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BRIEF COMMUNICATION

Sommerfeld's fine structure constant approximated as a series representation in e and π

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Abstract Sommerfeld in 1916 introduced the dimensionless fine structure constant, α, in to the context of atomic physics, in the course of working out the relativistic theory of the H atom, under the old quantum theory of Bohr. He was able to account for the fine structural detail of the atomic line spectrum of H by introducing this dimensionless constant which emerged naturally from his relativistic theory of the H atom. Since this time, the fine structure constant has emerged in several other contexts within experimental and theoretical physics. It has attained a status of being a mysterious number in physics that defies understanding as to its experimentally verified magnitude and identity. Being physically dimensionless, such a number invites a suggestion (or approximation) of its value in terms of mathematical constants in some formulation. Feynman most famously has conjectured that it might be possible to account for α in some type of series or product expression in "e", the base of natural logarithms, and "π" the familiar circular constant. Here we propose an infinite series in the product $e \cdot \pi$ that converges, within a few terms, to better than 9999 parts in 10,000 of the true value of α.

Keywords Fine structure constant α · Sommerfeld · Infinite series · e, π

1 Introduction

Sommerfeld introduced the dimensionless, so-called fine structure constant in to atomic physics contexts in 1916 [1]. He was motivated to define this dimensionless constant in order to account for the fine spectroscopic structural detail observed in

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Table 1 Physical constants and mathematical constants associated with Sommerfeld number

Symbol	Physical quantity	Value
k_c	Coulomb's law constant	8.98755×10^9 N m^2/C^2
q	Electric charge	1.60217×10^{-19} C
h/2π	Quantum of action	1.05457×10^{-34} J s
c	Speed of light	2.99792×10^8 m/s
π	Circle constant	3.14159
e	Euler constant	2.71828
α	Fine structure constant	0.0854245

the atomic spectrum of H. Sommerfeld's atomic model was based upon the notion of circular orbits in the atomic structure of H as described in the "old quantum theory" by Bohr's postulates [2]. He then introduced relativistic corrections to the electronic motion in H by generalizing the structure in terms of elliptical orbits. As part of this Sommerfeld relativistic analysis of atomic structure, there emerged naturally the occurrence of a dimensionless constant, accounting for the separation of the fine atomic emission lines in the H spectrum produced by an arc discharge lamp containing gaseous hydrogen. Sommerfeld thus defined this constant, initially as v/c, where "v" is the velocity of the electron in the 1st Bohr orbit and "c" is the speed of light. This constant is called the fine structure constant, represented as α, and is defined below in terms of the elementary charge, Planck's reduced quantum of action, the speed of light and the Coulomb's law constant [3]. Note that in Eq. (2) below, we define α to be $\sqrt{}$(v/c) to be consistent with the quotation due to Feynman in the discussion following (Table 1) [4].

$$\alpha^2 = \frac{(k_c)\,(q)^2}{(h/2\pi)\,(c)} \tag{1a}$$

$$\alpha^{-2} = 137.036\ldots \tag{1b}$$

$$\alpha = 0.0854245\ldots \tag{2}$$

2 Methods and discussion

In the years since Sommerfeld first identified this dimensionless constant, as mentioned above symbolized by the Greek letter "α", it has emerged in several different contexts in experimental and theoretical physics [4]. It has several definitions based on different combinations of natural constants including those definitions recently described by M.E. Tobar [5]. It has achieved a certain degree of mystery and beauty within the physics community, and among spectroscopists and theorists, including most famously Richard Feynman [4] who has described the status of "α" in the following quotation:

> There is a most profound and beautiful question associated with the observed coupling constant, e . . . .the amplitude for a real electron to emit or absorb a real

photon. It is a simple number that has been experimentally determined to be close to 0.08542455. (My physicist friends won't recognize this number, because they like to remember it as the inverse of its square: about 137.03597 with about an uncertainty of about 2 in the last decimal place. It has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists put this number up on their wall and worry about it.) Immediately you would like to know where this number for a coupling comes from: is it related to pi or perhaps to the base of natural logarithms? Nobody knows. It's one of the greatest damn mysteries of physics: a magic number that comes to us with no understanding by man. You might say the "hand of God" wrote that number, and "we don't know how he pushed his pencil." We know what kind of a dance to do experimentally to measure this number very accurately, but we don't know what kind of dance to do on the computer to make this number come out, without putting it in secretly!

From Feynman's quotation one can clearly see that he suggests, and is open to, the development of an expression for the dimensionless fine structure constant of Sommerfeld, in terms of the transcendental constants of mathematics including principally "e" the base of natural logarithms and "π" the ubiquitous circular constant. In this communication we propose one possible series representation in "e" and "π" that is relatively simple and that converges to better than 99.99% of the true value of "α" as defined in (1). Feynman in particular pointed out that the commonly accepted value of the fine structure constant is about 137.036... This is in fact the value of α^{-2}, where the value defined for α in this communication is close to that described in the Feynman quotation and is 0.0854245… and this alternative decimal representation will be used in the analysis that follows as shown in (2).

3 Results and conclusion

Equation (3a) shows the generic formula of the proposed series representation of "α". Note that the series is principally in terms of the product of "e" multiplied by "π" in a power series starting from the first power and successive terms are in the second power and third power etcetera... in the product "e.π". The actual series, out to third order (it converges at 4th order and beyond) is shown in Eq. (3b).

$$\alpha\prime = \sum_{n=0}^{\infty} \frac{(e \cdot \pi)^{n+1}}{(2 \cdot n + 1) \cdot (10^{4n+2})} \tag{3a}$$

$$\alpha\prime = \frac{(e \cdot \pi)^1}{1 \cdot 10^2} + \frac{(e \cdot \pi)^2}{3 \cdot 10^6} + \frac{(e \cdot \pi)^3}{5 \cdot 10^{10}} + \cdots \tag{3b}$$

Table 2 below shows the results of the calculations of the 1st order, 1st order + 2nd order and 1st order + 2nd order + 3rd order approximations to "α" from using Eq. (3b). One can see that even at 1st order the series returns a value of "α" beyond 99.9% accuracy. Thus the product "e $\cdot$ π" is nearly proportional to "α" by all appearances… and it is not easily clear why this should be the case. At 4th order and beyond the series converges off at slightly beyond 99.9967% agreement with the true value of the

Table 2 Approximations to the Feynman fine structure constant "√(v/c)"

Order	√(v/c)	√(v/c)'	√(v/c)'/√(v/c) × 100
0	0.0854245	–	100.000
1	0.0854245	0.0853973	99.9681
2	0.0854245	0.0854216	99.9966
3	0.0854245	0.0854217	99.9967

Table 3 Approximations to the Sommerfeld fine structure constant "v/c"

Order	(v/c)	(v/c)'	(v/c)'/(v/c) × 100
0th	0.00729735	–	100.000
1st	0.00729735	0.00729269	99.9361
2nd	0.00729735	0.00729684	99.9930
3rd	0.00729735	0.00729686	99.9932

fine structure constant defined in Eq. (2). Thus the series representation proposed here in "e" and "π" is obviously not the final word on approximations to "α", but such a series may be close to the final form that such an approximation can take on... and it is thus left to the reader to "fine tune" Eq. (3) in an attempt to make it more accurate.

The fine structure constant α is defined differently, usually, than in the preceding discussion based upon the Feynman quotation cited here [4]. The usual definition is that α is set equal to v/c, where "v" is the velocity of the electron in the first Bohr orbit of H and "c" is the speed of light. Thus in the usual definition it is taken as the square of the quantity identified by Feynman [4]. In the NIST CODATA compilation [3] the value of α (= v/c) is 0.0072973525664(17). Table 3 below indicates the accuracy of the squares of the approximations calculated using Eq. (3), to the latter definition of the fine structure constant. Note that out to 3rd order, in Table 3, the approximation based upon Eq. (3) is good to better than 99.9932%. Results in Tables 2 and 3 reflect the fact that the approximations calculated by Eq. (3) are accurate to only 4 significant figures, i.e. beyond 4 significant figures the numbers have uncertainty in them due to the level of approximation in Eq. (3), and for this reason the associated numbers are truncated at 6 significant figures in order to be consistent with the limitations of the approximation employed.

A final point to be made in this connection is that Eddington [6] identified 4 dimensionless constants formed out of various combinations of the natural constants... these are called the dimensionless "Eddington constants". Thus the fine structure constant is just one of several dimensionless physical constants, and the value of each one could potentially be fit to a suitable series or product representation using the mathematical constants such as e and π and perhaps the "golden ratio" identified as ϕ... or some other mathematical constants. The results of such an analysis might provide insight concerning deeper connections between the constants of mathematics and those natural constants. Such insights might have implications for Eddington's proposed "Fundamental Theory" in this regard.

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