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Unusual poles of the ζ-functions for some regular singular differential operators

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Abstract
We consider the resolvent of a system of first-order differential operators with a regular singularity, admitting a family of self-adjoint extensions. We find that the asymptotic expansion for the resolvent in the general case presents powers of λ which depend on the singularity, and can take even irrational values. The consequences for the pole structure of the corresponding ζ- and η-functions are also discussed.

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1. Introduction

It is well known that in quantum field theory under external conditions, quantities such as vacuum energies and effective actions, which describe the influence of boundaries or external fields on the physical system, are generically divergent and require a renormalization to get a physical meaning.

In this context, a powerful and elegant regularization scheme to deal with these problems is based on the use of the ζ-function [1, 2] or the heat-kernel (for recent reviews see, for example, [3–7]) associated with the relevant differential operators appearing in the quadratic part of the actions. In this way, ground-state energies, heat-kernel coefficients, functional determinants and partition functions for quantum fields can be given in terms of the corresponding ζ-function, where the ultraviolet divergent pieces of the one-loop contributions are encoded as poles of its holomorphic extension.

Thus, it is of major interest in physics to determine the singularity structure of ζ-functions associated with these physical models.

In particular [8], for an elliptic boundary value problem in a ν-dimensional compact manifold with boundary, described by a differential operator A of order ω, with smooth coefficients and a ray of minimal growth, defined on a domain of functions subject to local boundary conditions, the ζ-function

$$\zeta_A(s) \equiv \mathrm{Tr}\{A^{-s}\} \tag{1.1}$$

has a meromorphic extension to the complex s-plane whose singularities are isolated simple poles at $s = (\nu - j)/\omega$, with $j = 0, 1, 2, \ldots$.

In the case of positive definite operators, the ζ-function is related, via Mellin transform, to the trace of the heat-kernel of the problem, and the pole structure of $\zeta_A(s)$ determines the small-t asymptotic expansion of this trace [8, 9]:

$$\mathrm{Tr}\{\mathrm{e}^{-tA}\} \sim \sum_{j=0}^{\infty} a_j(A) t^{(j-\nu)/\omega} \tag{1.2}$$

where the coefficients are related to the residues by

$$a_j(A) = \mathrm{Res}|_{s=(\nu-j)/\omega} \Gamma(s)\zeta_A(s). \tag{1.3}$$

For operators $-(\mathrm{d}/\mathrm{d}x)^2 + V(x)$ with a singular potential $V(x)$ asymptotic to g/x^2 as $x \to 0$, this expansion is substantially different. If $g \geqslant 3/4$, the operator is essentially self-adjoint. This case has been treated in [10–12], where log terms are found, as well as terms with coefficients which are distributions concentrated at the singular point $x = 0$. For the case $g > -1/4$, the Friedrichs extension has been treated in [13] for operators in $\mathbf{L}_2(0, 1)$, and in [14] for operators in $\mathbf{L}_2(\mathbf{R}^+)$.

On the other hand, [15] gave the pole structure of the ζ-function of a second-order differential operator defined on the (non-compact) half-line $\mathbf{R}^+$, having a singular zeroth-order term given by $V(x) = gx^{-2} + x^2$. It showed that, for a certain range of real values of g, this operator admits nontrivial self-adjoint extensions in $\mathbf{L}_2(\mathbf{R}^+)$, for which the associated ζ-function presents isolated simple poles which (in general) do not lie at $s = (1 - j)/2$ for $j = 0, 1, \ldots$, and can even take irrational values.

A similar structure has been noticed in [16] for the singularities of the ζ-function of a system of first-order differential operators in the half line, appearing in a model of supersymmetric quantum mechanics with a singular superpotential $\sim x^{-1}$.

Let us mention that singular potentials $\sim 1/x^2$ have been considered in the description of several physical systems, such as the Calogero model [15, 17–19], conformal invariant quantum mechanical models [20–22] and, more recently, the dynamics of quantum particles in the asymptotic near-horizon region of black holes [23–27]. Moreover, singular superpotentials have been considered as possible agents of supersymmetry breaking in models of supersymmetric quantum mechanics [28–30] (see also [16]).

It is the aim of the present paper to analyse the behaviour of the resolvent and ζ- and η-functions of a system of first-order differential operators with a regular singularity in a compact segment, admitting a family of self-adjoint extensions.

We will show that the asymptotic expansion for the resolvent in the general case presents powers of λ which depend on the singularity, and can take even irrational values. The consequence of this behaviour on the corresponding ζ- and η-functions is the presence of simple poles lying at points which also depend on the singularity, with residues depending on the self-adjoint extension considered.

We first construct the resolvents for two particular extensions, for which the boundary condition at the singular point $x = 0$ is invariant under the scaling $x \to cx$. The resolvent expansion for these special extensions displays the usual powers, leading to the usual poles for the ζ-function. The resolvents of the remaining extensions are convex linear combinations of these special extensions, but the coefficients in the convex combination depend on the eigenvalue parameter λ. This dependence leads to unusual powers in the resolvent expansion, and hence to unusual poles for the zeta-function. These self-adjoint extensions are not invariant under the scaling $x \to cx$; as $c \to 0$ they tend (at least formally) to one of the invariant extensions, and as $c \to \infty$ they tend to the other. As $c \to 0$ the residues at the anomalous poles tend to zero, whereas as $c \to \infty$ these residues become infinite. The way these residues depend on the boundary condition is explained by a scaling argument in section 7.

The structure of the paper is as follows: in section 2, we define the operator and determine its self-adjoint extensions, and in section 3 we study their spectra. In section 4 we construct the resolvent for a general extension as a linear combination of the resolvent of two limiting cases, and in section 5 we consider the traces of these operators. The asymptotic expansion of these traces, evaluated in section 6, is used in section 7 to construct the associated ζ- and η-functions and study their singularities.

Finally, in section 8 we briefly describe similar results one can obtain for a second-order differential operator with a regular singularity, also admitting a family of self-adjoint extensions.

2. The operator and its self-adjoint extensions

Let us consider the differential operator

$$D_x = \begin{pmatrix} 0 & \tilde{A}_x \\ A_x & 0 \end{pmatrix} \tag{2.1}$$

with

$$A_x = -\partial_x + \frac{g}{x} = -x^g \partial_x x^{-g} \qquad \tilde{A}_x = \partial_x + \frac{g}{x} = x^{-g}\partial_x x^g \tag{2.2}$$

and $g \in \mathbb{R}$, defined on a domain of (two component) smooth functions with compact support in a segment, $\mathcal{D}(D_x) = \mathcal{C}_0^\infty(0, 1)$. It can be easily seen that D_x so defined is symmetric.

The adjoint operator D_x^*, which is the maximal extension of D_x, is defined on the domain $\mathcal{D}(D_x^*)$ of functions $\Phi(x) = \binom{\phi_1(x)}{\phi_2(x)} \in \mathbf{L}_2(0, 1)$, having a locally summable first derivative and such that

$$D_x\Phi(x) = \begin{pmatrix} \tilde{A}_x\phi_2(x) \\ A_x\phi_1(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \in \mathbf{L}_2(0, 1). \tag{2.3}$$

Lemma 2.1. *If $\Phi(x) \in \mathcal{D}(D_x^*)$ and $-\frac{1}{2} < g < \frac{1}{2}$, then*

$$|\phi_1(x) - C_1[\Phi]x^g| + |\phi_2(x) - C_2[\Phi]x^{-g}| \leqslant K_g \|D_x\Phi(x)\| x^{1/2} \tag{2.4}$$

for some constants $C_1[\Phi]$ and $C_2[\Phi]$, where $\|\cdot\|$ is the $\mathbf{L}_2$-norm.

Indeed, equations (2.3) and (2.2) imply

$$\begin{aligned} \phi_1(x) &= C_1[\Phi]x^g - x^g \int_0^x y^{-g} f_2(y)\,\mathrm{d}y \\ \phi_2(x) &= C_2[\Phi]x^{-g} + x^{-g} \int_0^x y^g f_1(y)\,\mathrm{d}y \end{aligned} \tag{2.5}$$

where $C_1[\Phi]$ and $C_2[\Phi]$ are integration constants which depend on the function $\Phi(x)$. Taking into account that

$$\begin{aligned} \left|\int_0^x y^g f_1(y)\,\mathrm{d}y\right| &\leqslant \frac{x^{g+1/2}}{\sqrt{1+2g}}\|f_1\| \\ \left|\int_0^x y^{-g} f_2(y)\,\mathrm{d}y\right| &\leqslant \frac{x^{-g+1/2}}{\sqrt{1-2g}}\|f_2\| \end{aligned} \tag{2.6}$$

we immediately get equation (2.4) with $K_g = (1-2g)^{-1/2} + (1+2g)^{-1/2}$.

Lemma 2.2. *Let* $\Phi(x) = \binom{\phi_1(x)}{\phi_2(x)}$, $\Psi(x) = \binom{\psi_1(x)}{\psi_2(x)} \in \mathcal{D}(D^*)$. *Then*

$$(D_x\Psi, \Phi) - (\Psi, D_x\Phi) = \{C_1[\Psi]^* C_2[\Phi] - C_2[\Psi]^* C_1[\Phi]\} + \{\psi_2(1)^*\phi_1(1) - \psi_1(1)^*\phi_2(1)\}. \tag{2.7}$$

In fact, from equation (2.2) one easily obtains

$$(D_x\Psi, \Phi) - (\Psi, D_x\Phi) = \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \partial_x\{x^g\psi_2(x)^* x^{-g}\phi_1(x) - x^{-g}\psi_1(x)^* x^g\phi_2(x)\}\,\mathrm{d}x \tag{2.8}$$

from which, taking into account the results in lemma 2.1, equation (2.7) follows directly.

Now, if $\Psi(x)$ in equation (2.7) belongs to the domain of the closure of D_x, $\overline{D}_x = (D_x^*)^*$,

$$\Psi(x) \in \mathcal{D}(\overline{D}_x) \subset \mathcal{D}(D_x^*) \tag{2.9}$$

then the right-hand side of equation (2.7) must vanish for any $\Phi(x) \in \mathcal{D}(D_x^*)$. Therefore,

$$C_1[\Psi] = 0 = C_2[\Psi] \qquad \text{and} \qquad \Psi(1) = 0. \tag{2.10}$$

On the other hand, if $\Psi(x)$, $\Phi(x)$ belong to the domain of a symmetric extension of D_x (contained in $\mathcal{D}(D_x^*)$), the right-hand side of equation (2.7) must also vanish.

Thus, the closed extensions of D_x correspond to the subspaces of $\mathbb{C}^4$ under the map $\Phi \to (C_1[\Phi], C_2[\Phi], \phi_1(1), \phi_2(1))$, and the self-adjoint extensions correspond to those subspaces $S \subset \mathbb{C}^4$ such that $S = S^\perp$, with the orthogonal complement taken in the sense of the symplectic form on the right-hand side of equation (2.7).

For definiteness, in the following we will consider self-adjoint extensions satisfying the local boundary condition

$$\phi_1(1) = 0. \tag{2.11}$$

Each such extension is determined by a condition of the form

$$\alpha C_1[\Phi] + \beta C_2[\Phi] = 0 \tag{2.12}$$

with $\alpha, \beta \in \mathbb{R}$, and $\alpha^2 + \beta^2 \neq 0$. We denote this extension by $D_x^{(\alpha,\beta)}$.

3. The spectrum

In order to determine the spectrum of the self-adjoint extensions of D_x, we need the solutions of

$$(D_x - \lambda)\Phi(x) = 0 \Rightarrow \begin{cases} \tilde{A}_x\phi_2(x) = \lambda\phi_1(x) \\ A_x\phi_1(x) = \lambda\phi_2(x) \end{cases} \tag{3.1}$$

satisfying the boundary conditions in equations (2.11) and (2.12).

The solution of the homogeneous equation for $\lambda = 0$ is

$$\Phi(x) = \begin{pmatrix} C_1 x^g \\ C_2 x^{-g} \end{pmatrix} \tag{3.2}$$

but the boundary conditions in equations (2.11) and (2.12) imply that $C_1 = 0$ and $C_2 = 0$, unless $\beta = 0$. Consequently, there are no zero modes except for the self-adjoint extension characterized by $\beta = 0$, $D_x^{(1,0)}$.

Applying $\tilde{A}$ to the second line in equation (3.1), and using the first one, one easily gets

$$\left\{\partial_x^2 - \frac{g(g-1)}{x^2} + \lambda^2\right\}\phi_1(x) = 0. \tag{3.3}$$

Then, for $\lambda \neq 0$, the solutions are of the form

$$\phi_1(x) = K_1\sqrt{X}J_{\frac{1}{2}-g}(X) + K_2\sqrt{X}J_{g-\frac{1}{2}}(X) \tag{3.4}$$

with $X = \tilde{\lambda}x$, where $\tilde{\lambda} = +\sqrt{\lambda^2}$ and K_1, K_2 are constants.

This implies for the lower component of $\Phi(x)$

$$\phi_2(x) = \sigma\left\{-K_1\sqrt{X}J_{-g-\frac{1}{2}}(X) + K_2\sqrt{X}J_{g+\frac{1}{2}}(X)\right\} \tag{3.5}$$

where $\sigma = \tilde{\lambda}/\lambda$.

Taking into account that

$$J_\nu(X) = X^\nu\left\{\frac{1}{2^\nu\Gamma(1+\nu)} + O(X^2)\right\} \tag{3.6}$$

we get

$$\alpha C_1[\Phi] + \beta C_2[\Phi] = \frac{\alpha K_2\tilde{\lambda}^g}{2^{g-\frac{1}{2}}\Gamma\left(\frac{1}{2}+g\right)} - \sigma\frac{\beta K_1\tilde{\lambda}^{-g}}{2^{-g-\frac{1}{2}}\Gamma\left(\frac{1}{2}-g\right)} = 0. \tag{3.7}$$

For $\alpha = 0$, equation (3.7) implies $K_1 = 0$. Therefore, $\phi_1(1) = 0 \Rightarrow J_{g-\frac{1}{2}}(\tilde{\lambda}) = 0$. Thus, the spectrum of this extension, $D_x^{(0,1)}$, is non-degenerate and symmetric with respect to the origin, with the eigenvalues given by

$$\lambda_{\pm,n} = \pm j_{g-\frac{1}{2},n} \qquad n = 1, 2, \ldots \tag{3.8}$$

where $j_{\nu,n}$ is the nth positive zero of the Bessel function $J_\nu(z)$.[5]

For $\alpha \neq 0$, from equation (3.7) we can write

$$\frac{K_2}{K_1} = \sigma\tilde{\lambda}^{-2g}\left[\frac{4^g\Gamma\left(\frac{1}{2}+g\right)}{\Gamma\left(\frac{1}{2}-g\right)}\right]\left(\frac{\beta}{\alpha}\right). \tag{3.10}$$

In this case, the boundary condition at $x = 1$ determines the eigenvalues as the solutions of the transcendental equation

$$\tilde{\lambda}^{2g}\frac{J_{\frac{1}{2}-g}(\tilde{\lambda})}{J_{g-\frac{1}{2}}(\tilde{\lambda})} = \sigma\rho(\alpha,\beta) \tag{3.11}$$

where we have defined

$$\rho(\alpha,\beta) := -\frac{4^g\Gamma\left(\frac{1}{2}+g\right)}{\Gamma\left(\frac{1}{2}-g\right)}\left(\frac{\beta}{\alpha}\right). \tag{3.12}$$

[5] Let us recall that large zeros of $J_\nu(\lambda)$ have the asymptotic expansion

$$j_{\nu,n} \simeq \gamma - \frac{4\nu^2-1}{8\gamma} + O\Big(\frac{1}{\gamma}\Big)^3 \tag{3.9}$$

with $\gamma = \left(n + \frac{\nu}{2} - \frac{1}{4}\right)\pi$.

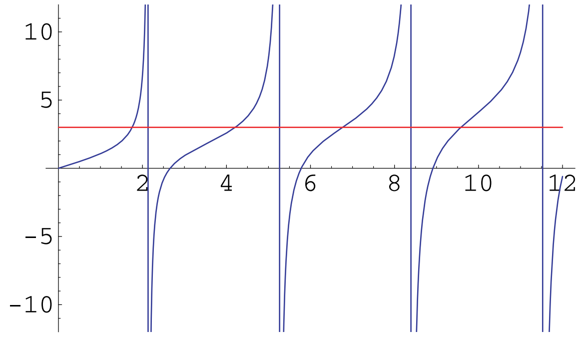

Figure 1. Plot for $F(\lambda) := \lambda^{2g} J_{\frac{1}{2}-g}(\lambda)/J_{g-\frac{1}{2}}(\lambda)$, with $g = 1/3$, and $\rho(\alpha, \beta) = 3$.

For the positive eigenvalues $\tilde{\lambda} = \lambda \Rightarrow \sigma = 1$, and equation (3.11) reduces to

$$F(\lambda) := \lambda^{2g} \frac{J_{\frac{1}{2}-g}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} = \rho(\alpha, \beta) \tag{3.13}$$

the relation plotted in figure 1 for particular values of $\rho(\alpha, \beta)$ and g.

On the other hand, for negative eigenvalues $\lambda = \mathrm{e}^{\mathrm{i}\pi}\tilde{\lambda} \Rightarrow \sigma = \tilde{\lambda}/\lambda = \mathrm{e}^{-\mathrm{i}\pi}$, and equation (3.11) reads

$$F(\tilde{\lambda}) = \mathrm{e}^{-\mathrm{i}\pi} \rho(\alpha, \beta) = \rho(\alpha, -\beta). \tag{3.14}$$

Therefore, the negative eigenvalues of $D_x^{(\alpha,\beta)}$ are $\mathrm{e}^{\mathrm{i}\pi}$ times the positive eigenvalues of $D_x^{(\alpha,-\beta)}$.

Note that the spectrum is always non-degenerate, and there is a positive eigenvalue between each pair of consecutive zeros of $J_{g-\frac{1}{2}}(\lambda)$.

Moreover, the spectrum is symmetric with respect to the origin only for the $\alpha = 0$ extension (which we call the 'D-extension' (see equation (3.8)), and for the $\beta = 0$ extension (which we call the 'N-extension'). Indeed, in this last case, from equations (3.11) and (3.12) one can see that the eigenvalues of $D_x^{(1,0)}$ are given by

$$\lambda_0 = 0 \qquad \lambda_{\pm,n} = \pm j_{\frac{1}{2}-g,n} \qquad n = 1, 2, \ldots \tag{3.15}$$

4. The resolvent

In this section we will construct the resolvent of D_x,

$$G(\lambda) = (D_x - \lambda)^{-1} \tag{4.1}$$

for its different self-adjoint extensions.

We will first consider the two limiting cases in equation (2.12), namely the 'D-extension', for which $\alpha = 0 \Rightarrow C_2[\Phi] = 0$, and the '$N$-extension', with $\beta = 0 \Rightarrow C_1[\Phi] = 0$. The resolvent for a general self-adjoint extension will be later evaluated as a linear combination of those obtained for these two limiting cases.

For the kernel of the resolvent

$$G(x, y; \lambda) = \begin{pmatrix} G_{11}(x, y; \lambda) & G_{12}(x, y; \lambda) \\ G_{21}(x, y; \lambda) & G_{22}(x, y; \lambda) \end{pmatrix} \tag{4.2}$$

we have

$$(D_x - \lambda) G(x, y; \lambda) = \delta(x, y) \mathbf{1}_2 \tag{4.3}$$

from which we straightforwardly get for the diagonal elements

$$\begin{aligned} \left\{\partial_x^2 - \frac{g(g-1)}{x^2} + \lambda^2\right\} G_{11}(x, y; \lambda) &= -\lambda\delta(x, y) \\ \left\{\partial_x^2 - \frac{g(g+1)}{x^2} + \lambda^2\right\} G_{22}(x, y; \lambda) &= -\lambda\delta(x, y) \end{aligned} \tag{4.4}$$

while for the non-diagonal ones we have

$$\begin{aligned} G_{21}(x, y; \lambda) &= \frac{1}{\lambda}\left\{-\partial_x + \frac{g}{x}\right\} G_{11}(x, y; \lambda) \\ G_{12}(x, y; \lambda) &= \frac{1}{\lambda}\left\{\partial_x + \frac{g}{x}\right\} G_{22}(x, y; \lambda) \end{aligned} \tag{4.5}$$

for $\lambda \neq 0$.

Since the resolvent is analytic in λ, it is sufficient to evaluate it on the open right half plane.

In so doing, we will need the upper and lower components of some particular solutions of the homogeneous equation (3.1).

Then, let us define

$$\begin{cases} L_1^D(X) = \sqrt{X} J_{g-\frac{1}{2}}(X) \\ L_2^D(X) = \sqrt{X} J_{g+\frac{1}{2}}(X) \\ L_1^N(X) = \sqrt{X} J_{\frac{1}{2}-g}(X) \\ L_2^N(X) = \sqrt{X} J_{-g-\frac{1}{2}}(X) \\ R_1(X; \lambda) = \sqrt{X}\big[J_{g-\frac{1}{2}}(\lambda) J_{\frac{1}{2}-g}(X) - J_{\frac{1}{2}-g}(\lambda) J_{g-\frac{1}{2}}(X)\big] \\ R_2(X; \lambda) = \sqrt{X}\big[J_{g-\frac{1}{2}}(\lambda) J_{-g-\frac{1}{2}}(X) + J_{\frac{1}{2}-g}(\lambda) J_{g+\frac{1}{2}}(X)\big]. \end{cases} \tag{4.6}$$

Note that $R_1(\lambda; \lambda) = 0$, and $\tilde{A}_x R_2(\lambda x; \lambda)|_{x=1} = 0$.

We will also need the Wronskians

$$\begin{cases} W\left[L_1^D(X), R_1(X; \lambda)\right] = -\dfrac{2}{\pi}\cos(g\pi) J_{g-\frac{1}{2}}(\lambda) = \dfrac{1}{\gamma_D(\lambda)} \\ W\left[L_2^D(X), R_2(X; \lambda)\right] = \dfrac{2}{\pi}\cos(g\pi) J_{g-\frac{1}{2}}(\lambda) = \dfrac{-1}{\gamma_D(\lambda)} \\ W\left[L_1^N(X), R_1(X; \lambda)\right] = -\dfrac{2}{\pi}\cos(g\pi) J_{\frac{1}{2}-g}(\lambda) = \dfrac{1}{\gamma_N(\lambda)} \\ W\left[L_2^N(X), R_2(X; \lambda)\right] = -\dfrac{2}{\pi}\cos(g\pi) J_{\frac{1}{2}-g}(\lambda) = \dfrac{1}{\gamma_N(\lambda)} \end{cases} \tag{4.7}$$

which vanish only at the zeros of $J_\nu(\lambda)$, for $\nu = \pm\left(\frac{1}{2} - g\right)$.

4.1. The resolvent for the D-extension

In this case, the function

$$\Phi(x) = \int_0^1 G_D(x, y; \lambda) \begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix} \mathrm{d}y \tag{4.8}$$

must satisfy $\phi_1(1) = 0$ and $C_2[\Phi] = 0$, for any functions $f_1(x), f_2(x) \in \mathbf{L}_2(0, 1)$.

This requires that

$$G^D_{11}(x,y;\lambda) = \gamma_D(\lambda) \times \begin{cases} L^D_1(X)R_1(Y;\lambda) & \text{for} \quad x \leqslant y \\ R_1(X;\lambda)L^D_1(Y) & \text{for} \quad x \geqslant y \end{cases} \tag{4.9}$$

and

$$G^D_{22}(x,y;\lambda) = -\gamma_D(\lambda) \times \begin{cases} L^D_2(X)R_2(Y;\lambda) & \text{for} \quad x \leqslant y \\ R_2(X;\lambda)L^D_2(Y) & \text{for} \quad x \geqslant y \end{cases} \tag{4.10}$$

with the other components, $G^D_{12}(x,y;\lambda)$ and $G^D_{21}(x,y;\lambda)$, given as in equation (4.5). The fact that the boundary conditions are satisfied, as well as $(D_x - \lambda)\Phi(x) = \binom{f_1(x)}{f_2(x)}$, can be straightforwardly verified from equations (4.6) and (4.7).

Indeed, from equations (4.8)–(4.10), (4.5)–(4.7), one gets

$$\phi_1(x) = C^D_1[\Phi]x^g + O(\sqrt{x}) \qquad \phi_2(x) = O(\sqrt{x}) \tag{4.11}$$

with

$$C^D_1[\Phi] = \frac{-\pi\lambda^{g+1}}{2^{\frac{1}{2}+g}\cos(g\pi)J_{g-\frac{1}{2}}(\lambda)\Gamma\left(\frac{1}{2}+g\right)} \int_0^1 [R_1(\lambda y;\lambda)f_1(y) - R_2(\lambda y;\lambda)f_2(y)]\,\mathrm{d}y \tag{4.12}$$

for λ not a zero of $J_{g-\frac{1}{2}}(\lambda)$.

Note that $C^D_1[\Phi] \neq 0$ if the integral in the right-hand side of equation (4.12) is non-vanishing.

4.2. *The resolvent for the N-extension*

In this case, the function

$$\Phi(x) = \int_0^1 G_N(x,y;\lambda)\begin{pmatrix} f_1(y) \\ f_2(y) \end{pmatrix}\mathrm{d}y \tag{4.13}$$

must satisfy $\phi_1(1) = 0$ and $C_1[\Phi] = 0$, for any functions $f_1(x), f_2(x) \in \mathbf{L}_2(0,1)$.

This requires that

$$G^N_{11}(x,y;\lambda) = \gamma_N(\lambda) \times \begin{cases} L^N_1(X)R_1(Y;\lambda) & \text{for} \quad x \leqslant y \\ R_1(X;\lambda)L^N_1(Y) & \text{for} \quad x \geqslant y \end{cases} \tag{4.14}$$

and

$$G^N_{22}(x,y;\lambda) = \gamma_N(\lambda) \times \begin{cases} L^N_2(X)R_2(Y;\lambda) & \text{for} \quad x \leqslant y \\ R_2(X;\lambda)L^N_2(Y) & \text{for} \quad x \geqslant y \end{cases} \tag{4.15}$$

with the other components, $G^N_{12}(x,y;\lambda)$ and $G^N_{21}(x,y;\lambda)$, given as in equation (4.5). These boundary conditions, as well as the fact that $(D_x - \lambda)\Phi(x) = \binom{f_1(x)}{f_2(x)}$, can be straightforwardly verified from equations (4.6) and (4.7).

This time, from equations (4.13)–(4.15), (4.5)–(4.7), one gets

$$\phi_1(x) = O(\sqrt{x}) \qquad \phi_2(x) = C^N_2[\Phi]x^{-g} + O(\sqrt{x}) \tag{4.16}$$

with

$$C_2^N[\Phi] = \frac{\pi\lambda^{1-g}}{2^{\frac{1}{2}-g}\cos(g\pi)J_{\frac{1}{2}-g}(\lambda)\Gamma\left(\frac{1}{2}-g\right)}\int_0^1 [R_1(\lambda y;\lambda)f_1(y) - R_2(\lambda y;\lambda)f_2(y)]\,\mathrm{d}y \tag{4.17}$$

for λ not a zero of $J_{\frac{1}{2}-g}(\lambda)$.

Note that $C_2^N[\Phi] \neq 0$ if the integral on the right-hand side of equation (4.17) (the same integral as the one appearing in the D-extension, equation (4.12)) is non-vanishing.

4.3. *The resolvent for a general self-adjoint extension of D_x*

For the general case, we can adjust the boundary conditions

$$\phi_1(1) = 0 \qquad \alpha C_1[\Phi] + \beta C_2[\Phi] = 0 \qquad \alpha,\beta \neq 0 \tag{4.18}$$

for

$$\Phi(x) = \int_0^1 G(x,y;\lambda)\begin{pmatrix} f_1(y) \\ f_2(y)\end{pmatrix}\mathrm{d}y \tag{4.19}$$

for any $f_1(x), f_2(x) \in \mathbf{L}_2(0,1)$, by taking a linear combination of the resolvent for the limiting cases,

$$G(x,y;\lambda) = [1-\tau(\lambda)]G_D(x,y;\lambda) + \tau(\lambda)G_N(x,y;\lambda). \tag{4.20}$$

Since the boundary condition at $x=1$ is automatically fulfilled, one must just impose

$$\alpha[1-\tau(\lambda)]C_1^D[\Phi] + \beta\tau(\lambda)C_2^N[\Phi] = 0. \tag{4.21}$$

Note that, in view of equations (4.12), (4.17) and (3.13),

$$\alpha C_1^D[\Phi] - \beta C_2^N[\Phi] = 0 \tag{4.22}$$

precisely when λ is an eigenvalue of $D_x^{(\alpha,\beta)}$. Therefore, from equation (4.21) we get the resolvent of $D_x^{(\alpha,\beta)}$ by setting

$$\tau(\lambda) = \frac{\alpha C_1^D[\Phi]}{\alpha C_1^D[\Phi] - \beta C_2^N[\Phi]} = \frac{1}{1-\frac{\rho(\alpha,\beta)}{F(\lambda)}} = 1 - \frac{1}{1-\frac{\lambda^{2g}}{\rho(\alpha,\beta)}\frac{J_{\frac{1}{2}-g}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)}} \tag{4.23}$$

for λ not a zero of $J_{g-\frac{1}{2}}(\lambda)$.

5. The trace of the resolvent

It follows from equation (4.20) that the resolvent of a general self-adjoint extension of D_x can be expressed in terms of the resolvents of the two limiting cases, $G_D(\lambda)$ and $G_N(\lambda)$. Moreover, since the eigenvalues of any extension grow linearly with n (see section 3), these resolvents are Hilbert–Schmidt operators and their λ-derivatives are trace class.

So, let us consider the relation

$$G^2(\lambda) = \partial_\lambda G(\lambda) = \partial_\lambda G_D(\lambda) - \tau'(\lambda)[G_D(\lambda) - G_N(\lambda)] - \tau(\lambda)[\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)] \tag{5.1}$$

from which it follows that the difference $G_D(\lambda) - G_N(\lambda)$ is a strongly analytic function of λ (except at the zeros of $\tau'(\lambda)$) taking values in the trace class operators ideal.

Since we have explicitly constructed $G_D(\lambda)$ and $G_N(\lambda)$ in the previous section (see equations (4.9), (4.10), (4.14) and (4.15)), we straightforwardly get (see appendix A for the details)

$$\begin{aligned}
\operatorname{Tr}\{\partial_\lambda G_D(\lambda)\} &= \int_0^1 \operatorname{Tr}\{\partial_\lambda G_D(x,x;\lambda)\}\,\mathrm{d}x \\
&= \partial_\lambda \left\{ \frac{J_{g+\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} \right\} = 1 - \frac{2g}{\lambda}\frac{J_{g+\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} + \frac{J^2_{g+\frac{1}{2}}(\lambda)}{J^2_{g-\frac{1}{2}}(\lambda)} \\
&= 1 - \frac{g^2}{\lambda^2} + \left(\frac{1}{2\lambda} + \frac{J'_{g-\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} \right)^2
\end{aligned} \tag{5.2}$$

where, in the last step, we have taken into account that

$$J_{\nu\pm 1}(z) = \frac{\nu}{z} J_\nu(z) \mp J'_\nu(z). \tag{5.3}$$

Similarly,

$$\begin{aligned}
\operatorname{Tr}\{G_D(\lambda) - G_N(\lambda)\} &= \frac{2g}{\lambda} + \frac{J_{g+\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} + \frac{J_{-g-\frac{1}{2}}(\lambda)}{J_{-g+\frac{1}{2}}(\lambda)} \\
&= -\frac{2g}{\lambda} - \frac{J'_{\frac{1}{2}-g}(\lambda)}{J_{\frac{1}{2}-g}(\lambda)} + \frac{J'_{g-\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)}.
\end{aligned} \tag{5.4}$$

Moreover, since

$$\partial_\lambda \operatorname{Tr}\{G_D(\lambda) - G_N(\lambda)\} = \operatorname{Tr}\{\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)\} \tag{5.5}$$

we get

$$\begin{aligned}
&\operatorname{Tr}\{\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)\} \\
&\quad = -\frac{2g}{\lambda^2} - \frac{2g}{\lambda}\left[\frac{J_{g+\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} + \frac{J_{-g-\frac{1}{2}}(\lambda)}{J_{-g+\frac{1}{2}}(\lambda)} \right] + \left[\frac{J^2_{g+\frac{1}{2}}(\lambda)}{J^2_{g-\frac{1}{2}}(\lambda)} - \frac{J^2_{-g-\frac{1}{2}}(\lambda)}{J^2_{-g+\frac{1}{2}}(\lambda)} \right] \\
&\quad = \frac{2g}{\lambda^2} + \left(\frac{1}{2\lambda} + \frac{J'_{\frac{1}{2}-g}(\lambda)}{J_{\frac{1}{2}-g}(\lambda)} \right)^2 - \left(\frac{1}{2\lambda} + \frac{J'_{g-\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} \right)^2.
\end{aligned} \tag{5.6}$$

Finally, we can also write

$$\operatorname{Tr}\{G^2(\lambda)\} = \operatorname{Tr}\{\partial_\lambda G_D(\lambda)\} - \partial_\lambda[\tau(\lambda)\operatorname{Tr}\{G_D(\lambda) - G_N(\lambda)\}]. \tag{5.7}$$

6. Asymptotic expansion for the trace of the resolvent

Using the Hankel asymptotic expansion for Bessel functions [32] (see appendix B), we get for the first term on the right-hand side of equation (5.7)

$$\begin{aligned}
\operatorname{Tr}\{\partial_\lambda G_D(\lambda)\} \sim \sum_{k=2}^{\infty} \frac{A_k(g,\sigma)}{\lambda^k} &= -\frac{g}{\lambda^2} + \mathrm{i}\sigma\frac{g(g-1)}{\lambda^3} - \frac{3}{2}\frac{g(g-1)}{\lambda^4} \\
&\quad + \mathrm{i}\sigma\frac{(g-3)(g-1)g(g+2)}{2\lambda^5} + O\left(\frac{1}{\lambda}\right)^6
\end{aligned} \tag{6.1}$$

where $\sigma = 1$ for $\mathrm{Im}(\lambda) > 0$, and $\sigma = -1$ for $\mathrm{Im}(\lambda) < 0$. The coefficients in this series can be straightforwardly evaluated from equations (B.7) and (B.17). Note that $A_k(g, -1) = A_k(g, 1)^*$, since $A_{2k}(g, 1)$ is real and $A_{2k+1}(g, 1)$ is pure imaginary.

Similarly, from equations (5.4), (5.6) and (B.20) we simply get

$$\mathrm{Tr}\{G_D(\lambda) - G_N(\lambda)\} \sim -\frac{2g}{\lambda} \tag{6.2}$$

and

$$\mathrm{Tr}\{\partial_\lambda G_D(\lambda) - \partial_\lambda G_N(\lambda)\} \sim \frac{2g}{\lambda^2}. \tag{6.3}$$

On the other hand, taking into account equation (B.9), we have

$$\begin{aligned}\tau(\lambda) &\sim 1 - \frac{1}{1 - \frac{\exp\left(\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)\lambda^{2g}}{\rho(\alpha,\beta)}} \\ &\sim \begin{cases} -\sum_{k=1}^{\infty}\left(\frac{\exp\left(\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)\lambda^{2g}}{\rho(\alpha,\beta)}\right)^k & \text{for } -\frac{1}{2} < g < 0 \\ \sum_{k=0}^{\infty}\left(\rho(\alpha,\beta)\,\exp\left(-\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)\lambda^{-2g}\right)^k & \text{for } 0 < g < \frac{1}{2} \end{cases}\end{aligned} \tag{6.4}$$

where $\sigma = 1$ ($\sigma = -1$) corresponds to $\mathrm{Im}(\lambda) > 0$ ($\mathrm{Im}(\lambda) < 0$). Note the appearance of non-integer, g-dependent, powers of λ in this asymptotic expansion.

Similarly

$$\begin{aligned}\tau'(\lambda) &\sim -\left(1 - \frac{\exp\left(\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)\lambda^{2g}}{\rho(\alpha,\beta)}\right)^{-2}\frac{\exp\left(\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)2g\lambda^{2g-1}}{\rho(\alpha,\beta)} \\ &\sim \begin{cases} -\frac{2g}{\lambda}\sum_{k=1}^{\infty} k\left(\frac{\exp\left(\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)\lambda^{2g}}{\rho(\alpha,\beta)}\right)^k & \text{for } -\frac{1}{2} < g < 0 \\ -\frac{2g}{\lambda}\sum_{k=1}^{\infty} k\left(\rho(\alpha,\beta)\,\exp\left(-\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)\lambda^{-2g}\right)^k & \text{for } 0 < g < \frac{1}{2} \end{cases}\end{aligned} \tag{6.5}$$

which are the term by term derivatives of the corresponding asymptotic series in equation (6.4).

Therefore, we have

$$\begin{aligned}&\partial_\lambda[\tau(\lambda)\,\mathrm{Tr}\{G_D(\lambda) - G_N(\lambda)\}] \\ &\sim \begin{cases} 2g\sum_{k=1}^{\infty}\left(\frac{\exp\left(\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)}{\rho(\alpha,\beta)}\right)^k (2gk-1)\lambda^{2gk-2} & \text{for } -\frac{1}{2} < g < 0 \\ 2g\sum_{k=0}^{\infty}\left(\rho(\alpha,\beta)\,\exp\left(-\sigma \mathrm{i}\pi\left(\frac{1}{2}-g\right)\right)\right)^k (2gk+1)\lambda^{-2gk-2} & \text{for } 0 < g < \frac{1}{2}. \end{cases}\end{aligned} \tag{6.6}$$

Note the g-dependent powers of λ appearing in these asymptotic expansions.

7. The ζ- and η-functions

The ζ-function for a general self-adjoint extension of D_x is defined, for $\mathrm{Re}(s) > 1$, as [8]

$$\zeta(s) = -\frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \frac{\lambda^{1-s}}{s-1}\,\mathrm{Tr}\{G^2(\lambda)\}\,\mathrm{d}\lambda \tag{7.1}$$

where the curve $\mathcal{C}$ encircles counterclockwise the spectrum of the operator, keeping to the left of the origin. According to equation (5.7), we have

$$\zeta(s) = \zeta^D(s) + \frac{1}{2\pi \mathrm{i}}\oint_{\mathcal{C}} \frac{\lambda^{1-s}}{s-1}\partial_\lambda[\tau(\lambda)\,\mathrm{Tr}\{G_D(\lambda) - G_N(\lambda)\}]\,\mathrm{d}\lambda \tag{7.2}$$

where $\zeta^D(s)$ is the ζ-function for the D-extension.

Since the negative eigenvalues of the self-adjoint extension of D_x characterized by the pair (α, β), $D_x^{(\alpha,\beta)}$, are minus the positive eigenvalues corresponding to the extension $D_x^{(\alpha,-\beta)}$ (as discussed in section 3), we define a partial ζ-function through a path of integration encircling the positive eigenvalues only,

$$\begin{aligned}\zeta_+^{(\alpha,\beta)}(s) &= \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty+0}^{\mathrm{i}\infty+0} \frac{\lambda^{1-s}}{s-1}\,\mathrm{Tr}\{G^2(\lambda)\}\,\mathrm{d}\lambda \\ &= \zeta_+^{D}(s) - \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty+0}^{\mathrm{i}\infty+0} \frac{\lambda^{1-s}}{s-1}\partial_\lambda[\tau(\lambda)\,\mathrm{Tr}\{G_D(\lambda)-G_N(\lambda)\}]\,\mathrm{d}\lambda \end{aligned} \tag{7.3}$$

where $\zeta_+^D(s)$ is the partial ζ-function for the D-extension.

We can also write

$$\begin{aligned}\zeta_+^{(\alpha,\beta)}(s) &= \frac{1}{2\pi}\int_1^\infty \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(1-s)}\frac{\mu^{1-s}}{s-1}\,\mathrm{Tr}\left\{G^2\left(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}\mu\right)\right\}\mathrm{d}\mu \\ &\quad + \frac{1}{2\pi}\int_1^\infty \exp\left(-\mathrm{i}\frac{\pi}{2}(1-s)\right)\frac{\mu^{1-s}}{s-1}\,\mathrm{Tr}\left\{G^2\left(\exp\left(-\mathrm{i}\frac{\pi}{2}\mu\right)\right)\right\}\mathrm{d}\mu + \frac{h_1(s)}{s-1}\end{aligned} \tag{7.4}$$

where $h_1(s)$ is an entire function. Therefore, in order to determine the poles of $\zeta_+^{(\alpha,\beta)}(s)$, we can subtract and add a partial sum of the asymptotic expansion obtained in the previous section to $\mathrm{Tr}\{G^2(\lambda)\}$ in the integrands on the right-hand side of equation (7.4).

In so doing, we get for the D-extension and for a real $s > 1$

$$\begin{aligned}\zeta_+^D(s) &= \frac{1}{2\pi(s-1)}\int_1^\infty \exp\left(\mathrm{i}\frac{\pi}{2}(1-s)\right)\mu^{1-s}\left\{\sum_{k=2}^{N}\exp\left(-\mathrm{i}\frac{\pi}{2}k\right)A_k(g,1)\mu^{-k}\right\}\mathrm{d}\mu \\ &\quad + \frac{1}{2\pi(s-1)}\int_1^\infty \exp\left(-\mathrm{i}\frac{\pi}{2}(1-s)\right)\mu^{1-s}\left\{\sum_{k=2}^{N}\exp\left(\mathrm{i}\frac{\pi}{2}k\right)A_k(g,1)^*\mu^{-k}\right\}\mathrm{d}\mu \\ &\quad + \frac{h_2(s)}{s-1} = \frac{1}{\pi(s-1)}\sum_{k=2}^{N}\frac{1}{s-(2-k)}\mathrm{Re}\left\{\exp\left(\mathrm{i}\frac{\pi}{2}(1-s-k)\right)A_k(g,1)\right\} + \frac{h_2(s)}{s-1}\end{aligned} \tag{7.5}$$

where $h_2(s)$ is an analytic function in the open half plane $\mathrm{Re}(s) > 1-N$.

Consequently, the meromorphic extension of $\zeta_+^D(s)$ presents a simple pole at $s = 1$ (see equation (7.3)), with a residue given by (see equation (5.2))

$$\mathrm{Res}\,\zeta_+^D(s)|_{s=1} = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty+0}^{\mathrm{i}\infty+0}\lambda^0\partial_\lambda\left\{\frac{J_{g+\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)}\right\}\mathrm{d}\lambda = \frac{1}{\pi} \tag{7.6}$$

where we have used equations (B.10) and (B.11).

It also presents simple poles at $s = 2-k$, for $k = 2, 3, \ldots$, with residues given by

$$\mathrm{Res}\,\zeta_+^D(s)|_{s=2-k} = \frac{\mathrm{Re}\{\mathrm{i}A_k(g,1)\}}{(k-1)\pi} \tag{7.7}$$

with the coefficients $A_k(g,1)$ given in equation (6.1). In particular, note that these residues vanish for even k.

For a general self-adjoint extension $D_x^{(\alpha,\beta)}$, we must also consider the singularities coming from the asymptotic expansion of $\partial_\lambda[\tau(\lambda)\,\mathrm{Tr}\{G_D(\lambda)-G_N(\lambda)\}]$ in equation (6.6). For definiteness, let us consider in the following the case $-\frac{1}{2} < g < 0$ (the case $0 < g < \frac{1}{2}$ leads to similar results).

From equation (7.3), and taking into account equation (7.4), for real $s > 1$ we can write

$$\zeta_+^{(\alpha,\beta)}(s) - \zeta_+^D(s) = \frac{h_3(s)}{s-1} - \frac{g}{\pi(s-1)} \int_1^\infty \exp\left(\mathrm{i}\frac{\pi}{2}(-s-1)\right)\mu^{1-s}$$
$$\times \left\{\sum_{k=1}^N \left(\frac{\exp\left(\mathrm{i}\frac{\pi}{2}\right)}{\rho(\alpha,\beta)}\right)^k (2gk-1)\mu^{2gk-2}\right\} \mathrm{d}\mu - \frac{g}{\pi(s-1)}$$
$$\times \int_1^\infty \exp\left(-\mathrm{i}\frac{\pi}{2}(-s-1)\right)\mu^{1-s} \left\{\sum_{k=1}^N \left(\frac{\exp\left(-\mathrm{i}\frac{\pi}{2}\right)}{\rho(\alpha,\beta)}\right)^k (2gk-1)\mu^{2gk-2}\right\} \mathrm{d}\mu$$
$$= \frac{-2g}{\pi(s-1)} \sum_{k=1}^N \left(\frac{2gk-1}{s-2gk}\right) \mathrm{Re}\left\{\frac{\exp\left(\mathrm{i}\frac{\pi}{2}(k-s-1)\right)}{\rho(\alpha,\beta)^k}\right\} + \frac{h_3(s)}{s-1} \tag{7.8}$$

where $h_3(s)$ is holomorphic for $\mathrm{Re}(s) > 2g(N+1)$.

Therefore, $\left(\zeta_+^{(\alpha,\beta)}(s) - \zeta_+^D(s)\right)$ has a meromorphic extension which presents a simple pole at $s = 1$, with a vanishing residue,

$$\mathrm{Res}\left(\zeta_+^{(\alpha,\beta)}(s) - \zeta_+^D(s)\right)\Big|_{s=1} = -\frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty+0}^{\mathrm{i}\infty+0} \lambda^0 \partial_\lambda[\tau(\lambda)\,\mathrm{Tr}\{G_D(\lambda) - G_N(\lambda)\}]\,\mathrm{d}\lambda = 0 \tag{7.9}$$

as follows from equations (6.2) and (6.4).

Note also the presence of simple poles located at negative non-integer, g-dependent positions, $s = 2gk = -2|g|k$, for $k = 1, 2, \ldots$, with residues which also depend on the self-adjoint extension, given by

$$\mathrm{Res}\left\{\zeta_+^{(\alpha,\beta)}(s) - \zeta_+^D(s)\right\}\Big|_{s=2gk} = \frac{-2g}{\pi\rho(\alpha,\beta)^k} \sin\left[\left(\frac{1}{2} - g\right)k\pi\right]. \tag{7.10}$$

Now, taking into account our comment after equation (3.14), we get for the complete ζ-function

$$\zeta^{(\alpha,\beta)}(s) = \zeta_+^{(\alpha,\beta)}(s) + \mathrm{e}^{-\mathrm{i}\pi s}\zeta_+^{(\alpha,-\beta)}(s). \tag{7.11}$$

In particular, for the $\alpha = 0$ extension we get

$$\zeta^D(s) = (1 + \mathrm{e}^{-\mathrm{i}\pi s})\zeta_+^D(s) \tag{7.12}$$

since the spectrum of $D_x^{(0,1)}$ is symmetric with respect to the origin (see equation (3.8)). Then one concludes that $\zeta^D(s)$ has vanishing residues. Indeed, from equation (7.7), the residue at $s = 2 - k$ vanishes for k even, and for $k = 2l + 1$, with $l = 0, 1, 2, \ldots$, we have

$$\mathrm{Res}\{\zeta^D(s)\}|_{s=1-2l} = (1 + \mathrm{e}^{-\mathrm{i}\pi(1-2l)})\,\mathrm{Res}\left\{\zeta_+^D(s)\right\}\Big|_{s=1-2l} = 0. \tag{7.13}$$

On the other hand, for a general self-adjoint extension, the singularities of $\zeta^{(\alpha,\beta)}(s)$ are simple poles located at $s = 2gk < 0$, for $k = 1, 2, \ldots$, with residues

$$\mathrm{Res}\{\zeta^{(\alpha,\beta)}(s) - \zeta^D(s)\}|_{s=2gk}$$
$$= \mathrm{Res}\left\{\left[\zeta_+^{(\alpha,\beta)}(s) - \zeta_+^D(s)\right] + \mathrm{e}^{-\mathrm{i}\pi s}\left[\zeta_+^{(\alpha,-\beta)}(s) - \zeta_+^D(s)\right]\right\}\Big|_{s=2gk}$$
$$= (-1)^k \frac{2g}{\pi} \frac{\sin(2gk\pi)}{\rho(\alpha,\beta)^k} \exp\left(\mathrm{i}\pi\left(\frac{1}{2} - g\right)k\right) \tag{7.14}$$

where we have used $\rho(\alpha,-\beta) = -\rho(\alpha,\beta)$, from equation (3.12).

Similarly, for the spectral asymmetry [31] we have

$$\eta^{(\alpha,\beta)}(s) = \zeta_+^{(\alpha,\beta)}(s) - \zeta_+^{(\alpha,-\beta)}(s). \tag{7.15}$$

In particular, $\eta^{(0,1)}(s) \equiv 0 \equiv \eta^{(1,0)}(s)$, since these spectra are symmetric (see equations (3.8) and (3.15)).

For a general self-adjoint extension and $-\frac{1}{2} < g < 0$, $\eta^{(\alpha,\beta)}(s)$ presents simple poles at $s = 2gk$, for $k = 1, 2, \ldots$, with residues given by

$$\operatorname{Res}\{\eta^{(\alpha,\beta)}(s)\}|_{s=2gk} = [(-1)^k - 1]\frac{2g}{\pi}\frac{\sin\left[\left(\frac{1}{2} - g\right)k\pi\right]}{\rho(\alpha,\beta)^k} \tag{7.16}$$

which vanish for even k.

For the case $0 < g < \frac{1}{2}$, an entirely similar calculation shows that $\left(\zeta_+^{(\alpha,\beta)}(s) - \zeta_+^D(s)\right)$ has a meromorphic extension which presents simple poles at negative non-integer, g-dependent positions, $s = -2gk$, for $k = 1, 2, \ldots$, with residues depending on the self-adjoint extension, given by

$$\operatorname{Res}\left\{\zeta_+^{(\alpha,\beta)}(s) - \zeta_+^D(s)\right\}\Big|_{s=-2gk} = -\frac{2g}{\pi}\rho(\alpha,\beta)^k \sin\left[\left(\frac{1}{2} - g\right)k\pi\right]. \tag{7.17}$$

From this result, one can immediately get the residues for the η- and ζ-functions. One gets the same expressions as on the right-hand sides of equations (7.14) and (7.16), with $\rho(\alpha,\beta)$ and $\exp\left(\mathrm{i}\pi\left(\frac{1}{2} - g\right)k\right)$ replaced by their inverses.

Let us remark that when neither α nor β is 0, the residue of $\zeta_+^{(\alpha,\beta)}$ at $s = -2|g|k$ is a constant times $(\beta/\alpha)^{k\,\mathrm{sign}(g)}$. This is consistent with the behaviour of D_x under the scaling isometry $Tu(x) = c^{1/2}u(cx)$ taking $\mathbf{L}_2(0, 1) \to \mathbf{L}_2(0, 1/c)$. The extension $D_x^{(\alpha,\beta)}$ is unitarily equivalent to the operator $(1/c)\dot{D}_x^{(\alpha',\beta')}$ similarly defined on $\mathbf{L}_2(0, 1/c)$, with $\alpha' = c^{-g}\alpha$ and $\beta' = c^g\beta$:

$$TD_x^{(\alpha,\beta)} = \frac{1}{c}\dot{D}_x^{(\alpha',\beta')}T. \tag{7.18}$$

Note that only for the extensions with $\alpha = 0$ or $\beta = 0$ the boundary condition at the singular point $x = 0$, equation (2.12), is left invariant by this scaling.

Therefore, we have for the partial ζ-function of the scaled problem

$$\dot{\zeta}_+^{(\alpha',\beta')}(s) = c^{-s}\zeta_+^{(\alpha,\beta)}(s) \tag{7.19}$$

and for the residues

$$\operatorname{Res}\left\{\dot{\zeta}_+^{(\alpha',\beta')}(s)\right\}\Big|_{s=-2|g|k} = c^{2|g|k}\operatorname{Res}\left\{\zeta_+^{(\alpha,\beta)}(s)\right\}\Big|_{s=-2|g|k}. \tag{7.20}$$

The factor $c^{2|g|k}$ exactly cancels the effect the change in the boundary condition at the singularity has on $\rho(\alpha,\beta)$,

$$\rho(\alpha,\beta)^{k\,\mathrm{sign}(g)} = c^{-2|g|k}\rho(\alpha',\beta')^{k\,\mathrm{sign}(g)}. \tag{7.21}$$

Thus the difference between the intervals (0, 1) and (0, $1/c$) has no effect on the structure of these residues, which presumably are determined locally in a neighbourhood of $x = 0$.

Finally, let us point out that these anomalous poles are not present in the $g = 0$ case. Indeed, in this case $\tau(\lambda)$ in equation (4.23) has a constant asymptotic expansion, while $\operatorname{Tr}\{G_D(\lambda) - G_N(\lambda)\} \sim 0$ (see equation (6.2)). Moreover, the residues of the poles coming from $\zeta_+^D(s)$ are all zero (see equations (7.7) and (6.1)), except for the one at $s = 1$, with residue $1/\pi$ (see equation (7.6)).

Consequently, the presence of poles in the spectral functions located at non-integer positions is a consequence of the singular behaviour of the zeroth-order term in D_x near the origin, together with a boundary condition which is not invariant under scaling.

8. Comments on the second-order case

In this section we briefly describe similar results one can obtain for the self-adjoint extensions of the second-order differential operator

$$\Delta_x = -\partial_x^2 + \frac{g(g-1)}{x^2} \tag{8.1}$$

with $-\frac{1}{2} < g < \frac{1}{2}$, defined on a set of functions satisfying $\phi(1) = 0$ and behaving as

$$\phi(x) = C_1 x^g + C_2 x^{1-g} + O(x^{3/2}) \tag{8.2}$$

where the coefficients $C_{1,2}$ are constrained as in equation (2.12).

It can be shown that the spectrum of the self-adjoint extension $\Delta_x^{(\alpha,\beta)}$ is determined by a relation similar to equation (3.13):

$$\mathcal{F}(\mu) := \frac{1}{\mu} F(\mu) = \varrho(\alpha, \beta) \tag{8.3}$$

where the constant

$$\varrho(\alpha, \beta) := \left(\frac{\beta}{\alpha}\right) 2^{2g-1} \frac{\Gamma\left(\frac{1}{2} + g\right)}{\Gamma\left(\frac{3}{2} - g\right)}. \tag{8.4}$$

Also in this case, $\alpha = 0$ and $\beta = 0$ correspond to two scale invariant boundary conditions at the singularity. For these two limiting extensions, it is easily seen from equations (4.4), (4.11) and (4.16) that the entry $G_{11}(x, y; \mu)$ in the resolvent of our first-order operator $D_x^{(\alpha,\beta)}$ is μ times the corresponding resolvent of $\Delta_x^{(\alpha,\beta)}$ at $\lambda = \mu^2$,

$$\mathcal{G}_{D,N}(x, y; \mu^2) = \frac{1}{\mu} G_{11}^{D,N}(x, y; \mu). \tag{8.5}$$

The resolvent for a general self-adjoint extension $\Delta_x^{(\alpha,\beta)}$ is constructed as a convex linear combination of $\mathcal{G}_D(\mu^2)$ and $\mathcal{G}_N(\mu^2)$ as in (4.20), with a coefficient

$$\tau(\mu) = \frac{1}{1 - \frac{\varrho(\alpha,\beta)}{\mathcal{F}(\mu)}}. \tag{8.6}$$

Following the methods employed for the first-order case, one can show that the ζ-function associated with $\Delta_x^{(\alpha,\beta)}$ also displays anomalous poles located at $s = -\left(\frac{1}{2} - g\right)k$, with $k = 1, 2, \ldots$, which implies the presence of anomalous powers $t^{(\frac{1}{2}-g)k}$ in the heat trace small-t asymptotic expansion. The residues at these poles, and the corresponding heat trace coefficients are similarly evaluated. More details on this calculation will be reported elsewhere.

Note added in Proof. It has come belatedly to our attention that the article by Edith A Mooers, 1999 Heat kernel asymptotics on manifolds with conic singularities, *J. Anal. Math.* **78** 1–36, gives the first 'unusual' term in the expansion of the Laplacian on the half line with a domain which is not scaling invariant. That article also gives a construction which in principle would give the complete expansion in the case of a manifold with isolated conic singularities, for an arbitrary self-adjoint realization of the Laplacian. For the case considered here, by contrast, the present results are simpler and more complete, as they treat the first-order case and the eta invariant, and give more explicit coefficients.

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Appendix A. Evaluation of the traces

In this appendix, we briefly describe the evaluation of the traces appearing in section 5.

We need to compute

$$\operatorname{Tr}\{G_D(\lambda) - G_N(\lambda)\} = \int_0^1 [\operatorname{Tr}\{G_D(x, x; \lambda)\} - \operatorname{Tr}\{G_N(x, x; \lambda)\}]\,\mathrm{d}x. \qquad \text{(A.1)}$$

Let us first consider the contribution of $G_N(\lambda)$ to this integral. From equations (4.14) and (4.15) we get for the matrix trace of $G_N(\lambda)$ on the diagonal

$$\begin{aligned}\operatorname{Tr}\{G_N(x, x; \lambda)\} = -\frac{\pi x\lambda \sec(g\pi)}{2J_{\frac{1}{2}-g}(\lambda)} \big\{ & J_{-\frac{1}{2}-g}(x\lambda)^2 J_{-\frac{1}{2}+g}(\lambda) \\ & + J_{\frac{1}{2}-g}(x\lambda)\big[J_{\frac{1}{2}-g}(x\lambda)J_{-\frac{1}{2}+g}(\lambda) - J_{\frac{1}{2}-g}(\lambda)J_{-\frac{1}{2}+g}(x\lambda)\big] \\ & + J_{-\frac{1}{2}-g}(x\lambda)J_{\frac{1}{2}-g}(\lambda)J_{\frac{1}{2}+g}(x\lambda)\big\}\end{aligned} \qquad \text{(A.2)}$$

an integrable expression behaving as

$$\operatorname{Tr}\{G_N(x, x; \lambda)\} = x^{-2g}\left\{-\left(\frac{4^g\pi\,\sec(g\pi)J_{g-\frac{1}{2}}(\lambda)}{\lambda^{2g}\Gamma\left(\frac{1}{2}-g\right)^2 J_{\frac{1}{2}-g}(\lambda)}\right) + O(x)\right\} + O(x) \qquad \text{(A.3)}$$

near the origin.

Therefore, it is sufficient to know the primitives [33]

$$\int x J_\nu^2(\lambda x)\,\mathrm{d}x = \frac{x^2}{2}\{J_\nu(x\lambda)^2 - J_{\nu-1}(x\lambda)J_{\nu+1}(x\lambda)\} \qquad \text{(A.4)}$$

and

$$\int x J_\nu(\lambda x)J_{-\nu}(\lambda x)\,\mathrm{d}x = \frac{-\nu^2}{\lambda^2\Gamma(1-\nu)\Gamma(1+\nu)}[{}_1F_2(\{-1/2\}, \{-\nu, \nu\}, -x^2\lambda^2) - 1] \qquad \text{(A.5)}$$

where

$$\begin{aligned}{}_1F_2(\{-1/2\}, \{-\nu, \nu\}, -x^2\lambda^2) = -\frac{\pi x^2\lambda^2 \operatorname{cosec}(\pi\nu)}{4\nu}\{ & J_{-1-\nu}(x\lambda)J_{-1+\nu}(x\lambda) \\ & + 2J_{-\nu}(x\lambda)J_\nu(x\lambda) + J_{1-\nu}(x\lambda)J_{1+\nu}(x\lambda)\}.\end{aligned} \qquad \text{(A.6)}$$

These primitives, together with the relation

$$J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z}J_\nu(z) \qquad \text{(A.7)}$$

necessary to simplify the intermediate results, eventually lead to

$$I_N(\lambda) := \int_0^1 \operatorname{Tr}\{G_N(x, x; \lambda)\}\,\mathrm{d}x = -\frac{2g}{\lambda} - \frac{J_{-g-\frac{1}{2}}(\lambda)}{J_{-g+\frac{1}{2}}(\lambda)}. \qquad \text{(A.8)}$$

Similarly, for the matrix trace of $G_D(\lambda)$ on the diagonal we have

$$\begin{aligned}\operatorname{Tr}\{G_D(x, x; \lambda)\} = \frac{\pi x\lambda \sec(g\pi)}{2J_{-\frac{1}{2}+g}(\lambda)}\big\{ & -J_{\frac{1}{2}-g}(x\lambda)J_{-\frac{1}{2}+g}(\lambda)J_{-\frac{1}{2}+g}(x\lambda) \\ & + J_{-\frac{1}{2}-g}(x\lambda)J_{-\frac{1}{2}+g}(\lambda)J_{\frac{1}{2}+g}(x\lambda) + J_{\frac{1}{2}-g}(\lambda)\big(J_{-\frac{1}{2}+g}(x\lambda)^2 + J_{\frac{1}{2}+g}(x\lambda)^2\big)\big\}\end{aligned} \qquad \text{(A.9)}$$

which behaves as

$$\mathrm{Tr}\{G_D(x,x;\lambda)\} = x^{2g}\left\{\frac{\pi\lambda^{2g}\sec(g\pi)J_{\frac{1}{2}-g}(\lambda)}{4^g\Gamma\left(\frac{1}{2}+g\right)^2 J_{g-\frac{1}{2}}(\lambda)} + O(x)\right\} + O(x). \tag{A.10}$$

The same argument as before leads to

$$I_D(\lambda) := \int_0^1 \mathrm{Tr}\{G_D(x,x;\lambda)\}\,\mathrm{d}x = \frac{J_{g+\frac{1}{2}}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)}. \tag{A.11}$$

Therefore, we get

$$\mathrm{Tr}\{G_D(\lambda) - G_N(\lambda)\} = I_D(\lambda) - I_N(\lambda) \tag{A.12}$$

as in equation (5.4).

On the other hand, we have

$$\partial_\lambda \mathrm{Tr}\{G_D(x,x;\lambda)\} = O(x) + x^{2g}\left\{\frac{2^{1-2g}\lambda^{-1+2g}\left[1 + g\pi J_{\frac{1}{2}-g}(\lambda)J_{g-\frac{1}{2}}(\lambda)\sec(g\pi)\right]}{J_{g-\frac{1}{2}}(\lambda)^2\Gamma\left(\frac{1}{2}+g\right)^2} + O(x)\right\}. \tag{A.13}$$

Then,

$$\mathrm{Tr}\{\partial_\lambda G_D(\lambda)\} = \int_0^1 \partial_\lambda\,\mathrm{Tr}\{G_D(x,x;\lambda)\}\,\mathrm{d}x = \partial_\lambda I_D(\lambda) \tag{A.14}$$

in agreement with equation (5.2).

Appendix B. The Hankel expansion

In order to develop an asymptotic expansion for the trace of the resolvent, we use the Hankel asymptotic expansion for the Bessel functions: for $|z| \to \infty$, with ν fixed and $|\arg z| < \pi$, we have [32]

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\{P(\nu,z)\cos\chi(\nu,z) - Q(\nu,z)\sin\chi(\nu,z)\} \tag{B.1}$$

where

$$\chi(\nu,z) = z - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi \tag{B.2}$$

$$P(\nu,z) \sim \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(\frac{1}{2}+\nu+2k\right)}{(2k)!\Gamma\left(\frac{1}{2}+\nu-2k\right)}\frac{1}{(2z)^{2k}} \tag{B.3}$$

and

$$Q(\nu,z) \sim \sum_{k=0}^{\infty}\frac{(-1)^k\Gamma\left(\frac{1}{2}+\nu+2k+1\right)}{(2k+1)!\Gamma\left(\frac{1}{2}+\nu-2k-1\right)}\frac{1}{(2z)^{2k+1}}. \tag{B.4}$$

Moreover, $P(-\nu,z) = P(\nu,z)$ and $Q(-\nu,z) = Q(\nu,z)$, since these functions depend only on ν^2 (see [32], p 364).

Therefore, for z in the upper open half plane,

$$J_\nu(z) \sim \frac{\mathrm{e}^{-\mathrm{i}z}\,\mathrm{e}^{\mathrm{i}\pi\left(\frac{\nu}{2}+\frac{1}{4}\right)}}{\sqrt{2\pi z}}\{P(\nu,z) - \mathrm{i}Q(\nu,z)\} \tag{B.5}$$

while for z in the lower open half plane we get

$$J_\nu(z) \sim \frac{\mathrm{e}^{\mathrm{i}z}\,\mathrm{e}^{-\mathrm{i}\pi(\frac{\nu}{2}+\frac{1}{4})}}{\sqrt{2\pi z}}\{P(\nu,z)+\mathrm{i}Q(\nu,z)\}. \tag{B.6}$$

In these equations,

$$P(\nu,z)\pm\mathrm{i}Q(\nu,z) \sim \sum_{k=0}^{\infty}\langle\nu,k\rangle\left(\frac{\pm\mathrm{i}}{2z}\right)^k \tag{B.7}$$

where the coefficients

$$\langle\nu,k\rangle = \frac{\Gamma\left(\frac{1}{2}+\nu+k\right)}{k!\Gamma\left(\frac{1}{2}+\nu-k\right)} = \langle-\nu,k\rangle \tag{B.8}$$

are the Hankel symbols.

In particular, the quotient

$$\frac{J_{\frac{1}{2}-g}(\lambda)}{J_{g-\frac{1}{2}}(\lambda)} \sim \mathrm{e}^{\pm\mathrm{i}\pi(\frac{1}{2}-g)}\frac{P\left(\frac{1}{2}-g,\lambda\right)\mp\mathrm{i}Q\left(\frac{1}{2}-g,\lambda\right)}{P\left(g-\frac{1}{2},\lambda\right)\mp\mathrm{i}Q(g-\frac{1}{2},\lambda)} = \mathrm{e}^{\pm\mathrm{i}\pi(\frac{1}{2}-g)} \tag{B.9}$$

for $\mathrm{Im}(\lambda) > 0$ and $\mathrm{Im}(\lambda) < 0$, respectively, since $P(\nu,z)$ and $Q(\nu,z)$ are even in ν.

For the quotient of two Bessel functions we have

$$\frac{J_{\nu_1}(z)}{J_{\nu_2}(z)} \sim \mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}(\nu_1-\nu_2)}\frac{P(\nu_1,z)\mp\mathrm{i}Q(\nu_1,z)}{P(\nu_2,z)\mp\mathrm{i}Q(\nu_2,z)} \tag{B.10}$$

where the upper sign is valid for $\mathrm{Im}(\lambda) > 0$, and the lower one for $\mathrm{Im}(\lambda) < 0$. The coefficients of these asymptotic expansions can be easily obtained, to any order, from equation (B.7),

$$\frac{P(\nu_1,z)\pm\mathrm{i}Q(\nu_1,z)}{P(\nu_2,z)\pm\mathrm{i}Q(\nu_2,z)} \sim 1+(\langle\nu_1,1\rangle-\langle\nu_2,1\rangle)\left(\frac{\pm\mathrm{i}}{2z}\right)+O\left(\frac{1}{z^2}\right). \tag{B.11}$$

Similarly, the derivative of the Bessel function has the following asymptotic expansion [32] for $|\arg z| < \pi$:

$$J_\nu'(z) \sim -\frac{2}{\sqrt{2\pi z}}\{R(\nu,z)\sin\chi(\nu,z)+S(\nu,z)\cos\chi(\nu,z)\} \tag{B.12}$$

where

$$R(\nu,z) \sim \sum_{k=0}^{\infty}(-1)^k\frac{\nu^2+(2k)^2-1/4}{\nu^2-(2k-1/2)^2}\frac{\langle\nu,2k\rangle}{(2z)^{2k}} \tag{B.13}$$

and

$$S(\nu,z) \sim \sum_{k=0}^{\infty}(-1)^k\frac{\nu^2+(2k+1)^2-1/4}{\nu^2-(2k+1-1/2)^2}\frac{\langle\nu,2k+1\rangle}{(2z)^{2k+1}}. \tag{B.14}$$

Then,

$$J_\nu'(z) \sim \mp\mathrm{i}\frac{\mathrm{e}^{\mp\mathrm{i}z}\,\mathrm{e}^{\pm\mathrm{i}\pi(\frac{\nu}{2}+\frac{1}{4})}}{\sqrt{2\pi z}}\{R(\nu,z)\mp\mathrm{i}S(\nu,z)\} \tag{B.15}$$

where the upper sign is valid for $\mathrm{Im}(\lambda) > 0$, and the lower one for $\mathrm{Im}(\lambda) < 0$. We have also

$$R(\nu,z)\pm\mathrm{i}S(\nu,z) = P(\nu,z)\pm\mathrm{i}Q(\nu,z)+T_\pm(\nu,z) \tag{B.16}$$

with

$$T_\pm(\nu,z) \sim \sum_{k=1}^{\infty}(2k-1)\langle\nu,k-1\rangle\left(\frac{\pm\mathrm{i}}{2z}\right)^k. \tag{B.17}$$

Therefore, we get

$$\frac{J'_\nu(z)}{J_\nu(z)} \sim \mp\mathrm{i}\left\{1 + \frac{T_\mp(\nu, z)}{P(\nu, z) \mp \mathrm{i}Q(\nu, z)}\right\} \tag{B.18}$$

where the upper sign is valid for $\mathrm{Im}(\lambda) > 0$, and the lower one for $\mathrm{Im}(\lambda) < 0$. The coefficients of the asymptotic expansion in the right-hand side of equation (B.18) can be easily obtained from equations (B.7) and (B.17),

$$\frac{T_\pm(\nu, z)}{P(\nu, z) \pm \mathrm{i}Q(\nu, z)} = \left(\frac{\pm\mathrm{i}}{2z}\right) + O\left(\frac{1}{z^2}\right). \tag{B.19}$$

Finally, since the Hankel symbols are even in ν (see equation (B.8)), from equations (B.7), (B.17) and (B.18) we have

$$\frac{J'_\nu(z)}{J_\nu(z)} \sim \frac{J'_{-\nu}(z)}{J_{-\nu}(z)}. \tag{B.20}$$

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