Comment on "Spontaneous breaking of permutation symmetry in pseudo-Hermitian quantum mechanics"

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Li and Miao [Phys. Rev. A **85**, 042110 (2012)] proposed a non-Hermitian Hamiltonian that is neither Hermitian nor PT symmetric but exhibits real eigenvalues for some values of the model parameters. In order to explain this fact, they resorted to PT-pseudo Hermiticity and to a so-called permutation symmetry. Here we show that the spectrum of this Hamiltonian can be easily analyzed in the usual way in terms of exact or broken antiunitary symmetries that appear to be more relevant than the permutation symmetry. In addition, we show why the authors' Hamiltonian and the well-known Pais-Uhlenbeck oscillator lead to the same fourth-order differential equation for the coordinates.

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Some time ago, Li and Miao [1] proposed a non-Hermitian and non-PT-symmetric Hamiltonian with real spectrum for some values of the model parameters. They argued that their model is equivalent to the Pais-Uhlenbeck oscillator and discussed the spontaneous breaking of the so-called permutation symmetry. They suggested extending the well-known η-pseudo-Hermiticity [2–5] to the case in which η is the antilinear and antiunitary operator PT (the parity-time operator).

The purpose of this comment is twofold. First, we show that this model exhibits two antiunitary symmetries [6] that explain the transition from real to complex spectra in a simple, straightforward way in terms of broken antiunitary symmetry. Second, we obtain slightly more general conditions for real eigenvalues that encompass the Hermitian and non-Hermitian regimes. As a by-product, we show that many properties of Hamiltonians that are quadratic functions of the coordinates and momenta can be derived by means of an algebraic method [7,8] in a simple and straightforward way.

The oscillator proposed by Li and Miao [1] is

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + a_1^2 x_1^2 + a_2^2 x_2^2\right) + b p_1 p_2,$$
$$b = \frac{i a_3}{2 a_1 a_2}, \tag{1}$$

where x_i and p_i, $i = 1,2$ are the coordinates and conjugate momenta, respectively, and the model parameters a_i, $i = 1,2,3$ are real. Since this Hamiltonian is neither Hermitian nor PT symmetric, the authors based a good deal of their discussion on its invariance with respect to the permutation transformations $(a_1,a_2,x_1,x_2,p_1,p_2) \to (a_2,a_1,x_2,x_1,p_2,p_1)$.

However, the spectrum of the Hamiltonian (1) can be more easily analyzed in terms of the two antiunitary symmetries given by the antilinear operators $A_1 = U_1 T$ and $A_2 = U_2 T$, where $U_1 : (x_1,x_2,p_1,p_2) \to (-x_1,x_2,-p_1,p_2)$ and $U_2 : (x_1,x_2,p_1,p_2) \to (x_1,-x_2,p_1,-p_2)$ are unitary transformations. Obviously, $A_i^{-1} = A_i$ and $A_i H A_i = H$ for $i = 1,2$. This kind of antiunitary symmetry has recently been called partial PT symmetry [10,11] and is an example of a larger class of antiunitary symmetries in quadratic Hamiltonians [12]. It is well known that if A is an antiunitary symmetry of H and $|\psi\rangle$ and eigenvector with eigenvalue E then $HA|\psi\rangle = AH|\psi\rangle = AE|\psi\rangle = E^* A|\psi\rangle$. Therefore, when A is an exact symmetry $A|\psi\rangle = a|\psi\rangle$, then $E = E^*$. It is clear that the transition from real to complex spectrum can be discussed as in any PT-symmetric Hamiltonian.

In order to derive more general conditions for real eigenvalues, we allow b to be either real or imaginary. When $b^* = -b$, we have a non-Hermitian Hamiltonian with the two antiunitary symmetries just mentioned. On the other hand, when $b^* = b$, the Hamiltonian is Hermitian. In order to obtain the conditions for real eigenvalues, it is necessary neither to solve the Schrödinger equation for H nor to resort to lengthy transformations of the coordinates and momenta. We simply make use of the algebraic method proposed recently that yields the natural frequencies of any Hamiltonian, which is a quadratic function of coordinates and momenta [7,8]. Although the approach was originally developed for Hermitian Hamiltonians, most of the results in those articles apply to non-Hermitian operators as well [9].

According to the algebraic method, the two natural frequencies of the operator (1) are the eigenvalues of the regular or adjoint matrix [7–9,13,14]

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & i a_1^2 & 0 \\ 0 & 0 & 0 & i a_2^2 \\ -i & -ib & 0 & 0 \\ -ib & -i & 0 & 0 \end{pmatrix}. \tag{2}$$

The characteristic polynomial

$$\det(\mathbf{H} - \lambda \mathbf{I}) = \lambda^4 - \lambda^2\left(a_1^2 + a_2^2\right) + a_1^2 a_2^2 (1 - b^2) = 0, \tag{3}$$

where $\mathbf{I}$ is a 4×4 identity matrix, shows that if λ is a root, then $-\lambda$ is also a root. Therefore, there are just two natural frequencies that are the positive square roots of

$$\xi_\pm = \tfrac{1}{2}\left[a_1^2 + a_2^2 \pm \sqrt{\left(a_1^2 - a_2^2\right)^2 + 4 a_1^2 a_2^2 b^2}\right]. \tag{4}$$

These two frequencies are real, provided that

$$-\frac{\left(a_1^2 - a_2^2\right)^2}{4 a_1^2 a_2^2} < b^2 < 1. \tag{5}$$

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In the Hermitian regime, there are real eigenvalues only if $0 < b^2 < 1$. When $b^2 \geqslant 1$, the Hermitian operator does not have eigenfunctions in the Hilbert space where it is defined [9]. The transition from real to complex spectrum takes place at the exceptional points $b = \pm 1$ [15–18] where the Hermiticity breaks down [7–9]. When $a_1 = a_2$, there are real eigenvalues only in the Hermitian region $0 < b^2 < 1$. For this reason, Li and Miao [1] restricted their analysis to the case $a_1 \neq a_2$. Equations (3)–(5) are invariant under permutation of a_1 and a_2 and show that it is the relative magnitude of these parameters that determines the main features of the spectrum.

At the other two exceptional points $b^2 = -(a_1^2 - a_2^2)^2/4a_1^2a_2^2$, the characteristic polynomial reduces to $(2\lambda^2 - a_1^2 - a_2^2)^2 = 0$ and the two natural frequencies are equal. The form of the characteristic polynomial (3) clearly shows that the coordinates satisfy the differential equation

$$\frac{d^4x_j}{dt^4} + \left(a_1^2 + a_2^2\right)\frac{d^2x_j}{dt^2} + a_1^2a_2^2(1 - b^2) = 0, \quad j = 1,2, \tag{6}$$

that reduces to the one derived by Li and Miao [1] when $b = ia_3/(2a_1a_2)$. The present equation is more general because it also applies to the Hermitian regime $0 < b^2 < 1$. It may be rewritten as

$$\frac{d^4x_j}{dt^4} + (\xi_+ + \xi_-)\frac{d^2x_j}{dt^2} + \xi_+\xi_- = 0, \quad j = 1,2, \tag{7}$$

that is exactly the one associated to the Pais-Uhlenbeck oscillator because $\xi_\pm = \omega_{1,2}^2$. It can be proved that any Hamiltonian that is a quadratic function of two coordinates and their conjugate momenta leads to a fourth-order differential equation like (7) [9]; therefore, it is not surprising that the Hamiltonian operator (1) and the Pais-Uhlenbeck one are associated with the same differential equation for the coordinates.

The regular or adjoint matrix $\mathbf{H}$ is closely related to the fundamental matrix $\mathbf{F}$ [19] in the following way: $\mathbf{F} = i\mathbf{H}^T/2$, where the subscript T stands for transpose. It has been proved that "a *PT*-symmetric elliptic quadratic differential operator with real spectrum is similar to a self-adjoint operator precisely when the associated fundamental matrix has no Jordan blocks" (p. 444007). This statement is consistent with the fact that the eigenvalues of the Hamiltonian (1) are real when the eigenvalues of the adjoint or regular matrix are real.

In summary, although the non-Hermitian Hamiltonian (1) is not *PT* symmetric, we can analyze its spectrum exactly in the same way in terms of exact or broken antiunitary symmetry (the A_1 and A_2 shown above). It is not necessary to resort to the permutation symmetry in order to explain any of its properties because the Hamiltonian (1) is just a variant of the *PT*-symmetric class of Hamiltonians. The transition from real to complex spectrum takes place in the usual way at some exceptional points where the antiunitary symmetry is broken and the adjoint or regular matrix representation is not diagonalizable.

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