



# Quark–antiquark potential and generalized Borel transform<sup>☆</sup>

L.N. Epele, H. Fanchiotti, C.A. García Canal, M. Marucho

*Laboratorio de Física Teórica, Departamento de Física, Universidad Nacional de La Plata, C.C. 67, 1900 La Plata, Argentina*

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## Abstract

The heavy quark potential and particularly the one proposed by Richardson to incorporate both asymptotic freedom and linear confinement is analyzed in terms of a generalized Borel transform recently proposed. We were able to obtain, in the range of physical interest, an approximate analytical expression for the potential in coordinate space valid even for intermediate distances. The deviation between our approximate potential and the numerical evaluation of the Richardson's one is much smaller than  $\Lambda$  of QCD. The  $c\bar{c}$  and  $b\bar{b}$  quarkonia energy levels agree reasonably well with experimental data for  $c$  and  $b$  masses in good agreement with the values obtained from experiments. © 2001 Published by Elsevier Science B.V.

Among the different proposals for describing quark–antiquark interactions, the Richardson's potential [1], due to its simple structure, has been the subject of continuous interest [2]. This potential requires, in the author words, the minimal number of parameters. In fact, the only parameter entering the potential is the QCD related scale  $\Lambda$ . This potential, designed in order to present both asymptotic freedom and linear quark confinement includes the single dressed gluon exchange amplitude, namely

$$V(q) = \frac{4}{3} \frac{\alpha_s(q^2)}{q^2}. \quad (1)$$

Asymptotic freedom is present as soon as one adopts for  $\alpha_s(q^2)$  the effective running coupling constant provided by the renormalization group. Quark confinement is imposed by requiring that for  $q$  small  $V(q)$  behaves as the inverse four power of  $q$  that guarantees a linear behavior in  $r$ . Then, the Richardson's potential

reads

$$V^R(q) = -\frac{16\pi^2 C_F}{\beta_0} \frac{1}{q^2 \ln(1 + q^2/\Lambda^2)}, \quad (2)$$

where  $C_F$  is a group coefficient.

It is clear that the explicit calculation of the QCD coupling constant in position space is crucial when the Richardson's potential is to be applied in a concrete case. This is because the Fourier transform of (2) provides the configuration space expression

$$\begin{aligned} \bar{V}^R(r) &= -\frac{C_F}{r} \frac{2\pi}{\beta_0} \bar{\alpha}(1/r) \\ &= -\frac{C_F}{r} \frac{2\pi}{\beta_0} [a(1/r) - \Lambda^2 r^2]. \end{aligned} \quad (3)$$

Here

$$a(1/r) = 1 - 4f(r), \quad (4)$$

$$f(r) \equiv \int_1^\infty \frac{dq}{q} \frac{e^{-q\Lambda r}}{[\ln(q^2 - 1)]^2 + \pi^2}. \quad (5)$$

This expression was only computed numerically. There exist some analytical results corresponding to some

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E-mail address: [afa@fisica.unlp.edu.ar](mailto:afa@fisica.unlp.edu.ar) (M. Marucho).

asymptotic conditions. For example, for  $\Lambda r \ll 1$ , the Richardson's potential was shown to behave softer than the Coulomb interaction [4], namely

$$\bar{V}^R(r) \rightarrow \frac{1}{\Lambda r \ln(\Lambda r)}, \quad (6)$$

and for  $\Lambda r \gg 1$  provides linear confinement.

Our main point in this Letter is the calculation of the strong coupling constant in position space starting from the last integral representation Eq. (5). In so doing, we provide either an input for the Richardson's potential in configuration space or to any other alternative proposal for the quark potential including the original QCD running coupling constant or any alternative expression [5]. To this end, we fully analyze the analytic structure of the integral in the Borel plane [6]. Then, we are able to obtain the potential behaviour as a function of  $r$ , including intermediate distances.

Any perturbative analysis starts from the general relationship between  $V(q)$  and  $\bar{V}(r)$  that ends with the corresponding relation between the couplings  $\alpha_s(q)$  and  $\bar{\alpha}(1/r)$ . Notice that in the perturbative calculation,  $\bar{\alpha}(1/r)$  coincides with  $a(1/r)$ . Moreover, it has been shown [7] that one can write, for any static potential,

$$\bar{\alpha}(1/r) = \sum_n f_n \left[ -\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right]^n \alpha_s(q = \kappa/r), \quad (7)$$

where  $f_n$  are known constants,  $\beta(\alpha_s) \equiv \mu^2 \partial \alpha_s(q) / \partial \mu^2$  and  $\kappa = \exp(\gamma_E)$  a constant. In the case of the Richardson's potential, this series is asymptotically ( $q^2 \gg 1$ ) factorial divergent and its Borel sum does not exist, namely

$$\bar{\alpha}(1/r) \sim \alpha_s(q = \kappa/r) \sum_n f_n [\beta_0 \alpha_s(q = \kappa/r)]^n n!$$

Certainly, this expression provides sensible results only for very small distances because at increasing distances the non-perturbative contributions start to be important.

It is worth mentioning that the analytic behaviour of  $\bar{\alpha}(1/r)$  has been studied [8] by summing the divergent asymptotic series by using the standard Borel formalism. Clearly, being the expression no Borel summable, the Borel transform  $B(s)$  has singularities for different

values  $s_k$  on the integration path of

$$\bar{\alpha}(1/r) = \int_0^\infty \exp[-s/\alpha(q = \kappa/r)] B(s) ds.$$

Consequently, it can be defined only in principal value, showing ambiguities coming from the exponential in this integral. This approach implies that the non-perturbative contribution is considered of the same order of magnitude as the ambiguity inherent to the method [9]. An additional problem coming from the use of a perturbative  $\alpha_s$  is the presence of the Landau pole, conditioning the validity of any amplitude representing any physical observable to a finite range of energy. In this respect, there is an alternative proposal [5] that starts from a modification of the  $\alpha_s$  definition that avoids the Landau pole but retains the standard properties. Nevertheless, this change implies a modification in the linear confinement behaviour loosing the standard connection with the string tension.

All the previous mentioned problems can be avoided by using the generalized Borel transform (GBT) that was introduced in Ref. [10]. This version of the Borel transform was originally defined on a finite lattice but it can be readily adapted to the continuum, preserving all of its characteristics.

The main vantage of this proposal comes from the fact that its analytic properties have no ambiguities. Moreover, it allows to perform computations in terms of a real and positive arbitrary parameter  $\lambda$  avoiding the implementation of perturbative expansions. The approach generally ends with non-perturbative calculations of the saddle point type, when  $\lambda$  takes large arbitrary values. This is possible because the generalization implies the definition of a valid Borel transformation for each value of the parameter  $\lambda$ . Then, the better adapted value for each particular problem can be chosen. In other words, when using the GBT, a function of  $B_\lambda(s)$ , a whole class of transformations is performed. It is found that, as it should be, the results do not depend on  $\lambda$ . For the particular case of the present Letter, this can be summarize as

$$a(1/r) = T_\lambda^{-1} [T_\lambda(a(1/r))],$$

where

$$T_\lambda(a(1/r)) \equiv B_\lambda(s), \quad \text{for } 0 < r < \infty.$$

We start by presenting our previous generalization [10] of the Borel transform of a function  $f(r)$ , namely

$$B_\lambda(s) = \int_0^\infty \exp[s/\eta(r)] \left[ \frac{1}{\lambda\eta(r)} + 1 \right]^{-\lambda s} \times f(r) d(1/\eta(r)), \quad \text{Re}(s) < 0. \quad (8)$$

Among the properties of this definition we want to notice that it is valid for any analytic function  $\eta(r)$  in the interval  $0 < r < \infty$  that allows to define

$$u_\lambda(r) \equiv \frac{1}{\eta(r)} - \lambda \ln \left[ \frac{1}{\lambda\eta(r)} + 1 \right], \quad (9)$$

being monotonically increasing in the same interval if  $\eta(r)$  is. Consequently, the integral transform can be written as

$$B_\lambda(s) = \int_0^\infty \exp[su] f[r_\lambda(u)] \times \{1 + \lambda\eta[r_\lambda(u)]\} du, \quad \text{Re}(s) < 0, \quad (10)$$

where  $r_\lambda(u)$  is the inverse coming from the change of variables. From the last expression it is clear that

$$B_\lambda(s) = \int_0^\infty \exp(su) L_\lambda[r(u)] du, \quad \text{Re}(s) < 0, \quad (11)$$

is the Laplace transform of the function  $L_\lambda[r(u)]$  implicitly defined. That definition implies that  $\eta(r)$  gives rise to an analytic transformation in the negative Borel half-plane, such that its extension to the other half-plane is also analytic with a cut on the real positive axis. From this observation it is clear that  $f(r)$  can be unambiguously expressed in terms of the inverse Laplace transform integrated on the above mentioned cut (for details see Ref. [10])

$$f(r) = \frac{1}{\lambda\eta(r) + 1} \times \int_0^\infty \exp[-s/\eta(r)] \left[ \frac{1}{\lambda\eta(r)} + 1 \right]^{\lambda s} \Delta B_\lambda(s) ds. \quad (12)$$

As it was said before, the parameter  $\lambda$  can take any real positive non-zero value generating a continuous family of transformations. A large value of  $\lambda$  could be useful because in this case asymptotic techniques can be used in the calculations.

From this point on, a series of almost trivial calculations follows. Let us only indicate the most important steps. Using  $1/\eta(r) = \lambda[\exp(\Delta r/\lambda) - 1]$  which is well-defined for  $0 < r < \infty$ , the function  $u_\lambda(r)$  results

$$u_\lambda(r) = \lambda[\exp(\Delta r/\lambda) - 1] - \Delta r.$$

Consequently, one can write

$$B_\lambda(s) = \int_1^\infty dq H(q) \times \int_0^\infty \exp(s\lambda v)(v + 1)^{-\lambda(s+q)} dv, \quad (13)$$

with

$$H(q) = \frac{1}{[\ln(q^2 - 1)]^2 + \pi^2} \frac{1}{q}, \quad (14)$$

where the change of variable  $1/[\lambda\eta(r)] = v$  was introduced. Notice that the last integral, for  $\text{Re}(s) < 0$ , represents the confluent hypergeometric function [11]  $G(1, -\lambda s - \lambda q + 2, -s\lambda)$ . Consequently, in this region  $B_\lambda(s)$  is an analytic function and when an analytic continuation to the positive half-plane of  $s$  is performed, a cut appears. Introducing the discontinuity of the  $G$  function, one gets

$$\Delta B_\lambda(s) = 2\pi\lambda \exp(-\lambda s) \times \int_1^\infty dq H(q) \frac{(\lambda s)^{\lambda(s+q)-1}}{\Gamma[\lambda(s+q)]}. \quad (15)$$

We can now transform back to obtain  $f(r)$  from Eq. (12)

$$f(r) = \frac{1}{\pi} \lambda^2 A_\lambda(r) \times \int_{-\infty}^\infty \int_{-\infty}^\infty \exp[G(w, t, r, \lambda)] dw dt, \quad (16)$$

where

$$A_\lambda(r) = 1 - \exp[-\Delta r/\lambda],$$

and

$$G(w, t, r, \lambda) = -\lambda v(t)u_\lambda(r) + t - \pi w - \ln[w^2 + 1] - 2 \ln(q(w)) - \ln\{\Gamma[\lambda(\lambda v(t) + q(w))]\} + \lambda q(w) \ln[\lambda^2 v(t)] - \lambda^2 v(t) + (\lambda^2 v(t) - 1) \ln[\lambda^2 v(t)], \quad (17)$$

with

$$q(w) = [1 + e^{-\pi w}]^{1/2}, \quad v(t) = e^t.$$

The next step is to look for the asymptotic contribution in  $\lambda$  of the double integral in Eq. (16). To this end one can use the steepest descent technique in the combined variables  $(t, w)$ . Consequently one first computes the saddle points  $t_0(r)$  and  $w_0(r)$  and then one checks the positivity condition [12], in particular when the discriminant  $D(t_0, w_0)$  of the second derivatives of  $G$  at this point is positive. In so doing one obtains

$$f(r) \simeq 2\lambda^2 A_\lambda(r) \exp[G(w_0(r), t_0(r), r, \lambda)] \times \left[ \frac{\partial^2 G}{\partial w^2} \frac{\partial^2 G}{\partial t^2} - \left( \frac{\partial^2 G}{\partial w \partial t} \right)^2 \right]^{-1/2}, \quad (18)$$

the saddle point being

$$t_0 = \ln \left[ \frac{q_0(r)}{F(q_0(r))} \right] = t_0(r),$$

$$w_0 = -\frac{1}{\pi} \ln[q_0^2(r) - 1] = w_0(r), \quad (19)$$

where  $q_0(r)$  is the solution of the implicit equation coming from the extremes of the function  $G$ , namely

$$r^2 = \frac{F(q_0)}{\Lambda^2} \left[ F(q_0) + \frac{1}{q_0} \right], \quad (20)$$

with

$$F(q_0) = \frac{2}{q_0[q_0^2 - 1]} \left[ 1 - \frac{2q_0^2 \ln(q_0^2 - 1)}{[\ln(q_0^2 - 1)]^2 + \pi^2} \right]. \quad (21)$$

Notice that  $F(q_0)$  should be positive and consequently  $q_0 < 2.130156$ . On the other hand, from Eq. (19),  $q_0 > 1$ . In fact, moving  $q_0$  between these values, the variable  $r$  covers all the positive real axis in a biunivocal way. Moreover, the condition  $F(q_0) \neq 0$  implies that  $r = 0$  is excluded from the analysis. This is clearly not a drawback of the method.

Going now to the expression (18) of  $f(r)$ , one finally finds, in the saddle point approximation ( $\lambda \rightarrow \infty$ )

$$f_{\text{Ap}}(r) = f(q_0(r)) \simeq \frac{e^{-1/2} \sqrt{2} \sqrt{(F(q_0) + 1/q_0)}}{2\sqrt{\pi D(q_0)}} \frac{1}{[q_0]^{3/2}} [q_0^2 - 1] \times \frac{\exp[-q_0 F(q_0)]}{\left( \left[ \frac{1}{\pi} \ln(q_0^2 - 1) \right]^2 + 1 \right)}, \quad (22)$$

Table 1

Comparison between potentials for  $\Lambda = 0.398$  GeV

$r$ (GeV $^{-1}$ )	$V_{\text{Rich}}$ (GeV)	$V_{\text{Ap}}$ (GeV)	$ \delta V $
7.0	0.899941	0.899923	0.000018
5.0	0.554276	0.554244	0.000032
3.0	0.149355	0.149637	0.000281
1.0	-0.571347	-0.554343	0.017004
0.5	-1.114402	-1.062477	0.051926
0.2	-2.208255	-2.284687	0.086730

where

$$D(q_0) = \frac{\pi^2(q_0^2 - 1)}{2q_0^4} \left[ 1 + \frac{q_0^3 F(q_0)}{2} \right] + \frac{1 - \left[ \frac{1}{\pi} \ln(q_0^2 - 1) \right]^2}{\left[ \left[ \frac{1}{\pi} \ln(q_0^2 - 1) \right]^2 + 1 \right]^2} - \left[ \frac{\pi(q_0^2 - 1)F(q_0)}{2q_0} \right]^2. \quad (23)$$

Finally, the approximated expression for the potential is

$$V_{\text{Ap}}(r) = -\frac{2\pi C_F}{\beta_0} \left[ \frac{1 - 4f_{\text{Ap}}(r)}{r} - \Lambda^2 r \right]. \quad (24)$$

In order to test the precision provided by the GBT, we have compared the behavior in the coordinate space of our approximated analytical formula (24) with the numerical integration of the exact expression (3). This comparison, within the range of physical interest  $0.1 < r < 1$  fm (see, for example, [5,13]), has been performed for the Richardson's potential free parameter  $\Lambda = 0.398$  GeV which he found to be the best to fit the charmonium and bottomonium lowest bound states. The corresponding results are exhibited in Table 1, being the relative error 5% at most.

We have also evaluated, the charmonium and bottomonium spectra following the Richardson's algorithm. For this purpose, with the help of (24), we have adjusted  $m_c$  and  $\Lambda$  to reproduce the best measured charmonium energy levels  $\psi(1S)$  and  $\psi(2S)$  [14]. The resulting charm quark mass is  $m_c = 1.363$  GeV and the scale size  $\Lambda = 0.434$  GeV (to be compared with  $m_c = 1.491$  GeV and  $\Lambda = 0.398$  GeV obtained by Richardson). In the bottomonium case we kept  $\Lambda = 0.434$  GeV assuming, like Richardson, that the

Table 2  
System  $c\bar{c}$

$l$	$E_{\text{Exp}}$ (GeV)	$E_{\text{Rich}}$ (GeV)	$E_{\text{Ap}}$ (GeV)
0	3.096	3.096	3.096
0	3.684	3.684	3.684
0	4.040 (4.160)*	4.096	4.127
0	4.415	4.440	4.506
1	3.522	3.514	3.494
2	3.770	3.799	3.786

\* See [17] for an alternative level assignment.

Our parameters are ( $\Lambda = 0.434$  GeV,  $m_c = 1.363$  GeV) and the corresponding Richardson's ones ( $\Lambda = 0.398$  GeV,  $m_c = 1.491$  GeV).

Table 3  
System  $b\bar{b}$

$l$	$E_{\text{Exp}}$ (GeV)	$E_{\text{Rich}}$ (GeV)	$E_{\text{Ap}}$ (GeV)
0	9.460	9.460	9.522
0	10.023	10.016	10.023
0	10.355	10.343	10.351
0	10.580	10.607	10.620
1	9.900	9.896	9.910
1	10.260	10.249	10.252

Our parameters used are ( $\Lambda = 0.434$  GeV,  $m_b = 4.780$  GeV) and the corresponding Richardson's ones ( $\Lambda = 0.398$  GeV,  $m_b = 4.888$  GeV).

scale size should be a fundamental scale of the theory independent from the quarkonium system. The bottom mass reproducing the best measured bottomonium energy state  $\Upsilon(2S)$  [14] results in  $m_b = 4.780$  GeV. Notice that Richardson has fitted the bottomonium fundamental level  $\Upsilon(1S)$  obtaining  $m_b = 4.888$  GeV. The results for both spectra and the corresponding experimental values are presented in Tables 2 and 3, respectively.

In summary, we have obtained an analytic expression for the quark potential valid for any value of  $r$ . Our results, in the leading order of the saddle point approximation, give rise to absolute values of the uncertainties always much smaller than  $\Lambda$  of QCD ( $\sim 0.4$  GeV). It is worth mentioning that any perturbation based calculation obtained by means of the standard Borel transform, ends with uncertainties at least of the order of  $\Lambda$  [7]. This result shows the ability and

potentiality of the previously introduced generalized Borel transform.

As regards the quarkonia energy levels, our spectra prediction results reasonably well when compared with the spectra obtained by Richardson. The deviations of our results with respect to the corresponding experimental values, are certainly smaller than the scale size  $\Lambda$  (see Tables 2 and 3). Let us remark that our adjustment leads to quark masses, when compared with those of Richardson, in best agreement with the values reported in the particle data table [14]. The resulting  $\Lambda$  parameter falls in the range of values expected from perturbative QCD.

There exist in the literature other simple potentials in the coordinate space [3,15] that can compete with our approximation and that give similar results for the spectra. However these potentials are strictly phenomenological, not having a direct connection with perturbative QCD and in general, they have more free parameters to be adjusted.

Finally, our  $Q\bar{Q}$  approximate potential, due to computing simplicities, is clearly useful in any further calculations of interesting physical quantities [16].

## References

- [1] J.L. Richardson, Phys. Lett. B 82 (1979) 272.
- [2] M. Beneke, Phys. Lett. B 434 (1998) 115.
- [3] C. Quigg, J.L. Rosner, Phys. Rep. 56 (1979) 167.
- [4] M. Peter, Nucl. Phys. B 501 (1997) 471.
- [5] A.V. Nesterenko, Phys. Rev. D 62 (2000) 094028, hep-ph/0010257.
- [6] M. Beneke, Phys. Rep. 317 (1999) 1.
- [7] M. Jezabek, M. Peter, Y. Sumino, Phys. Lett. B 428 (1998) 352.
- [8] M. Pindor, hep-th/9903151.
- [9] U. Aglietti, Z. Ligeti, Phys. Lett. B 364 (1995) 75; M. Beneke, V.M. Braun, Nucl. Phys. B 426 (1994) 301.
- [10] L.N. Epele, H. Fanchiotti, C.A. García Canal, M. Marucho, Nucl. Phys. B 583 (2000) 454.
- [11] H. Bateman, Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, 1953.
- [12] H. Jeffrey, B.S. Jeffrey, Methods of Mathematical Physics, Cambridge, 1966, p. 187.
- [13] N. Brambilla, A. Vairo, hep-ph/9904330.
- [14] Particle Data Group, Eur. Phys. J. C 15 (2000) 1.
- [15] E. Eichten, K. Gottfried, T. Kinoshita, K.D. Lane, T.M. Yan, Phys. Rev. D 21 (1980) 203.
- [16] M. Jezabek, J.H. Kühn, M. Peter, Y. Sumino, T. Teubner, Phys. Rev. D 58 (1998) 014006.
- [17] R.N. Faustov, V.O. Galkin, A.V. Tatarintsev, A.S. Vshivtsev, Int. J. Mod. Phys. A 15 (2000) 209.