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From the hypergeometric differential equation to a non-linear Schrödinger one

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ABSTRACT

We show that the q-exponential function is a hypergeometric function. Accordingly, it obeys the hypergeometric differential equation. We demonstrate that this differential equation can be transformed into a non-linear Schrödinger equation (NLSE). This NLSE exhibits both similarities and differences vis-a-vis the Nobre–Rego-Monteiro–Tsallis one.

1. Introduction

In 2011, Nobre, Rego-Monteiro and Tsallis (NRT) [1–5] introduced an intriguing new version of the nonlinear Schrödinger equation (NLSE), an interesting proposal that one may regard as part of a project to explore non-linear versions of some of the fundamental equations of physics, a research venue actively visited in recent times [6,7]. Earlier non-linear versions of the SE have found application in diverse areas (fiber optics and water waves, for instance) [7]. A most studied NLSE involves a cubic nonlinearity in the wave function. In quantum settings the NLSE usually rules the behavior of a single-particle's wave function that, in turn, provides an effective, mean-field description of a quantum many-body system. An important case is the Gross–Pitaevskii equation, employed in researching Bose–Einstein condensates [8]. The cubic nonlinear term appearing in the Gross–Pitaevskii equation describes short-range interactions between the condensate's constituents. The NLSE for the system's (effective) single-particle wave function is found assuming a Hartree–Fock-like form for the global many-body wave function, with a Dirac's delta form for the inter-particle potential.

The NRT equation derives from the thermo-statistical formalism based upon the Tsallis S_q non-additive, power-law information measure. Applications of the functional S_q involve diverse physical systems and processes, having attracted much attention in the last 20 years (see, for example, [9–17], and references therein). In particular, the S_q entropy has proved to be useful for the analysis of diverse problems in quantum physics [18–26].

In this paper we traverse a totally different road. We start from the differential equation that governs hypergeometric functions and derive from it a new NLSE that is different from, but exhibits some similarities with, the NRT.

2. A new non-linear Schrödinger equation

The q-exponential e_q is defined as $e_q(x) = [1+(q-1)x]_+^{\frac{1}{1-q}}$, that is,

$$\begin{aligned} e_q(x) &= [1+(q-1)x]_+^{\frac{1}{1-q}} \\ &= [1+(q-1)x]^{\frac{1}{1-q}} \text{ if } 1+(q-1)x > 0 \\ e_q(x) &= 0, \quad \text{otherwise} \quad (\text{with } q \in \mathcal{R}). \end{aligned} \tag{2.1}$$

A search in [27] reveals that

$$F(-\alpha, \gamma; \gamma; -z) = (1+z)^{\alpha}, \tag{2.2}$$

which yields for $e_q[(i/\hbar)(px - Et)] \equiv e_q(Y)$ the relation (with $E = \frac{p^2}{2m}$)

$$\begin{aligned} &\left[1 + \frac{i}{\hbar}(1-q)(px - Et)\right]^{\frac{1}{1-q}} \\ &\quad = F\left[\frac{1}{q-1}, \gamma; \gamma; \frac{i}{\hbar}(q-1)(px - Et)\right], \end{aligned} \tag{2.3}$$

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which is a fundamental result for us. We consider below derivatives F' and F'' of F with respect to Y.

Now, according to [28], the hypergeometric function obeys the following, differential equation (primes denote derivatives with respect to Y)

$$z(1-z)F''(\alpha,\beta;\gamma;z)+ \\ +[\gamma-(\alpha+\beta+1)z]F'(\alpha,\beta;\gamma;z)-\alpha\beta F(\alpha,\beta;\gamma;z)=0, \tag{2.4}$$

so that, specializing things for our instance (2.3) we encounter

$$\frac{i}{\hbar}(q-1)(px-Et)\left[1-\frac{i}{\hbar}(q-1)(px-Et)\right] \\ \times F''\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ +\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{i}{\hbar}(q-1)(px-Et)\right] \\ \times F'\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ -\frac{\gamma}{q-1}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]=0. \tag{2.5}$$

This allows for a relation between the derivative with respect to the argument and the partial derivative with respect to time, for this hypergeometric function

$$F'\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ =\frac{i\hbar}{(q-1)E}\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]. \tag{2.6}$$

In analogous fashion we obtain, for the second partial derivative with respect to the position

$$F''\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ =-\frac{\hbar^2}{(q-1)^2p^2}\frac{\partial^2}{\partial x^2}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]. \tag{2.7}$$

Replacing (2.6) and (2.7) into (2.5), this last equation adopts the appearance

$$-\frac{i}{\hbar}(q-1)(px-Et)\left[1-\frac{i}{\hbar}(q-1)(px-Et)\right] \\ \times\frac{\hbar^2}{(q-1)^2p^2}\frac{\partial^2}{\partial x^2}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ +\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{i}{\hbar}(q-1)(px-Et)\right] \\ \times\frac{i\hbar}{(q-1)E}\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ -\frac{\gamma}{q-1}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]=0, \tag{2.8}$$

that can be recast in the fashion

$$-\frac{i}{\hbar}(q-1)(px-Et)\left[1-\frac{i}{\hbar}(q-1)(px-Et)\right] \\ \times\frac{\hbar^2}{(q-1)m^2}\frac{\partial^2}{\partial x^2}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ +\left[\gamma-\left(\frac{1}{q-1}+\gamma+1\right)\frac{i}{\hbar}(q-1)(px-Et)\right] \\ \times i\hbar\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ -\gamma EF\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]=0. \tag{2.9}$$

Deriving (2.3) with respect to time we obtain:

$$-\gamma EF\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right] \\ =-i\hbar\gamma\left\{F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]\right\}^{(1-q)} \\ \times\frac{\partial}{\partial t}F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]. \tag{2.10}$$

For simplicity, let us abbreviate

$$F\equiv F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right]. \tag{2.11}$$

Using now (2.10), Eq. (2.9) becomes

$$-\frac{\hbar^2}{2m(q-1)}\left[1-F^{(1-q)}\right]F^{(1-q)}\frac{\partial^2}{\partial x^2}F \\ +i\hbar\left\{\gamma+\left(\frac{1}{q-1}+\gamma+1\right)\left[F^{(1-q)}-1\right]\right\}\frac{\partial}{\partial t}F \\ -i\hbar\gamma F^{(1-q)}\frac{\partial}{\partial t}F=0. \tag{2.12}$$

Simplifying things in this last relation we arrive at

$$-\frac{\hbar^2}{2m}F^{(1-q)}\frac{\partial^2}{\partial x^2}F-i\hbar q\frac{\partial}{\partial t}F=0, \tag{2.13}$$

that can be rewritten as

$$i\hbar q\frac{\partial}{\partial t}F=F^{(1-q)}H_0F, \tag{2.14}$$

where H_0 is the free particle Hamiltonian, note that, for $q=1$, one reobtains Schrödinger's free particle equation. Now, if instead of (2.3) we deal just with

$$F(x,t)=A\left[1+\frac{i}{\hbar}(1-q)(px-Et)\right]^{\frac{1}{1-q}}, \tag{2.15}$$

then $F(0,0)=A$ and (2.14) becomes

$$i\hbar q\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]=\left[\frac{F(x,t)}{F(0,0)}\right]^{(1-q)}H_0\left[\frac{F(x,t)}{F(0,0)}\right], \tag{2.16}$$

or, equivalently,

$$i\hbar q\left[\frac{F(x,t)}{F(0,0)}\right]^{(q-1)}\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]=H_0\left[\frac{F(x,t)}{F(0,0)}\right], \tag{2.17}$$

that, in turn can be recast as

$$i\hbar\frac{\partial}{\partial t}\left[\frac{F(x,t)}{F(0,0)}\right]^q=H_0\left[\frac{F(x,t)}{F(0,0)}\right]. \tag{2.18}$$

At this stage we realize that this last equation could be 'generalized' to any Hamiltonian H as

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\psi(x,t)}{\psi(0,0)}\right]^q=H\left[\frac{\psi(x,t)}{\psi(0,0)}\right]. \tag{2.19}$$

With the change of variables $[\psi(x,t)]^q=\phi(x,t)$, Eq. (2.19) takes the form

$$i\hbar\frac{\partial}{\partial t}\left[\frac{\phi(x,t)}{\phi(0,0)}\right]=H\left[\frac{\phi(x,t)}{\phi(0,0)}\right]^{\frac{1}{q}}, \tag{2.20}$$

which trivially reduces to the ordinary Schrödinger equation for $q=1$.

3. Separation of variables and free particle case

Consider now Eq. (2.20) for a time-independent H. A separable situation ensues. Let

$$\frac{\psi(x,t)}{\psi(0,0)}=\frac{f(t)}{f(0)}\frac{g(x)}{g(0)}. \tag{3.1}$$

Then, (2.20) becomes

$$i\hbar\left[\frac{g(x)}{g(0)}\right]^q\frac{d}{dt}\left[\frac{f(t)}{f(0)}\right]^q=\left[\frac{f(t)}{f(0)}\right]H\left[\frac{g(x)}{g(0)}\right]. \tag{3.2}$$

Rewrite (3.2) as

$$i\hbar\left[\frac{f(t)}{f(0)}\right]^{-1}\frac{d}{dt}\left[\frac{f(t)}{f(0)}\right]^q=\left[\frac{g(x)}{g(0)}\right]^{-q}H\left[\frac{g(x)}{g(0)}\right]=\lambda, \tag{3.3}$$

from which we gather that $\lambda = constant$ and

$$i\hbar\frac{d}{dt}\left[\frac{f(t)}{f(0)}\right]^q=\lambda\left[\frac{f(t)}{f(0)}\right], \tag{3.4}$$

$$H\left[\frac{g(x)}{g(0)}\right]=\lambda\left[\frac{g(x)}{g(0)}\right]^q. \tag{3.5}$$

Indeed, spatial and temporal variables have been decoupled. Pass now to the free-particle case. Set in (3.4)–(3.5) $\psi(0,0)=1$. We have

$$i\hbar\frac{d}{dt}[f(t)]^q=\lambda f(t) \tag{3.6}$$

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}g(x)=\lambda[g(x)]^q \tag{3.7}$$

It is straightforward to ascertain that a possible solution is

$$E=\frac{p^2}{2m}=\lambda \tag{3.8}$$

$$f(t)=\left[1+\frac{i}{\hbar}\frac{(1-q)}{q}Et\right]^{\frac{1}{q-1}} \tag{3.9}$$

$$g(x)=\left[1+\frac{i}{\hbar}\frac{(1-q)}{\sqrt{2(q+1)}}px\right]^{\frac{2}{1-q}} \tag{3.10}$$

We have two free-particle solutions. For $q\to 1$ both solutions tend to the usual solution for the habitual Schrödinger equation.

4. The NRT equation

For comparison purposes, we remember that NRT equation reads [1,2]

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\psi(\vec{x},t)}{\psi(0,0)}\right]=H_0\left[\frac{\psi(\vec{x},t)}{\psi(0,0)}\right]^{2-q}. \tag{4.1}$$

We introduce here the change of variables $\phi=\psi^{2-q}$ to obtain

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right]^{\frac{1}{2-q}}=H_0\left[\frac{\phi(\vec{x},t)}{\phi(0,0)}\right]. \tag{4.2}$$

We thus see that our present equation is, for the free particle case, equivalent to the NRT equation.

The last NRT equation is amenable of generalization. Thus, for an arbitrary Hamiltonian H one may write

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\phi(x,t)}{\phi(0,0)}\right]^{\frac{1}{2-q}}=H\left[\frac{\phi(x,t)}{\phi(0,0)}\right], \tag{4.3}$$

so that

$$i\hbar(2-q)\frac{\partial}{\partial t}\left[\frac{\psi(x,t)}{\psi(0,0)}\right]=H\left[\frac{\psi(x,t)}{\psi(0,0)}\right]^{2-q}. \tag{4.4}$$

We show next that NRT-separation of variables is feasible.

5. NRT separation of variables

We show that, for a time independent H, (4.4) can be separated. For this set

$$\frac{\psi(x,t)}{\psi(0,0)}=\frac{f(t)}{f(0)}\frac{g(x)}{g(0)}, \tag{5.1}$$

so that (4.4) becomes

$$i\hbar(2-q)\left[\frac{g(x)}{g(0)}\right]\frac{d}{dt}\left[\frac{f(t)}{f(0)}\right]=\left[\frac{f(t)}{f(0)}\right]^{2-q}H\left[\frac{g(x)}{g(0)}\right]^{2-q}, \tag{5.2}$$

that can be rewritten as

$$i\hbar\left[\frac{f(t)}{f(0)}\right]^{q-2}\frac{d}{dt}\left[\frac{f(t)}{f(0)}\right]=\left[\frac{g(x)}{g(0)}\right]^{-1}H\left[\frac{g(x)}{g(0)}\right]^{2-q}=\lambda, \tag{5.3}$$

with $\lambda = constant$. Accordingly:

$$i\hbar(2-q)\frac{d}{dt}\left[\frac{f(t)}{f(0)}\right]=\lambda\left[\frac{f(t)}{f(0)}\right]^{2-q}, \tag{5.4}$$

$$H\left[\frac{g(x)}{g(0)}\right]^{2-q}=\lambda\left[\frac{g(x)}{g(0)}\right]. \tag{5.5}$$

In the free particle case one has

$$i\hbar(2-q)\frac{d}{dt}f(t)=\lambda[f(t)]^{2-q}, \tag{5.6}$$

$$-\frac{\hbar^2}{2m}\frac{d}{dx^2}[g(x)]^{2-q}=\lambda g(x), \tag{5.7}$$

with solution

$$E=\frac{p^2}{2m}=\lambda, \tag{5.8}$$

$$f(t)=\left[1+\frac{i}{\hbar}\frac{(1-q)}{2-q}Et\right]^{\frac{1}{q-1}}, \tag{5.9}$$

$$g(x)=\left[1+\frac{i}{\hbar}\frac{(1-q)}{\sqrt{2(2-q)(3-q)}}px\right]^{\frac{2}{1-q}}. \tag{5.10}$$

This solution does NOT coincide with the solution obtained in Section 3, except for the case $q\to 1$.

Note that a similar treatment has been made in reference [31], in which the authors employ a constant quantity ϵ, defined in their Eq. (16), that relates to our energy E as

$$\epsilon=\frac{E}{2-q}=\frac{p^2}{2m(2-q)}. \tag{5.11}$$

6. Conclusions

We have noticed, first of all, an important feature of the q-exponential function, namely,

$$\left[1+\frac{i}{\hbar}(1-q)(px-Et)\right]^{\frac{1}{1-q}}$$
$$=F\left[\frac{1}{q-1},\gamma;\gamma;\frac{i}{\hbar}(q-1)(px-Et)\right], \tag{6.1}$$

that it is a hypergeometric function. From such result, it is clear that the q-exponential function obeys the hypergeometric differential equation (2.4).

Now, suitably manipulating equation (2.4) we have reached a non-linear Schrödinger equation that resembles the NRT one introduced in [1]. We have seen that, for time independent Hamiltonians, separation of spatial from temporal variables ensues, as in the ordinary Schrödinger equation. We have also seen that this separation can be made in the NRT case.

It is worth mentioning that, in contexts unrelated to the present one, connections between hypergeometric functions and q-statistics' applications have been encountered. See, for examples, [29,30].

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