adequate convergence for the relevant magnitudes is obtained. The renormalization group calculation was carried out in (1+1) and (2+1) dimensions whereas the variational method results are presented for (1+1) dimensions.

The BC model is defined, in a d -dimensional lattice, through the hamiltonian

$$H = J \sum_{\langle ij \rangle} s_i s_j - \Delta \sum_i (s_i)^2 \quad (1)$$

where s_i , the spin-one variable, takes the values 0, +1, -1. The coupling constants are measured in units of kT and Δ could be thought as a chemical potential for the state of $s=0$. Notice that the first sum runs over nearest-neighbours sites.

The Hamiltonian of the quantum version of the model, obtained in the standard way [6], reads

$$H = \sum_i [\gamma(s_x)_i + \delta(s_z)_i^2] - \sum_{\langle ij \rangle} (s_z)_i (s_z)_j \quad (2)$$

with

$$s_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad s_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3)$$

In the following section we present the real space renormalization group calculation of the model properties. Section III is devoted to the variational method and its results. Finally, Sect. IV contains some comments and conclusions.

II. Renormalization Group Calculation

The renormalization group method used is a standard block-spin procedure that develops recursion relations for the coupling constants of the model that can be analytically obtained [9]. The spin block chosen (see Fig. 1) with a central spin participating in all the internal block interactions, gives rise to a decoupling of the peripheral variables of the block if the eigenstates of the operator s_z corresponding to the central spin are considered. In this way, the renormalization group equations are easily obtained as a function of the coordination number of the lattice, ζ . In other words, the block hamiltonian can be written as

$$H_B = \bigoplus_{l=1}^3 \left\{ \sum_{i=1}^{\zeta} W_{i_l} \right\} \quad (4)$$

where

$$W_{i_l} = W_l = (2-l)s_z + \delta(s_z)^2 + \gamma s_x. \quad (5)$$

Fig. 1. Spin blocks for $d=2$, $\zeta=4$. The interpenetrated sublattices are indicated with dashed lines

The eigenvalues, λ_l , of this last matrix satisfy

$$\lambda_l [(\delta - \lambda_l)^2 - |2-l|] - \gamma^2 (\delta - \lambda_l) = 0 \quad (6)$$

and the corresponding eigenvectors are

$$|\lambda_l\rangle = \frac{|(\delta - \lambda_l)^2 - |2-l||}{a_l} \begin{pmatrix} \gamma/\sqrt{2} \\ \delta - \lambda_l + l - 2 \\ 1 \\ \gamma/\sqrt{2} \\ \delta - \lambda_l + 2 - l \end{pmatrix} \quad (7)$$

with

$$a_l = \{[(\delta - \lambda_l)^2 - |2-l|]^2 + \gamma^2 [(\delta - \lambda_l)^2 + |2-l|]\}^{1/2}. \quad (8)$$

Finally, the truncation of the Hilbert space is achieved by keeping only the eigenstates corresponding to the three lowest eigenvalues of each block (two of them are degenerate). These states are

$$|l\rangle_B = |l\rangle_0 \prod_{i=1}^{\zeta} |\lambda_l\rangle; \quad l=1, 2, 3. \quad (9)$$

The next step is to rebuild a total hamiltonian acting on the truncated Hilbert space. To this end we consider the matrix elements in the truncated basis of the remaining operator

$$V_{\langle j, j' \rangle} = \gamma s_x^j(0) - (\zeta - 1) s_z^j s_z^{j'} + \delta (s_z^j)^2(0) \quad (10)$$

where $\langle j, j' \rangle$ indicates two nearest-neighbour blocks and (0) the central spin. The final hamiltonian takes the original form but with renormalized coupling constants

$$H' = \sum_j \epsilon'_0 \mathbb{1}^j + \gamma' \sum_j s_x^j - \sum_{\langle j, j' \rangle} s_z^j s_z^{j'} - \delta' \sum_j (s_z^j)^2 \quad (11)$$

where

$$\epsilon'_0 = \frac{\zeta a_1^4 \lambda_2}{4(\zeta - 1) \gamma^4 (\delta - \lambda_1)^2} \quad (12)$$

and

$$\gamma' = \frac{a_1^4 \{[(\delta - \lambda_2)((\delta - \lambda_1)^2 - 1) + \gamma^2(\delta - \lambda_1)]^2 / a_1^2 [\gamma^2 + (\delta - \lambda_2)^2]\}^{\zeta/2}}{4(\zeta - 1) \gamma^3 (\delta - \lambda_1)^2} \quad (13a)$$

$$\delta' = \frac{a_1^4 [\zeta(\lambda_1 - \lambda_2) + \delta]}{4(\zeta - 1) \gamma^4 (\delta - \lambda_1)^2}. \quad (13b)$$

The fixed points of these equations were numerically obtained. Tables 1 and 2 summarized our results for the tricritical point and the corresponding critical exponents for the model in (1+1) dimensions and they are compared with some previous results. We have found also a discontinuity fixed point at $\gamma = 0$, $\delta = 1$ satisfying the Nienhuis-Nauenberg criteria

Table 1. Tricritical point coordinates for the quantum BC model in (1+1) dimensions. Previous results are included for comparison

	γ_{TC}	δ_{TC}
ORG	0.25	0.83
FSS [6]	0.42	0.91
RG [7]	0.32	0.83
MFA [7]	1.0	0.68
OV	0.35	0.83

Table 2. Critical exponents for the BC quantum model in (1+1) dimensions. The exponents for the BC classical model in 2 dimensions are also included

	ν_1	ν_2
Conjectured value [19]	1.8	0.8
ORG $d=1+1$	1.85	0.64
FSS [6] $d=1+1$	1.8	0.7
RG [7] $d=1+1$	1.81	0.93
RG [8] $d=1+1$	1.76	0.60
RG [15] $d=2$	1.84	0.92
RG Variational [16] $d=2$	1.77	0.52

Table 3. Tricritical point coordinates for the quantum BC model in (2+1) dimensions and different coordination numbers of the lattice

	γ_{TC}	δ_{TC}
$\zeta = 3$	0.28	0.96
$\zeta = 4$	0.28	1.04

Table 4. Critical exponents for the BC quantum model in (2+1) dimensions and for the BC classical one in 3 dimensions

	ν_1	ν_2
Gaussian	2	1
ORG $d=2+1$, $\zeta=3$	1.58	0.46
ORG $d=2+1$, $\zeta=4$	1.42	0.38
RG Variational [18] $d=3$	2.01	0.5
3- ε expansion [17] $d=3$	1.97	1.2

[12] in agreement with the result of [7]. Notice also that the ferromagnetic phase presents a sink at $\gamma = 0$, $\delta = -\infty$.

Tables 3 and 4 contain the corresponding results for (2+1) dimensions and different coordination numbers.

The phase diagram shown in Fig. 2, corresponding to the BC model in (1+1) dimensions, was obtained by using the renormalization group transformation flux.

Our particular renormalization group realization, defines two interpenetrated sublattices, in connection to the possible spin orientation, as shown in Fig. 1 and allows us to simultaneously study the antiferromagnetic properties of the BC model without including higher order couplings. The corresponding tricritical point and critical exponents are shown in Table 5 where one can notice that the critical exponents coincide with those of the ferromagnetic phase. This is due to the fact that a change of sign in the interaction coupling constant, giving the antiferromagnetic behaviour, can be absorbed by a

Fig. 2. Phase diagram ((γ, δ) plane) for the BC quantum model in (1+1) dimensions. MFA stands for mean field approximation [7], RG are previous renormalization group results [7], FSS labels the finite size scaling values [6], while ORG and OV indicate our renormalization group and variational method results, respectively. The corresponding tricritical points are also indicated

Table 5. Tricritical point coordinates and critical exponents for the antiferromagnetic phase in (1+1) and (2+1) dimensions

	γ_{TC}^A	δ_{TC}^A	y_1^A	y_2^A
$\zeta=2$	-0.25	-0.83	1.85	0.64
$\zeta=3$	-0.28	-0.96	1.58	0.46
$\zeta=4$	-0.28	-1.04	1.42	0.38

change of sign of spins in one of the interpenetrated sublattices. In fact, the antiferromagnetic phase diagram is deduced from the ferromagnetic one by using the symmetry of the renormalization group realization under the change of the δ and γ signs together with the inversion of the block hamiltonian spectrum as can be traced from Eqs. (7), (13a) and (13b).

III. Variational Method

A variational calculation is another simple way to analyze the ferromagnetic properties of the Blume-Capel model. We have selected a method [11] that is, in a sense, closely related to the real space renormalization group technique used above. It starts by solving the problem of only one site. With the obtained eigenstates, the problem of two sites is considered. From its solution only the lowest states are retained for the next step. Now the procedure is iterated, taking into account blocks of increasing number of sites, until the desired convergence for the order parameters, the free energy density and the energy gap are obtained.

In the present case of the BC model in (1+1) dimensions, the hamiltonian of one isolated site, i , reads

$$h_i = \gamma(s_x)_i + \delta(s_z)_i^2 \quad (14)$$

giving rise to three eigenstates after diagonalization.

Next, the two sites block hamiltonian

$$H_1 = h_1 + h_2 - (s_z)_1(s_z)_2 \quad (15)$$

is solved in a basis

$$|k\rangle = |i\rangle_1 |j\rangle_2; \quad \begin{cases} 1 \leq i \leq 3 \\ 1 \leq j \leq 3 \\ 1 \leq k \leq 9 \end{cases} \quad (16)$$

product of the eigenstates of h_1 and h_2 . Then, the matrix elements of H_1 are calculated on this basis and when diagonalized, the three lowest symmetric eigenstates are retained. The procedure is iterated

obtaining after $N+1$ steps a variational approach to the 2^N -sites problem.

We have applied the variational method just sketched to the quantum BC model in (1+1) dimensions. The phase diagram of Fig. 2 was obtained looking for the degeneracy of the variational ground state of the hamiltonian. In other words, the transition line was determined by searching the points in the parameter space (γ, δ) where the first energy gap begins to be zero. The tricritical point was identified as the point where simultaneously vanish both the first and the second energy gap. Clearly, the accuracy of this determination is not so good as that coming from the renormalization group method. We include in Table 1 the tricritical point parameters coming from the present variational approach.

The variational study of the (2+1) dimensional case is not included because it does not provide new information and is much involved and unstable from the calculational point of view.

IV. Final Comments

Both, the previously proposed renormalization group realization and the variational method used here for studying the properties of the BC model seems to be satisfactory. They provide very reasonable results for the entire phase diagram and particularly for the tricritical parameters. It is interesting to notice that the renormalization group chosen is able to consider simultaneously the ferromagnetic and antiferromagnetic phases. In this respect we remark that it provides the first values reported for the antiferromagnetic parameters of interest.

It seems that, at least for the first quantitative analysis of the quantum version of spin models, the renormalization group realization employed here and the variational technique implemented are very useful tools not only due to their simplicity but also for providing results in good agreement with those previously reported for the ferromagnetic case.

We warmly thank A. Fernandez Pacheco, whose visit to Argentina was financed by OAS, for introducing us to the variational techniques used here and very illuminating discussions.

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C.M. Arizmendi
Facultad de Ingeniería
Universidad Nacional de Mar del Plata
J.B. Justo 4302
RA-7600 – Mar del Plata
Argentina

L.N. Epele
H. Fanchiotti
C.A. García Canal
Laboratorio de Física Teórica
Departamento de Física
Universidad Nacional de La Plata
C.C. 67
RA-1900 – La Plata
Argentina