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# On Multifractal Rigidity

**Alejandro M. Mesón · Fernando Vericat**

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**Abstract** We analyze when a multifractal spectrum can be used to recover the potential. This phenomenon is known as multifractal rigidity. We prove that for a certain class of potentials the multifractal spectrum of local entropies uniquely determines their equilibrium states. This leads to a classification which identifies two systems up to a change of variables.

**Keywords** Multifractal spectrum · Free energy · Gibbs states

**Mathematics Subject Classifications (2010)** 37C45 · 37C30

## 1 Introduction

The *multifractal analysis* has its genesis in the physical ambient [13, 14]. In the study of chaotic behaviors, invariant sets with a complex mathematical structure are frequently found. These sets can be decomposed into subsets - with some scaling property. This kind of partition is called a *multifractal decomposition*. To reveal complete information about these level sets a rigorous

A. M. Mesón · F. Vericat (✉)
Instituto de Física de Líquidos y Sistemas Biológicos (IFLYSIB),
CONICET–UNLP, La Plata, Argentina
e-mail: vericat@iflysib.unlp.edu.ar

A. M. Mesón
e-mail: meson@iflysib.unlp.edu.ar

A. M. Mesón · F. Vericat
Grupo de Aplicaciones Matemáticas y Estadísticas de la Facultad de Ingeniería (GAMEFI),
UNLP, La Plata, Argentina

mathematical description is needed. A first attempt in this way was to consider an attractor $A$ carrying an invariant measure $\mu$ which scales with an exponent $\alpha$ in a scale level $r$. More specifically in [1, 2] was performed a multifractal decomposition of the attractor $A$ in sets

$$K_\alpha = \{x : \mu(B_r(x)) \sim r^\alpha \text{ as } r \to 0\}. \tag{1}$$

where $B_r(x)$ denotes the ball of centre $r$ and radius $\varepsilon$.

A complete description of the multifractal analysis of invariant measures was done by Pesin and Weiss in [24]. In that work all the results known until that moment about smooth conformal maps were extended. The general idea of multifractal analysis was introduced in [4] as follows: Given a set $X$ and a map $g : X \to [-\infty, +\infty]$ the level sets

$$K_\alpha = K_\alpha(g) = \{x : g(x) = \alpha\},$$

and the decomposition $X = \left(\bigcup_\alpha K_\alpha\right) \cup Y$, where $Y$ is the set in which $g$ is not defined, are considered. If $G$ is a function defined on sets, and $F(\alpha) = G(K_\alpha)$, then the map $F$ is called *the multifractal spectrum* specified by the pair $(g, G)$. When $g(x)$ is the dimension of the measure $D_\mu(x)$ and $F(\alpha)$ the Hausdorff dimension $\dim_H K_\alpha$ of the set $K_\alpha$, then this spectrum is called the dimension multifractal spectrum. The function $F(\alpha)$ gives a description of the fine-scale property of the part of $X$ where the measure $\mu$ is concentrated. The dimension multifractal spectrum was previously studied for particular cases in [8, 14] and further generalized in the above mentioned articles.

Another interesting example is the local entropies spectrum which is obtained with $g$ as the local entropy of a dynamical map $f$ and $F(\alpha)$ as the Bowen topological entropy (for non-compact sets) of the level sets. The Hausdorff dimension and the topological entropy are special cases of "characteristic dimensions" in metric spaces. Thus there is a close relationship between the fields of multifractal analysis and dimension theory of dynamical systems. The knowledge of adequate dimensions of the multifractal decomposition sets is necessary to investigate the complexity of them.

The classification of multifractal spectra is done by using families of measures $\{\mu_\alpha\}_{\alpha\in\mathbf{R}}$ such that $\mu_\alpha(K_\alpha) = 1$. Two multifractal spectra $(g_1, G_1)$ and $(g_2, G_2)$ are said to be *equivalent* with respect to the families of full measures $\{\mu_\alpha^1\}_{\alpha\in\mathbf{R}}$ and $\{\mu_\alpha^2\}_{\alpha\in\mathbf{R}}$ if there is a bijection $\sigma : [-\infty, +\infty] \to [-\infty, +\infty]$ such that $\mu_\alpha^1 = \mu_{\sigma(\alpha)}^2$ for every real $\alpha$. When the spectrum is defined from a potential $\varphi : X \to \mathbf{R}$ and dynamics $f : X \to X$, like the entropies spectrum, a one-parameter family of measures $\{\mu_q\}_{q\in\mathbf{R}}$ is introduced as the Gibbs state for each member of a certain family of potentials $\{\varphi_q\}$. Then a parametrization $\alpha(q)$ with $\mu_q\left(K_{\alpha(q)}\right) = 1$ and $\mu_q(K_\alpha) = 0$ if $\alpha \neq \alpha(q)$ is defined. Therefore, there is a correspondence between the level sets of the decomposition and the family of full measures $\{\mu_q\}$. The parametrization is obtained by setting $\alpha(q) := -T'(q)$, where $T$ is the "free energy" in Ruelle's thermodynamic formalism terminology, whereas $q$ is interpreted as the inverse of the temperature, so $\alpha$ may be the internal energy per volume. In the most known and used spectra (for instance the dimension, entropy or the Lyapunov spectra), the free energy map is, under certain conditions, a convex differentiable map whose Legendre transform is $F(\alpha)$, thus multifractal spectra can be classified by the dynamics and equilibrium states.

One interesting problem is to study when the spectrum determines the potential, a phenomenon called *multifractal rigidity*. In other words the issue is to analyze when the multifractal classification works as a complete invariant of dynamical systems as well as of equilibrium states. This classification fits better to a physical interpretation than the topological and measure-theoretic ones, because multifractal classification identifies two systems up to a bijection between variables.

A remarkable result in this direction was obtained in [4], where the authors established multifractal rigidity for the full shift in two symbols and for special potentials. Specifically they proved that if two Bernoulli schemes, with probabilities $p_i, \widehat{p_i}, i = 1, 2$, have the same dimension spectrum, then there is a homeomorphism between the respective phase spaces and the probabilities are uniquely determined by each multifractal spectrum

A meaningful step was then done by Pollicott and Weiss [25] who demonstrated that for the special class of *generic locally constant* potentials the free energy determines the potential. By locally constant potentials it must be understood those that depend on a finite number of coordinates, or finite range observables in the physical language. The genericity is a matrix property, which must be verified by the matrices associated to the potentials. The matrices with this property are in the complement of an algebraic variety of dimension one. In the above mentioned article examples of systems with locally potentials which have the same free energy but non-equivalent were presented. Also they established a local multifractal rigidity for symbolic dynamical systems and Hölder continuous potentials.

In this article we establish the existence of multifractal rigidity for larger classes of potentials than in the mentioned articles. If $(X, d)$ is a compact metric space and $f : X \to X$ an homeomorphism the local entropies spectrum is given from the decomposition $K_\alpha = \{x : h_\mu(x, f) = \alpha\}$ where $h_\mu(x, f)$ is the local entropy: $h_\mu(x, f) = \lim_{\varepsilon \to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu\left(B_{n,\varepsilon}(x)\right)$, with $B_{n,\varepsilon}(x)$ the ball of centre $x$ and radius $\varepsilon$ in the metric $d_n(x, y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0, 1, ..., n-1\right\}$. The map $\mathcal{E}(\alpha)$ defined on level sets is $\mathcal{E}(\alpha) = h_{\text{top}}(f, K_\alpha)$, with $h_{\text{top}}(f, Z)$ the Bowen topological entropy for non-compact nor invariant sets [6], and the free energy for this spectrum with potential $\varphi$ is the map $T(q) = P(q\varphi) - qP(\varphi)$ ($P$ the topological pressure) whose Legendre transform is precisely $\mathcal{E}(\alpha)$. The description of this multifractal spectrum for a class of potentials broader than Hölder continuous maps and for expansive homeomorphisms with specification was performed by Takens and Verbitski [30]. The lattice spin systems used in classical Statistical Mechanics are mathematically modelled by the Markov systems $\Sigma_A = \left\{x = (x_i)_{i\in\mathbf{Z}} : x_i \in \Omega, \forall i \in \mathbf{Z}, A_{x_i,x_{i+1}} = 1\right\}$, where $A$ is a $k \times k$ matrix with 0, 1 entries and $\Omega = \{0, 1, 2, ..., k-1\}$. The integers $i$ are called the *sites* and the corresponding coordinate $x_i$ the *spin* at the site $i$. The matrix $A$ indicates which configurations, i.e. which sequences $x = (x_i)_{i\in\mathbf{Z}}$, are allowed.

We prove, for Markov systems and an adequate class of potentials depending on infinite coordinates, the following result: $\mathcal{E}_{\varphi_1} = \mathcal{E}_{\varphi_2}$ implies $\mu_{\varphi_1} = \mu_{\varphi_2}$, where $\mu_\varphi$ is the Gibbs state associated to the potential $\varphi$. We use an approach based on transfer operators which also works for spin lattice models with infinite range potentials, i.e in which the potential depends on all the coordinates, The scheme followed is similar than [25], where stochastic matrices are used. We firstly prove that the multifractal spectrum determines the Fredholm determinant of the corresponding transfer operator (it plays the role of the matrix in the finite range case), then for the special class of potentials considered the determinant is related with the zeta function associated to the potential and finally since the zeta function uniquely determines the equilibrium states we are done. This last result will be actually proved in a much general context than symbolic systems.

We also study the variational properties of perturbations on the local entropies spectrum in order to get a local rigidity result. For this we shall consider for a fixed dynamical map $f : X \to X$ a family of potentials $\Phi = \{\varphi_\lambda\}_{\lambda\in(-\delta,\delta)}$ and study the variation of the entropies spectrum, by computing the first and second derivatives, with respect to the perturbative parameter $\lambda$, of the function $\tau(\lambda, q) := P(q\varphi_\lambda) - qP(\varphi_\lambda)$ which is in turn a perturbation of the function $T(q) = P(q\varphi_0) - qP(\varphi_0)$. The estimate of the influence of the perturbations and how numerical results could be affected by small perturbations is very useful for numerical computations. Results about first variational formulae for dimension spectra were obtained in [3] and [33] and for the second variation, also for dimension spectrum, in [15]. In all these cases the results are valid for hyperbolic diffeomorphisms. In [20] were calculated the first derivative of $\tau(\lambda, q)$ but under much weaker hypothesis than hyperbolicity and Hölder continuous potentials, we compute here the second derivatives of $\tau(\lambda, q)$ under these same hypothesis. The local rigidity result that we present herein is: If $\lambda \mapsto \mathcal{E}_{\varphi_\lambda}$ is constant for $\lambda \in (-\delta, \delta)$ then $\mu_{\varphi_\lambda}$ is constant for $\lambda \in (-\delta, \delta)$, whose validity is established for expansive homeomorphism with specification, conditions much weaker than the existence of Markov partitions, and for a class which includes on hyperbolic sets.

## 2 Basic Definitions and Previous Results

We begin by recalling the description of the local entropies multifractal spectrum, some of whose main aspects were sketched in the introduction: Let $(X, d)$ be a compact metric space, and $f : X \to X$ a continuous map. Let

$d_n(x, y) = \max\left\{d\left(f^i(x), f^i(y)\right) : i = 0, 1, ..., n-1\right\}$. We denote by $B_{n,\varepsilon}(x)$ the ball of centre $x$ and radius $\varepsilon$ in the metric $d_n$. If $\mu$ a $f-$invariant measure, the *upper and lower local entropies* are

$$\overline{h_\mu}(x, f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu\left(B_{n,\varepsilon}(x)\right)$$

$$\underline{h_\mu}(x, f) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu\left(B_{n,\varepsilon}(x)\right).$$

Then (Brin–Katok theorem [7]), the local entropy does exist, i.e. $\overline{h_\mu}(x, f) = \underline{h_\mu}(x, f) := h_\mu(x, f)$, for $\mu - a.e.\ x \in X$. Now the *local entropies spectrum of* $f$ is that specified by the pair $\left(h_\mu(f, x), \mathcal{E}(\alpha)\right)$ with $\mathcal{E}(\alpha) := h_{\text{top}}(f, K_\alpha)$. The set $X$ is naturally decomposed as

$$X = \bigcup_{\alpha=-\infty}^{\infty} K_\alpha \cup (X - Y),$$

where $Y$ is the set in which the local entropy map does not exist and is usually called the *irregular part* of the spectrum. By the Brin–Katok theorem $\mu(X - Y) = 0$, for any $f-$invariant measure $\mu$.

Next we collect a few definitions from the Ruelle thermodynamic formalism [26]. The *topological pressure* associated to $f$ and to a potential $\varphi : X \to \mathbf{R}$, is the number

$$P(\varphi) = \sup_\mu \left\{h_\mu(f) + \int \varphi d\mu\right\},$$

where the supremum is taken over all the $f-$invariant Borel measures $\mu$ on $X$, and $h_\mu(f)$ is the usual Kolmogorov measure-theoretic entropy of $f$.

An *equilibrium state* for the potential $\varphi$ is a measure $\mu_\varphi$ for which:

$$P(\varphi) = h_{\mu_\varphi}(f) + \int \varphi d\mu_\varphi. \tag{2}$$

The set of equilibrium states for the potential $\varphi$ will be denoted by $\mathcal{M}_\varphi(X)$.

Under certain conditions imposed on the map $f$ and the potential $\varphi$ an equilibrium state can be constructed [16–30]. The *specification property* for a map $f : X \to X$ intuitively says that for specified orbit segments a periodic orbit approximating the trajectory can be found. This condition ensures abundance of periodic points. It is a concept introduced by R. Bowen [5]. Formally, a homeomorphism $f : X \to X$ has the s*pecification property* if given a finite disjoint collection of integer intervals $I_1, I_2 ..., I_k$ and $\varepsilon > 0$, there is an integer $M(\varepsilon)$ and a function $\Phi : I = \cup I_i \to X$, such that the following conditions are satisfied:

(i) $dist\left(I_i, I_j\right) > M(\varepsilon)$ (Euclidean distance)
(ii) $f^{n_1 - n_2}(\Phi(n_1)) = \Phi(n_2)$
(iii) $d(f^n(x), \Phi(n)) < \varepsilon$, for some $x : f^m(x) = x$, with $m \geq M(\varepsilon) + length(I)$ and for every $n \in I$.

A homeomorphism $f : X \to X$ is called *expansive* if there is a constant $\delta > 0$, such that $d\left(f^n(x), f^n(y)\right) < \delta$, for any integer $n$ implies $x = y$.

For a potential $\varphi$ we put

$$S_n(\varphi)(x) = \sum_{i=0}^{n-1} \varphi\left(f^i(x)\right) \tag{3}$$

which is called the *statistical sum.*

Following [16] or [30], we say that a potential $\varphi$ belongs to the class $\nu_f(X)$ if it satisfies the following condition:

There are constants $\varepsilon$, $K > 0$ such that

$$d_n(x, y) < \varepsilon \Longrightarrow |S_n(\varphi)(x) - S_n(\varphi)(y)| < K. \tag{4}$$

We also recall how an equilibrium state associated to a potential $\varphi \in \nu_f(X)$ can be defined. Let $P_n(f) = \{x : f^n(x) = x\}$, then we set

$$\mu_{\varphi,n}(A) = \frac{1}{\widetilde{Z}(f, \varphi, n)} \sum_{x \in P_n(f)} \exp\left(S_n(\varphi)(x)\right) \delta_x(A), \tag{5}$$

where $\widetilde{Z}(f, \varphi, n) = \sum\limits_{x \in P_n(f)} \exp\left(S_n(\varphi)(x)\right)$ and $\delta_x$ is the Dirac measure at $x$:

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

If $X$ is compact the sequence $\left\{\mu_{\varphi,n}\right\}$ has an accumulation point and under the above conditions it has a weak limit $\mu_\varphi$, i.e.:

$$\lim_{n\to\infty} \int \psi(x)\, d\mu_{\varphi,n} = \int \psi(x)\, d\mu_\varphi, \tag{6}$$

for every continuous $\psi$ [16, 26].

**Theorem** [16, 26] *Let $f$ be an expansive homeomorphism with the specification property and $\varphi$ a potential belonging to the class $\nu_f(X)$, then $\mu_\varphi$ is the unique equilibrium state associated to $\varphi$. Besides $\mu_\varphi$ is ergodic.*

The conditions of expansiveness and specification are much weaker than the existence of Markov partitions. Under these hypothesis Takens and Verbitski [30] developed a multifractal formalism for local entropies spectrum, we review here the main results: let $T(q) = P(q\varphi) - qP(\varphi)$, $q \in \mathbf{R}$, called the free energy of $\varphi$,

(i) the function $T(q)$ is convex and continuously differentiable. This map has a Legendre transform $\mathcal{E}(\alpha) = \inf\limits_{q\in\mathbf{R}} \{q\alpha - T(q)\}$. $\mathcal{E}(\alpha)$ describes local entropies spectrum $f$.

(ii) If $K_\alpha = \left\{x : h_{\mu_\varphi}(x, f) = \alpha\right\}$, ($\mu_\varphi \neq \mu_{\max}$, the measure maximal entropy), then $\mathcal{E}(\alpha) = h_{\text{top}}(f, K_\alpha)$. Besides

$$\mathcal{E}(\alpha(q)) = q\alpha(q) + T(q); \qquad \alpha(q) := -T'(q), \qquad q = \mathcal{E}'(\alpha). \tag{7}$$

Let $\alpha_i = \lim_{q\to\infty} \alpha(q) = \inf_{q\in\mathbf{R}} \{\alpha(q)\}$, $\alpha_s = \lim_{q\to-\infty} \alpha(q) = \sup_{q\in\mathbf{R}} \{\alpha(q)\}$, then $K_\alpha = \emptyset$, if $\alpha \notin (\alpha_i, \alpha_s)$, so that the domain of definition of $\mathcal{E}(\alpha)$ is the range of $T'(q)$.

**Definition** A $f$-invariant measure $\mu$ is a *Gibbs state* if for sufficiently small $\varepsilon > 0$, there are constants $A_\varepsilon,\ B_\varepsilon > 0$, such that for any $x \in X$ and for any positive integer $n$:

$$A_\varepsilon (\exp(S_n(\varphi)(x)) - nP(\varphi)) \le \mu\left(B_{n,\varepsilon}(x)\right) \le B_\varepsilon (\exp(S_n(\varphi)(x)) - nP(\varphi)), \quad (8)$$

where $S_n(\varphi)(x) = \sum_{i=0}^{n-1} \varphi\left(f^i(x)\right)$.

**Theorem** [16–26] *Let $f : X \to X$ be an expansive homeomorphism which have the specification property and $\varphi$ a potential belonging to the class $\nu_f(X)$, then $\mu_\varphi$ is an equilibrium state associated to $\varphi$, which is a Gibbs state. Besides it is ergodic.*

The multifractal spectrum of local entropies is thus described by the family of measures $\{\mu_q\}$ whose members are the Gibbs states associated to the potentials $q\varphi - qP(\varphi)$. One has $\mu_q\left(K_{\alpha(q)}\right) = 1$, with $\alpha(q) = -T'(q)$.

One important general result about classification of equilibrium states is the following:

**Theorem** [30] *Let $\varphi, \psi \in \nu_f(X)$ where $X$ is a compact metric space and $f$ an expansive homeomorphism with specification, then $\mu_\varphi = \mu_\psi$ if and only if there is a constant $C$ such that $S_n(\varphi)(x) = S_n(\psi + C)(x)$, for any $n$ and for every $x \in P_n(f) = \{x : f^n(x) = x\}$.*

A proof of the above claim for the particular case of hyperbolic systems in Riemannian manifolds and Hölder continuous potentials. appears in [16, pp. 636–637].

According to the nomenclature of [25] the *unmarked orbit spectrum*, the *weak orbit spectrum* and the *marked periodic spectrum* of the potential $\varphi$ are respectively

$$\mathcal{S}_\varphi = \{(S_n(\varphi)(x), n) : x \in P_n(f)\},$$

$$\mathcal{W}_\varphi = \{S_n(\varphi)(x) : x \in P_n(f)\},$$

$$\mathcal{H}_\varphi = \{(S_n(\varphi)(x), x) : x \in P_n(f)\}.$$

In [25] is made an interesting parallelism between these spectra and length spectra of geodesics in compact hyperbolic surfaces. For instance $\mathcal{S}_\varphi$ is the analogue of the unmarked length spectrum which consists of the length of all closed geodesics and $\mathcal{W}_\varphi$ is the analogue of the set of the lengths of all closed geodesics marked with the free homotopy class of the geodesic. In this way is established a comparison between multifractal rigidity and the Kac problem *can you hear the shape of a drum?,* a question which summarizes the problem about when the geodesic spectrum determines the manifold.

A "Hamiltonian" approach to the presented multifractal rigidity can be formulated as follows: let $f : X \to X$ be an expansive homeomorphism with the property of specification and a potential $\varphi$ in the class $\nu_f(X)$, so that it has a Gibbs state $\mu_\varphi$. In [19] we have introduced a Hamiltonian of the form

$$H_{n,\varepsilon}(x) = -\log \mu_\varphi \left[ B_{n,\varepsilon}(x) \right]$$

This Hamiltonian may be considered as a generalization to the Sinai's one [29]. In that case the measure is the probability associated with cylinders. It should be noticed that balls like $B_{n,\varepsilon}(x)$, in the particular case of symbolic spaces (with a certain metric), correspond to cylinders.

Physically the point $x$ in the Hamiltonian can be thought as a *microstate* whose energy is given by the interaction of the point $x$ with all the points of the ball $B_{n,\varepsilon}(x)$, i.e. with all the points that follows the trajectory of $x$ within $\varepsilon$-distance up to time $n$. The total interaction being given by the measure of the ball. The microstates we are interested in are the whole set of periodic points $P_n(f)$. In analogy with statistical mechanics, we introduce the canonical partition function ($q$ interpreted as the inverse of the temperature):

$$Z(q; n, \varepsilon) = \sum_{x \in P_m(f)} \exp\left[-q H_{n,\varepsilon}(x)\right] = \sum_{x \in P_n(f)} \left(\mu_\varphi \left[B_{n,\varepsilon}(x)\right]\right)^q \tag{9}$$

and a "free energy"

$$\mathcal{F}(q) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log Z(q; n, \varepsilon). \tag{10}$$

We have proved [19] that $\mathcal{F}(q) = P(q\varphi) - qP(\varphi)$, for every real $q$. So that this energy function agrees with that introduced by Takens and Verbitski for their multifractal formalism which will be used for.

## 3 Local Multifractal Rigidity

Let us begin considering a homeomorphism $f : X \to X$, with $X$ a compact metric space, and a $C^k$– family of potentials $\Phi = \{\varphi_\lambda\}_{\lambda \in (-\delta,\delta)} \subset \nu_f(X)$ seen as a perturbation of a fixed potential $\varphi_0$. The requirement for the value of $k$ will depend of the order of derivative that we wish to compute. Next we introduce the map $\tau(\lambda, q) = P(q\varphi_\lambda) - qP(\varphi_\lambda)$, where $P = P(\varphi_\lambda, f)$, $\lambda \in (-\delta, \delta)$. For the non-perturbed case the map $\tau(0, q) = P(q\varphi_0) - qP(\varphi_0)$ will be denoted directly by $T(q)$, the free energy of $\varphi_0$.

The following results were set in [20]:

(1) $\frac{\partial\tau(\lambda,q)}{\partial\lambda}\mid_{\lambda=0}= q\int\Theta d\mu_q$, where $\Theta := \frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0} - \frac{\partial P(\varphi_\lambda)}{\partial\lambda}\mid_{\lambda=0}= \frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0} - \int\frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0} d\mu_q$ and $\mu_q$ is an equilibrium state for $q\varphi_0$.

(2) If $\mu_0 := \mu_{\varphi_0} \neq \mu_{\max}$, then $\mu_{\varphi_\lambda} \neq \mu_{\max}$ for sufficiently small $|\lambda|$ .

The necessity of establishing a result of this nature is to ensure the differentiability of the map $\lambda \to \mathcal{E}_\lambda(\alpha)$. Indeed if $\mu = \mu_{\max}$, then [30] $\mathcal{E}(\alpha) = h_{\text{top}}(f, K_\alpha) = \begin{cases} h_{\text{top}} & \text{if } \alpha = h_{\text{top}} \\ 0 & \text{if } \alpha \neq h_{\text{top}} \end{cases}$. Now it must be checked that under small perturbations one cannot have this degenerate behavior if it does not occur in the non-perturbed case.

**Lemma 1** *Let $\varphi, \psi \in \nu_f(X)$, with $f$ a homeomorphism with specification, then $\frac{d^2P(\varphi+\lambda\psi)}{d\lambda^2}\mid_{\lambda=0}= C_\varphi(\psi) := \mu_\varphi(\psi^2) - (\mu_\varphi(\psi))^2$, seeing the measure as a functional by $\mu_\varphi(\psi) = \int\psi d\mu_\varphi$, and $\mu_\varphi$ the Gibbs state associated to $\varphi$.*

*Proof* By the multifractal formalism described in the earlier section: $\frac{dP(\varphi+\lambda\psi)}{d\lambda}\mid_{\lambda=0}= \mu_\varphi(\psi) = \int\psi d\mu_\varphi$. Let us denote $\mu_\lambda = \mu_{\varphi+\lambda\psi}$ and so we have $\frac{dP(\varphi+\lambda\psi)}{d\lambda} = \mu_\lambda(\psi)$ . Let us recall (c.f. (5) and (6)) that the Gibbs state for a potential $\varphi$ in the class $\nu_f(X)$ is defined as the weak limit $\mu_\varphi$ of the "Gibbs ensembles"

$$\mu_{\varphi,n}(\{y\}) = \frac{\exp(S_n(\varphi)(y))}{\sum\limits_{x\in P_n(f)}\exp(S_n(\varphi)(x))}. \tag{11}$$

By the compactness of $X$ this sequence has an accumulation point which can be interpreted as its "thermodynamic limit". Thus for obtaining the second derivative we must differentiate $\mu_{n,\lambda}(\psi)$. Doing this we have $\frac{d\mu_{n,\lambda}(\psi)}{d\lambda} = \frac{\sum\limits_{x\in P_n(f)}\psi^2\exp(S_n(\varphi+\lambda\psi)(x))}{\sum\limits_{x\in P_n(f)}\exp(S_n(\varphi+\lambda\psi)(x))} - \left[\frac{\sum\limits_{x\in P_n(f)}\psi(x)\exp(S_n(\varphi+\lambda\psi)(x))}{\sum\limits_{x\in P_n(f)}\exp(S_n(\varphi+\lambda\psi)(x))}\right]^2$ and then $\frac{d^2P(\varphi+\lambda\psi)}{d\lambda^2}\mid_{\lambda=0}= \mu_\varphi(\psi^2) - (\mu_\varphi(\psi))^2$. □

**Theorem 1** *Let $\Phi = \{\varphi_\lambda\}_{\lambda\in(-\delta,\delta)} \subset \nu_f(X)$ be a $C^2$-family, with $f$ a homeomorphism with the specification property, then*

$$\begin{aligned}\frac{\partial^2\tau(\lambda,q)}{\partial\lambda^2}\mid_{\lambda=0}= q&\left[\mu_{q\varphi_0}\left(\frac{\partial^2\varphi_\lambda}{\partial\lambda^2}\mid_{\lambda=0}\right) - \mu_{\varphi_0}\left(\frac{\partial^2\varphi_\lambda}{\partial\lambda^2}\mid_{\lambda=0}\right)\right]\\ &+ q^2\left[C_{q\varphi_0}\left(\frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0}\right) - C_{\varphi_0}\left(\frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0}\right)\right].\end{aligned}$$

*Proof* We start by calculating $\frac{d^2P(\varphi+\lambda\psi)}{d\lambda^2}\mid_{\lambda=0}$, where $\{\psi_\lambda\}_{\lambda\in(-\delta,\delta)}$ is a $C^2-$family. For this we must differentiate $\mu_{\varphi+\lambda\psi_\lambda,n}(\psi_\lambda)$ with respect to $\lambda$. Thus $\mu_{\varphi+\lambda\psi_\lambda,n}(\psi_\lambda)=\frac{\sum_{x\in P_n(f)}\psi_\lambda\exp(S_n(\varphi+\lambda\psi_\lambda)(x))}{\sum_{x\in P_n(f)}\exp(S_n(\varphi+\lambda\psi_\lambda)(x))}$, and so

$$\frac{d\mu_{\varphi+\lambda\psi_\lambda,n}(\psi_\lambda)}{d\lambda}$$
$$=\frac{\sum_{x\in P_n(f)}\left[\frac{\partial\psi_\lambda}{\partial\lambda}\exp\left(S_n\left(\varphi+\lambda\psi_\lambda\right)(x)\right)+\psi_\lambda\exp\left(S_n\left(\varphi+\lambda\psi_\lambda\right)(x)\right)\left(\lambda\frac{\partial\psi_\lambda}{\partial\lambda}+\psi_\lambda\right)\right]}{\sum_{x\in P_n(f)}\exp\left(S_n\left(\varphi+\lambda\psi_\lambda\right)(x)\right)}$$
$$-\frac{\left(\sum_{x\in P_n(f)}\psi_\lambda\exp\left(S_n\left(\varphi+\lambda\psi_\lambda\right)(x)\right)\right)\sum_{x\in P_n(f)}\exp\left(S_n\left(\varphi+\lambda\psi_\lambda\right)(x)\right)\left(\frac{\partial\psi_\lambda}{\partial\lambda}+\lambda\psi_\lambda\right)}{\left[\sum_{x\in P_n(f)}\exp\left(S_n\left(\varphi+\lambda\psi_\lambda\right)(x)\right)\right]^2}$$

evaluating in $\lambda=0$

$$\frac{\sum_{x\in P_n(f)}\left[\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0}\exp\left(S_n(\varphi)(x)\right)\right]}{\sum_{x\in P_n(f)}\exp\left(S_n(\varphi)(x)\right)}+\frac{\sum_{x\in P_n(f)}\left[\psi_0^2\exp\left(S_n(\varphi)(x)\right)\right]}{\sum_{x\in P_n(f)}\exp\left(S_n(\varphi)(x)\right)}$$
$$-\frac{\sum_{x\in P_n(f)}\left[\psi_0\exp\left(S_n(\varphi)(x)\right)\right]}{\sum_{x\in P_n(f)}\exp\left(S_n(\varphi)(x)\right)}\times\frac{\sum_{x\in P_n(f)}\left[\psi_0\exp\left(S_n(\varphi)(x)\right)\right]}{\sum_{x\in P_n(f)}\exp\left(S_n(\varphi)(x)\right)}$$
$$=C_\varphi\left(\psi_0\right)+\mu_\varphi\left(\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0}\right).$$

Now

$$\frac{\partial^2\tau(\lambda,q)}{\partial\lambda^2}\mid_{\lambda=0}=\frac{\partial^2P\left(q\varphi_\lambda\right)}{\partial\lambda^2}\mid_{\lambda=0}-q\frac{\partial^2P\left(\varphi_\lambda\right)}{\partial\lambda^2}\mid_{\lambda=0}$$
$$=\frac{\partial^2P\left(q\varphi_0+q\frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0}\lambda+q\frac{\partial^2\varphi_\lambda}{\partial\lambda^2}\mid_{\lambda=0}\lambda^2+o\left(\lambda^2\right)\right)}{\partial\lambda^2}\mid_{\lambda=0}$$
$$-q\frac{\partial^2P\left(\varphi_0+q\frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0}\lambda+q\frac{\partial^2\varphi_\lambda}{\partial\lambda^2}\mid_{\lambda=0}\lambda^2+o\left(\lambda^2\right)\right)}{\partial\lambda^2}\mid_{\lambda=0}$$
$$=q\left[\mu_{q\varphi_0}\left(\frac{\partial^2\varphi_\lambda}{\partial\lambda^2}\mid_{\lambda=0}\right)-\mu_{\varphi_0}\left(\frac{\partial^2\varphi_\lambda}{\partial\lambda^2}\mid_{\lambda=0}\right)\right]$$
$$+q^2\left[C_{q\varphi_0}\left(\frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0}\right)-C_{\varphi_0}\left(\frac{\partial\varphi_\lambda}{\partial\lambda}\mid_{\lambda=0}\right)\right].$$

□

If we define a map $D(\lambda)$ by $P(D(\lambda)\psi_\lambda) = 0$, then we can calculate from the above theorem $\frac{\partial^2 D(\lambda)}{\partial\lambda^2}\mid_{\lambda=0}$. The interest of such a computation resides in the fact that, for the particular case of hyperbolic systems with basic set $\Lambda$ we have, by the Bowen equation, $\dim_H \Lambda = D(0)$ ($\dim_H$ means Hausdorff dimension). Thus we can find a first and a second variational formula for a "like perturbed dimension" under the general hypothesis of Theorem 1. A formula of this style was supplied in [15], but under stronger conditions.

**Proposition 1** *Under the same conditions for the dynamics and the potential as in Theorem 1 and for $D(\lambda)$ defined as above, it holds:* $\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0}= \frac{-D(0)\mu(\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0})}{\mu(\psi_0)}$ *and* $\frac{\partial^2 D(\lambda)}{\partial\lambda^2}\mid_{\lambda=0}= \{-C_{D(0)\psi_0}(\psi_0\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0} + D(0)\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0}) \times -2\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0} \times \mu(\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0}) - D(0)\mu(\frac{\partial^2\psi_\lambda}{\partial\lambda^2}\mid_{\lambda=0})\} \times \frac{1}{\mu(\psi_0)}$, *where $\mu$ is the Gibbs state associated to $D(0)\psi_0$.*

*Proof* We have $0 = \frac{\partial P(D(\lambda)\psi_\lambda)}{\partial\lambda}\mid_{\lambda=0}= \mu_{D(0)\psi_0}(\frac{\partial(D(\lambda)\psi_\lambda)}{\partial\lambda}\mid_{\lambda=0}) = \mu_{D(0)\psi_0}\times (\psi_0\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0} + D(0)\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0})$, and so $\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0}= \frac{-D(0)\mu_{D(0)\psi_0}(\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0})}{\mu_{D(0)\psi_0}(\psi_0)}$.

For the second derivative formula $0 = \frac{\partial^2 P(D(\lambda)\psi_\lambda)}{\partial\lambda^2}\mid_{\lambda=0}= \mu_{D(0)\psi_0}(\frac{\partial^2(D(\lambda)\psi_\lambda)_=}{\partial\lambda^2}\mid_{\lambda=0}) + C_{D(0)\psi_0}(\frac{\partial^2(D(\lambda)\psi_\lambda)_=}{\partial\lambda^2}\mid_{\lambda=0}) = \mu_{D(0)\psi_0}(\psi_0\frac{\partial^2(D(\lambda)\psi_\lambda)}{\partial\lambda^2}\mid_{\lambda=0} + 2\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0}\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0} + D(0)\frac{\partial^2\psi_\lambda}{\partial\lambda^2}\mid_{\lambda=0}) + C_{D(0)\psi_0}\psi_0\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0} D(0)\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0}$. So that $\frac{\partial^2 D(\lambda)}{\partial\lambda^2}\mid_{\lambda=0}= -C_{D(0)\psi_0}(\psi_0\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0} + D(0)\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0}) - 2\frac{\partial D(\lambda)}{\partial\lambda}\mid_{\lambda=0}\times\mu(\frac{\partial\psi_\lambda}{\partial\lambda}\mid_{\lambda=0}) - D(0)\mu_{D(0)\psi_0}(\frac{\partial^2\psi_\lambda}{\partial\lambda^2}\mid_{\lambda=0})$. □

Finally we state our result of local multifractal rigidity

**Theorem 2** *Let $f : X \to X$ be an expansive homeomorphism in a compact metric space with the specification property. Let $\Phi = \{\varphi_\lambda\}_{\lambda\in(-\delta,\delta)} \subset \nu_f(X)$ be a $C^2$-family such that $\mathcal{E}_{\varphi_\lambda}(\alpha)$ is constant, then $\mu_{\varphi_\lambda}$ is also constant.*

*Proof* From the equality of the multifractal spectra we deduce that the map $\lambda \to \tau(\lambda, q)$ is constant, for each $q$ and for $|\lambda| < \delta$. Therefore $\int \Theta_\lambda dm_{\lambda,q} = 0$, where $\Theta_\lambda := \frac{\partial\varphi_\lambda}{\partial\lambda} - \frac{\partial P(\varphi_\lambda)}{\partial\lambda}$ and with $m_{\lambda,q}$ the Gibbs state associated to $\varphi_{\lambda,q} = q\varphi_\lambda - P(\varphi_\lambda)$, $\lambda \in (-\delta, \delta)$.

Let us recall the classical Mazur theorem about existence of tangent functionals in Banach spaces [9, p. 450]: if $V$ is a separable Banach space and $P : V \to \mathbf{R}$ is convex continuous then the set at which there is a unique tangent functional to $P$ contains a countable intersection of dense open sets, and so, because $V$ is a Banach space, by the Baire category theorem it is dense. This theorem can be applied with $P : C(X) \to \mathbf{R}$ the topological pressure and the tangent functionals at $\varphi$ defined as the set of the signed measures $\mu$ such that $P(\varphi + \psi) - P(\varphi) \geq \int \psi d\mu$, for any $\psi \in C(X)$. If the entropy map $\mu \to h_\mu(f)$ is upper semi-continuous then the set of tangent functionals at $\varphi$ agrees with the set of equilibrium states of $\varphi$ and if $f$ is an expansive continuous map in a compact metric space then the entropy map is upper semi-continuous [32]. Now under the hypothesis considered in this work it holds that there is a dense subset $A$ of $C(X)$ such that for any $\varphi \in A$ the set $\mathcal{M}_{\varphi}(X)$ has just one element. Based upon the above results we have that $\frac{d}{d\lambda}\int(\varphi_{\lambda} - P(\varphi_{\lambda}))d\mu = 0$ for any equilibrium measure $\mu$ associated to potentials in an open dense subset of $\nu_f(X)$. Thus $\int \varphi_{\lambda} - P(\varphi_{\lambda})d\mu = C$, for some constant $C$ and so $S_n(\varphi_{\lambda} - P(\varphi_{\lambda}))(x) = S_n(C\lambda)(x)$ is a small neighborhood of $\lambda = 0$. Therefore there is a small interval $(-\delta, \delta)$ such that $\mu_{\varphi_{\lambda} - P(\varphi_{\lambda})} = \mu_{\varphi_{\lambda}}$ is constant for $\lambda \in (-\delta, \delta)$. □

The above proposition generalizes a similar result of [4]. There was proved a local multifractal rigidity theorem, but for hyperbolic systems and for the dimension spectrum instead.

## 4 Multifractal Rigidity for Spin Lattice Systems

The next step is to address to the following multifractal rigidity problem: let $\mathcal{E}_{\varphi_i}(\alpha), i = 1, 2$ be two multifractal spectra of local entropies defined from potentials $\varphi_i$ which an unique associated Gibbs state and dynamics $f : X \to X$. Under adequate conditions these spectra are determined by the respective free energies $T_{\varphi_i}(q)$ since they are the Legendre transforms of $\mathcal{E}_{\varphi_i}(\alpha)$. Now the problem will be to find classes of potentials and dynamics for which the free energy determines the equilibrium states. In short the issue is to establish when the following implication is valid

$$\mathcal{E}_{\varphi_1}(\alpha) = \mathcal{E}_{\varphi_2}(\alpha) \Longrightarrow \mu_{\varphi_1} = \mu_{\varphi_2}. \tag{12}$$

We briefly describe the special case treated by Barreira et al. in [4]: they have considered a one-dimensional map $f : I \to I$ $(I = [0, 1])$ which can be "partitioned" in two maps $f_i : I_i \to I_i$, $i = 1, 2$, with $I_i \subset [0, 1]$ and $f_i(I_i) = [0, 1]$. If $J = \bigcap_{k=1}^{\infty} f^{-k}(I_1 \cup I_2)$ then $\{J \cap I_1, J \cap I_2\}$ is a Markov partition for $(J, f)$ and this dynamical system is topologically conjugated to the full shift of two symbols $\Sigma_2 = \{x = (x_i)_{i \in \mathbf{N}} : x_i \in \{0, 1\}\}$, which is a Bernoulli scheme with probabilities $p_i$, $i = 0, 1$, assigned to each $x_i$. The potential is $\varphi : \Sigma_2 \to \mathbf{R}$ defined by $\varphi(x) = \log p_i$ if $x \in I_i$, this map is in fact of the form $\varphi(x) = \psi(x_0)$ with the probabilities $p_i = \frac{\exp(\psi(i))}{\exp(\psi(0)) + \exp(\psi(1))}$, $i = 0, 1$, while the topological pressure is $P(\varphi) = \log(\exp(\psi(0)) + \exp(\psi(1)))$. Thus a direct calculation leads to $T_{\varphi}(q) = \log\left(\sum p_i^q\right)$. The Gibbs state associated to $\varphi$ is the product measure in $\Sigma_2$ of the measures $p_i$.

Let $f, \widehat{f}$ be one-dimensional Markov maps with invariant sets $J, \widehat{J}$ as above and let $\chi, \widehat{\chi} : \Sigma_2 \to \mathbf{R}$ be the coding maps giving the conjugations between each $J, \widehat{J}$ and $\Sigma_2$. In [4] it is then proved that there is a homeomorphism $\phi : J \to \widehat{J}$ such that $\phi \circ f = \widehat{f} \circ \phi$, so that the dynamical systems $(J, f)$ and $\left(\widehat{J}, \widehat{f}\right)$ are topologically conjugated, and there is an automorphism $\rho$ of $\Sigma_2$ such that

$\varkappa \circ \phi = \rho \circ \widehat{\chi}$. This was established by showing that the free energy uniquely determines the probabilities. Now in this special situation (12) holds.

As we mentioned in the introduction the problem on whether the free energy determines the potential was solved by Pollicott and Weiss for potentials depending on a finite number of coordinates (finite range potentials). Our aim herein is to establish the validity of (12) for a class which include infinite range potentials, i.e. depending on the entire configuration. One interesting example in this situation is the Kac model: let $\Omega = \{\pm 1\}$ with the transition matrix with all entries equal to 1 and the potential $\varphi(x) = Jx_0 \sum_{n=1}^{\infty} x_n \lambda^n$, with $\lambda \in (0, 1)$, $J \in \mathbf{R}$ is a coupling parameter.

In the case of finite range potentials can be defined a primitive matrix ($H$ is primitive if exists a positive integer $p$ such that $H^p$ has all its entries positive). Indeed if $\varphi : \Sigma_A \to \mathbf{R}$ depends on two coordinates let $\mathbf{L}_\varphi = \mathbf{L}_{i,j} = \begin{cases} 0 & \text{if} \quad A_{i,j} = 0 \\ \exp \varphi(x) & \text{if} \quad A_{i,j} = 1 \end{cases}$, with $x_0 = i$, $x_1 = j$, for instance in the Ising model $\varphi(x) = Jx_0 x_1$ and $\mathbf{L}_{i,j} = \exp\left(Jx_i x_j\right)$. If we consider a "partition function" $Z_n(\varphi) = \sum_{x \in P_n(\sigma)} \exp\left(S_n(\varphi)(x)\right)$ then

$$Z_n(\varphi) = Tr\left(\mathbf{L}^n\right). \tag{13}$$

On the other hand the "thermodynamic limit" $\lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi)$ does exist and equals $\log E_1(\mathbf{L})$, where $E_1$ is the leading positive eigenvalue of $\mathbf{L}$ [26]. The existence of such a leading eigenvalue is ensured by the Perron–Frobenius theorem, since the matrix is primitive. For Hölder continuous potentials is valid $P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi)$ [16].

If we are in the more general situation of not having always potentials depending on a finite number of coordinates we must work with other class of objects than matrices. They will be *transfer operators*, in the style of those introduced by Ruelle in his thermodynamic formalism, and the aim will be to obtain an analogous relationship to (13) with the trace of the operator instead of the matrix.

Next we shall write down such an operator: for

$$\Sigma_A^+ = \left\{x = (x_i)_{i\in\mathbf{N}} : x_i \in \Omega, \forall i \in \mathbf{N}, A_{x_i, x_{i+1}} = 1\right\}$$

and $\varphi \in C\left(\Sigma_A^+\right)$, let

$$\mathcal{L}_\varphi(\varkappa)(x) = \sum_{i\in\Omega} A_{i,\kappa_0} \exp\left(\varphi(i, x)\right) \chi\left((i, x)\right), \tag{14}$$

where $(i, x)$ is the configuration $\left(i, x_0, x_{1,\ldots}\right)$. The space of finite range potentials, i.e. depending on a finite number of coordinates, is left invariant by $\mathcal{L}$ and so the operator can be reduced in this subspace to a matrix like $\mathbf{L}$ for which the relationship (13) is satisfied.

Let us return to the Kac model, in this case the transfer operator reads:

$$\mathcal{L}_{\varphi}(\varkappa)(x)=\sum_{i=\pm 1}\exp\left(Jx_0\sum_{n=1}^{\infty}x_n\lambda^n\right)\chi((i,x)). \tag{15}$$

Next we consider the space of functions $\mathcal{A}_{\infty}(\Sigma_A^+):=\{\varphi\in C(\Sigma_A^+)$ : there exists a $\chi\in\mathcal{A}_{\infty}(D_R)$ with $\varphi(x)=\chi(\pi(x))\}$, where $D_R=\{z:|z|=R\}$ and $\pi$ is a projection $\pi:\Sigma_A^+\to D_R$ defined by the assignation $x\longmapsto\sum_{n=1}^{\infty}x_{n-1}\lambda^n$. The space $\mathcal{A}_{\infty}(U)$ is the space of complex functions holomorphic in $U$ and continuous in $\overline{U}$ (the closure of $U$), endowed with the norm $\|\chi\|=\sup_{z\in D_R}|\chi(z)|$. On $\mathcal{A}_{\infty}\left(\Sigma_A^+\right)$the operator $\mathcal{L}_{\varphi}$ induces another one acting on $\mathcal{A}_{\infty}(D_R)$, also denoted by $\mathcal{L}_{\varphi}$, in the following way:

Let $\psi_j:D_R\to D_R$, $\psi_j(z)=\lambda(j+z)$, $j=\pm1$, and thus

$$\mathcal{L}_{\varphi}(\varkappa)(z)=\sum_{j=\pm1}\exp(Jxz)\,\chi\left(\psi_j(z)\right), \tag{16}$$

for $\chi\in\mathcal{A}_{\infty}(D_R)$.

By using the trace formula deduced from [17] we have

$$Z_n(\varphi)=\left(1-\lambda^n\right)Tr\left(\mathcal{L}_{\varphi}^n\right)=Tr\left(\mathcal{L}_{\varphi}^n\right)-Tr\left(\widetilde{\mathcal{L}}_{\varphi}^n\right),\ \text{with } \widetilde{\mathcal{L}}=\lambda\mathcal{L}, \tag{17}$$

what we were looking for, i.e. a relationship in the style of (13) with the operator playing the role of the matrix.

Now the task will be to develop a more general approach to obtain a similar result. For this we shall work in spin lattice systems modeled by finite subshift type $\left(\Sigma_A^+,\sigma\right)$ with potentials $\varphi:\Sigma_A^+\to\mathbf{R}$ for which the following conditions be satisfied:

($C1$) There is a projection $\pi:\Sigma_A^+\to\mathbf{R}^d$, for some $d\geq1$, and open sets $\{W_i\}\subset\mathbf{R}^d$ such that $\pi\left(\Sigma_A^+\right)\subset\bigcup_i W_i$ and maps $\psi_i:\bigcup_{j\in\Omega_i}W_j\to W_i$ ($\Omega_j:=\left\{i\in\Omega:A_{i,j}=1\right\}$. Besides $\pi(i,x)=\psi_i(\pi(x))\in\Sigma_A^+$, recall that $(i,x)$ is the configuration $\left(i,x_0,x_{1,\ldots}\right)$.

($C2$) There are neighborhoods $U_i\subset\mathbf{C}^d$ of $W_i$ such that each $\psi_i$ extends holomorphically to $\bigcup_{j\in\Omega_i}U_j$ and applies $\bigcup_{j\in\Omega_i}U_j$ strictly inside itself. By "strictly inside itself" we understand: let $D$ be a bounded connected subspace of a Banach space $B$ and $\psi$ a holomorphic map defined on $D$. We say that $\psi$ applies $D$ strictly inside itself if $\inf_{z\in D,\,z'\in B-D}\left\|\psi(z)-z'\right\|\geq\delta>0$.

($C3$) There exists holomorphic functions $\varphi_i$ defined on $U_i$ such that $\varphi(i,x)=\varphi_i(\psi_i(\pi(x)))$, for any $x\in\Sigma_A^+$.

These conditions allow to define a transfer operator by:

$$\mathcal{L}_\varphi : \bigoplus_{i\in\Omega} \mathcal{A}_\infty (U_i) \to \bigoplus_{i\in\Omega} \mathcal{A}_\infty (U_i) \tag{18}$$
$$\left(\mathcal{L}_\varphi (\chi)\right)_i (z) = \sum_{j\in\Omega} A_{i,j} \exp\left(\varphi_j \left(\psi_j (z)\right)\right) \chi\left(\psi_j (z)\right)$$

A trace formula for such an operator, in the style of the Atiyah–Bott formula on Lefschetz fixed point, is displayed in [17] as:

$$Tr\left(\mathcal{L}_\varphi\right) = \sum_{i\in\Omega} A_{i,i} \exp\left(\varphi_i\left(\overline{z}_i\right)\right) \frac{1}{\det\left(1 - D\psi_i\left(\overline{z}_\kappa\right)\right)}, \tag{19}$$

where $\overline{z}_i$ is the fixed point of $\psi_i$ and $D\psi$ is the differential map of $\psi$, seen as a linear operator. It must be pointed out that, by the Earle–Hamilton theorem [10] a map $\psi$ applying strictly a domain $D$ inside itself has exactly a fixed point $\overline{z} \in D$ with $\|D\psi\left(\overline{z}\right)\| < 1$.

A relevant fact about these transfer operators is that they are *nuclear*. Let us recall that an operator $\mathcal{L}$ acting on a Banach space $B$ is nuclear if there exist sequences $(x_n) \subset B$, $(f_n) \subset B^*$ (the dual space of $B$) with $\|x_n\| = 1$, $\|f_n\| = 1$ and numbers $(\rho_n)$ with $\sum\limits_{n=0}^{\infty} |\rho_n| < \infty$ such that $\mathcal{L}(x) = \sum\limits_{n=0}^{\infty} \rho_n f_n(x) x_n$ for every $x \in B$. The nuclearity of operators similar to (18) and also for those corresponding to a continuous case was established in [21, 22]. These proofs can be easily adapted to operators (18) and so we will omit it.

Let us consider now the family of operators $\mathcal{L}_q$, which are the transfer operators associated to the family of potentials $\{q\varphi\}$. In this case the condition ($C3$) is formulated as follows: there exist holomorphic functions $\varphi_{i,q}$ defined on $U_i$ such that $q\varphi(i, x) = \varphi_{i,q}\left(\psi_i\left(\pi(x)\right)\right)$, for any $x \in \Sigma_A^+$. These operators will be denoted by $\mathcal{L}_q$.

By the Grothendieck theory for nuclear operators [11, 12] the *Fredholm determinant* $\det(1 - z\mathcal{L}_q)$ is an entire map in both variables $z, q$ and it has the expansion $\det(1 - z\mathcal{L}_q) = \exp(-\sum\limits_{n=1}^{\infty} \frac{z^n}{n} Tr(\mathcal{L}_q^n))$. If the charts $\psi_i$, defined in ($C1$)–($C3$) are constant then by the Mayer trace formula it holds $Z_n(q) := Z_n(q\varphi) = Tr(\mathcal{L}_q^n)$, this is the case, for instance, for the Ising model and many other statistical systems. If the $\psi_i$ are linear, like in the Kac-model, there is also a relationship between the partition function $Z_n(q)$ and the trace of $\mathcal{L}_q^n$ in the style of (17). The general relationship between partition function and trace is

$$Z_n(q) = \sum_{p=0}^{d} Tr\left[\left(\mathcal{L}_q^{(p)}\right)^n\right], \tag{20}$$

where $\mathcal{L}_q^{(p)}$ are operators defined on $\bigoplus_{\kappa\in\Omega}\bigwedge_p \mathcal{B}(U_\varkappa)$, where $\bigwedge_p \mathcal{B}(U_i)$ is the space of the differential $p$-forms holomorphic on $U_i$, as

$$\mathcal{L}_q^{(p)}:\bigoplus_{i\in\Omega}\bigwedge_p \mathcal{B}(U_i)\to\bigoplus_{i\in\Omega}\bigwedge_p \mathcal{B}(U_i),\, U_i\subset\mathbf{C}^d$$

$$\left(\mathcal{L}_q^{(p)}\left(w_p\right)\right)_i(z)=\sum_{j\in\Omega}A_{i,j}\exp\left(\varphi_{j,q}(z)\right)\bigwedge_p D\psi_j(z)\left(w_p\right)\left(\psi_j(z)\right),$$

here $w_p\in\bigwedge_p \mathcal{B}(Ui)$ and $\bigwedge_p D\psi$ is the $p$-fold exterior product of the differential map $D\psi$ (considered as a linear operator). We have $\mathcal{L}_q^{(0)}=\mathcal{L}_q$ and any $\mathcal{L}_q^{(p)}$ is nuclear, as a natural extension of the fact that $\mathcal{L}_q^{(0)}$ does. Thus the Fredholm determinant $D_p(z,q):=\det(1-z\mathcal{L}_q^{(p)})$ is entire in $z$ and $q$, for any $p$.

Now for $p=0$, $d=1$ and constant charts there is an obvious and direct relationship between the Fredholm determinant and the *Ruelle zeta function* [26] which is defined as

$$\zeta(z,q)=\zeta_\varphi(z,q)=\exp\left(\sum_{n=1}^{\infty}\frac{z^n}{n}Z_n(q)\right).$$

We have then $\zeta(z,q)=\frac{1}{D_0(z,q)}$. If the charts are linear we obtain an expression of the partition function as the difference of $Tr(\mathcal{L}_q^n)$ and a constant by $Tr(\mathcal{L}_q^n)$, like in (17) for the Kac-model. So that in this case are also related the determinant and zeta. For $d\geq 2$ the connection comes from (20).

Another result about the transfer operators $\mathcal{L}_q$ is the relationship between the spectral radius $\rho(\mathcal{L}_q)$ and the topological pressure, which is $\rho(\mathcal{L}_q)=\exp(P(q\varphi))$. This was proved by Ruelle for the operators (14) and for operator similar to (17) in the above quoted references. To obtain an expression in terms of the free energy $T(q)$ we just consider renormalized operators $\exp(-qP(\varphi))\mathcal{L}_q$ and so the leading eigenvalue results $\exp(T(q))$. For simplicity, we also denote the renormalized operators by $\mathcal{L}_q$. In [21, 22] it was established the analyticity of the map $q\longmapsto\rho(\mathcal{L}_q)$, provided conditions in the style of $(C1)$–$(C3)$ were fulfilled, and consequently the absence of phase transitions.

The following proposition will be useful to obtain a description of the transfer operators spectrum.

**Proposition 2** *The spectrum of the operators $\mathcal{L}=\phi C_\psi$, where $C_\psi$ is the composition operator $C_\psi(\chi)(z)=(\chi\circ\psi)(z)$, acting on the space of functions $\mathcal{A}_\infty(U)$ is discrete. It consists in eigenvalues $E_n=\left\{\phi(\overline{z})\left(D\psi(\overline{z})\right)^n\right\}$ where $\overline{z}$ is a fixed point of $\psi$ together with 0 as unique accumulation point.*

*Proof* The fact that the operators $\mathcal{L}=\phi C_\psi$ have discrete spectrum is actually due to [17]. Let $\psi\in\mathcal{A}_\infty(D)$, we have the eigenvalue equation $\mathcal{L}\chi(z)=\phi(z)\chi(\psi(z))=E\chi(z)$. Clearly if $\chi(\overline{z})\neq 0$ then an eigenvalue of $\mathcal{L}$ is $E=$

$\phi(\overline{z})$, where $\overline{z}$ is a fixed point of $\psi$. If $\chi(\overline{z}) = 0$ then differentiating, with respect to $z$, the above eigenvalue equation is obtained.

$$D\phi(\overline{z}) \times \chi(\overline{z}) + \phi(\overline{z}) \times D\chi(\overline{z})\, D\psi(\overline{z}) = ED\psi(\overline{z}).$$

Thus if $D\phi(\overline{z}) \neq 0$ then $E = \phi(\overline{z})\, D\psi(\overline{z})$. Now the set of eigenvalues of $\mathcal{L}$ (recall that the spectrum is discrete) is

$$E_n = \left\{\phi(\overline{z})\,(D\psi(\overline{z}))^n\right\}.$$

Recall that by the Earle-Hamilton theorem $\|D\psi(\overline{z})\| < 1$, therefore 0 is the only point of accumulation .

Notice that $Tr(\mathcal{L}) = \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \phi(\overline{z})\,(D\psi(\overline{z}))^n = \frac{\phi(\overline{z})}{\det(1-D\psi(\overline{z}))}$, the Mayer trace formula. □

*Remark* The above result describes indeed the spectrum of the transfer operators since they are finite sums of composite ones.

Now we shall show that the Ruelle zeta function determines the equilibrium state for a broader class of potentials than in [25].

**Proposition 3** *Let $f : X \to X$ be an expansive homeomorphism in a compact metric space with the specification property and let $\varphi_1, \varphi_2 \in \nu_f(X)$. Under these conditions holds $\zeta_{\varphi_1}(z, q) = \zeta_{\varphi_2}(z, q) \Longrightarrow \mathcal{S}_{\varphi_1} = \mathcal{S}_{\varphi_2}$ ($\mathcal{S}_{\varphi_1}, \mathcal{S}_{\varphi_2}$ are the unmarked orbit spectra of the potentials $\varphi_1, \varphi_2$ as defined at the end of Section 2).*

*Proof* We have $\zeta_\varphi(z, q) = \exp(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(q))$, with $Z_n(q) = \sum_{x \in P_n(f)} \exp(S_n(q\varphi)(x))$. The power expansion determines an analytical function in the disc $|z| < \exp(T(q))\exp(-qP(\varphi))$. If we have an expression of the form $B(q) = \sum_{i=1}^{N} \lambda_i^q$, $\lambda_i > 0$, then from Newton identities we deduce that $B(q)$ uniquely determines the $\lambda_i$, it just needs to know $B(1)$, $B(2)$,..., $B(N)$. This can be applied to the finite sum $\sum_{x \in P_n(f)} \left[\exp(S_n(\varphi(x)))\right]^q$ and so $Z_n(q)$ uniquely determines the terms $S_n(\varphi(x))$, in turn the coefficients $Z_n(q)$ are recovered from the expansion by differentiation with respect to $q$. In this way the spectrum $\mathcal{S}_\varphi$ is uniquely determined from the zeta function. □

**Definition** A matrix $H = (a_{i,j})$ is *typical* when the numbers $\log a_{i,j}$ are rationally independent, or equivalently if no non trivial product of powers of the $a_{ij}$'s (with integer exponents) is equal to 1.

Now we state the main result of this section:

**Theorem 3** *For spin lattice systems and potentials for which conditions* $(C1)$–$(C3)$ *are fulfilled, let* $H = (a_{i,j})$ *be a* $N \times N$ *matrix with* $\prod_{i=1}^{N} E_i = \sum_{\sigma \in P_n} a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}$, *where* $E_i = E_i(q)$ *are the eigenvalues of the transfer operator* $\mathcal{L}_q = \mathcal{L}_{q\varphi}$ *(see (18) for the definition). If H is a typical matrix, then the phenomenon of multifractal rigidity is verified, i.e., the multifractal spectrum* $\mathcal{E}_\varphi(\alpha)$ *(c.f. (7)) determines the spectrum* $\mathcal{S}_\varphi$.

*Proof* The scheme of proof is as follows. Firstly it is naturally established that the multifractal spectrum of local entropies determines the free energy $T_\varphi(q)$, since it is the Legendre transform of the spectrum map $\mathcal{E}_\varphi(\alpha)$. Then we consider the Fredholm determinant $D(z,q)$ and the map $\beta(q) = \frac{1}{\rho(\mathcal{L}_q)} = \exp(-T(q))$, so that $D(\beta(q), q) = 0$. Let $P(z)$ be an analytic map such that $P(\beta(q)) = 0$ and with $\beta(q)$ determining $P$. We show that $P(z)$ is a factor of $D(z,q)$, but we also will prove that it is not possible to write $D(z,q) = P(z,q)\,Q(z,q)$, where $P, Q$ are non-constant maps. So that the Fredholm determinant is in some sense "minimal", and then $\beta(q)$ determines the Fredholm determinant. By the relationship of $D(z,q)$ with the zeta function and by Proposition 3, the claim of the theorem will be proved.

For the above procedure we use an approach based on Tuncel developments which combines algebraic and dynamic technics [31]. Let $\mathcal{R} = \{\sum_{i=0}^{k} n_i a_i^q : n_i \in \mathbf{Z},\, a_i > 0\}$, if we set $\exp = \left\{a^q : a \in \mathbf{R}^+\right\}$ then $\mathbf{Z}\left[\exp\right] = \mathcal{R}$, i.e. $\mathcal{R}$ is the ring of integral combinations of elements in exp, or we can write

$\mathcal{R} = \{\beta : \mathbf{R} \to \mathbf{R} : \beta(q) = \sum_{i=0}^{k} n_i a_i^q\}$. If the potential $\varphi$ depends on a finite number of coordinates, for instance $\varphi = \varphi\left(x_i, x_j\right)$, then it can be defined a family of matrices $H(q)$ with coefficients in $\mathcal{R} = \mathbf{Z}\left[\exp\right]$ by $H(q) = \begin{cases} 0 & \text{if} \quad A_{i,j} = 0 \\ \exp^q \varphi(x) & \text{if} \quad A_{i,j} = 1 \end{cases}$, with $x_0 = i$, $x_1 = j$. If $\beta(q) = \beta_A(q) = \rho(A(q))$, it is proved in [31] that $\beta(q)$is analytic and $\beta_A(1) = \log E_1$, where $E_1$ is the leading eigenvalue of $A = A(1)$, existing by the Perron–Frobenius theorem.

In our case with a potential which in general depends on the whole configuration we shall take $\beta(q) = \frac{1}{\rho(\mathcal{L}_q)} = \exp(-T(q))$, which as we point out was proved to be analytic and verifies $D(\beta(q), q) = 0$. Recall that by Proposition 2 the transfer operators have discrete spectrum and so we can put $D(z,q) = \det\left(1 - z\mathcal{L}_q\right) = \prod_{n=1}^{\infty}(1 - zE_n(q))$, where $E_1(q) = \exp(T(q))$, so that the $z-$zeros of the Fredholm determinant are the inverses of the eigenvalues of $\mathcal{L}_q$.

As we anticipate at the beginning of the proof we consider a map $P(z,q)$ with $P(\beta(q), q) = 0$, analytic in $z$ and whose expansion has coefficients in $\mathcal{R}$. Let $\mathcal{F}$ be the field of fractions of $\mathcal{R}$ and let $\mathcal{G}$ be the set of expansions of analytic maps with coefficients in $\mathcal{F}$. We consider an ideal $\mathcal{I}$ in $\mathcal{G}$ given by $F \in \mathcal{I}$ if and only if $F$ can be expressed as $F = Q/R$ where $Q = Q(z,q)$is an analytic map in $z$ with expansion with coefficients in $\mathcal{R}$ and $Q(\beta(q), q) = 0$ for some analytic function $\beta(q)$ and $R \in \mathcal{R}$. By the analyticity of $\beta(q)$ the choice does not depend on $R$. So $\mathcal{I} = \{F : \ F \text{ can be expanded with coefficients in } \mathcal{F}, \text{and } F(\beta(q), q) = 0\}$. Let $\mathcal{I} = P\mathcal{G}$ for some $P$ with coefficients in $\mathcal{F}$, we shall show that the expansion has actually coefficients in $\mathcal{R}$. We have that the Fredholm determinant belongs to $\mathcal{I}$ and so it can be written: $D(z,q) = P(z,q)\,Q(z,q)$, where $P$ and $Q$ have coefficients in $\mathcal{F}$ and $D$ has expansion in $\mathcal{R}$. We then have

$$D = \sum_{n=0}^{\infty} a_n z^n, \text{ with } a_n = \sum_{i_n \in I_n} M_{i_n} A_{i_n}^q \in \mathcal{R},\ I_n \text{ finite},$$

$$P = \sum_{n=0}^{\infty} b_n z^n, \text{ with } b_n = \frac{\sum_{j_n \in J_n} N_{j_n} B_{i_n}^q}{\sum_{j_n \in J_n} N_{j_n}' B_{i_n}'^q} \in \mathcal{F},\ J_n \text{ finite},$$

$$Q = \sum_{n=0}^{\infty} c_n z^n, \text{ with } c_n = \frac{\sum_{\ell_n \in L_n} U_{\ell_n} C_{\ell_n}^q}{\sum_{\ell_n \in L_n} U_{i_n}' C_{i_n}'^q} \in \mathcal{F},\ L_n \text{ finite}.$$

For any positive integer $n$ let $S_n$ be the subgroup of $\mathbf{R}^+$ generated by $A_{i_n}, B_{j_n} B_{j_n}', C_{\ell_n}, C_{i_n}'$ and $\mathbf{Z}[S_n]$ is a unique factorization domain. We have $a_0 + a_1 z + \ldots + a_n z^n = (b_0 + b_1 z + \ldots + a_r z^r)\left(c_0 + c_1 z + \ldots + c_{n-r} z^{n-r}\right)$, then each $b_i$ can be expressed as $b_i = \widetilde{b}_i / b$ with $\widetilde{b}_i \in \mathbf{Z}[S_n]$ as well as any $c_i = \widetilde{c}_i / c$ with $\widetilde{c}_i \in \mathbf{Z}[S_n]$ and for some $b, c$ such that $\left(b, \widetilde{b}_1, \ldots, \widetilde{b}_r\right) = 1$, $(c, \widetilde{c}_1, \ldots, \widetilde{c}_{n-r}) = 1$. Hence the following expression is an equation in $\mathbf{Z}[S_n]\, b\, c\, (a_0 + a_1 z + \ldots + a_n z^n) = \left(\widetilde{c}_0 + \widetilde{c}_1 z + \ldots + \widetilde{c}_{n-r} z^{n-r}\right)\left(\widetilde{b}_0 + \widetilde{b}_1 z + \ldots + \widetilde{b}_r z^r\right)$, since $\mathbf{Z}[S_n]$ is a unique factorization domain each factor of $bc$ must divide all the $\widetilde{b}_i$ or all the $\widetilde{c}_i$, and besides is invertible. Thus $c$ is a "monomial" and so $P$ has actually coefficients in $\mathcal{R}$. Therefore if $P(z,q)$ has coefficients in $\mathcal{R}$ and $\beta(q)$ is a $z-$zero of $P$ then this map is a factor of the Fredholm determinant $D(z,q)$.

Next we prove the "minimality" of the Fredholm determinant, we consider a "truncation" $D_N(z,q) := \prod_{n=1}^{N} (1 - zE_n(q)) \in \mathcal{R}[z]$. In this way $D_N(z,q) = 1 + (\sum_i E_i) z + (\sum_{i,j} E_i E_j) z^2 + \ldots + [(-1)^n \prod_i E_i] z^N$. Another expression for the Fredholm determinant is $D(z,q) = 1 + \sum_{n=1}^{\infty} D_n(q)\, z^n$, where $D_n(q) = \sum_{\substack{(i_1,\ldots,i_m)\\ i_1+\ldots+i_m=n}} \frac{(-1)^m}{m!} \prod_{j=1}^{m} \frac{1}{i_j} Tr(\mathcal{L}_q^{i_j})$, so that $D_N(z,q) = 1 + Tr(\mathcal{L}_q)z + Tr(\mathcal{L}_q^2)z + \ldots + \Big[ \sum_{\substack{(i_1,\ldots,i_m)\\ i_1+\ldots+i_m=n}} \frac{(-1)^m}{m!} \prod_{j=1}^{m} \frac{1}{i_j} Tr(\mathcal{L}_q^{i_j}) \Big] z^N$.

Let us assume that $D(z,q) = P(z,q)\,Q(z,q)$, as we have seen $P, Q$ have expansions with coefficients in $\mathcal{R}$ if $D(z,q)$ does. We compare the coefficients in each $N-$truncation of $D$ and $P.Q$. Thus $D_N(z,q) = 1 + (\sum_i E_i)z + (\sum_{i,j} E_i E_j)z^2 + \ldots + [(-1)^n \prod_i E_i]z^N = [\sum_{j_0\in J_0} N_{j_0} B_{i_0}^q + (\sum_{j_1\in J_1} N_{j_1} B_{i_1}^q)z + \ldots + (\sum_{j_r\in J_r} N_{j_r} B_{i_r}^q)z^r] \times [\sum_{\ell_0\in L_0} U_{\ell_0} C_{\ell_0}^q + (\sum_{\ell_1\in L_1} U_{\ell_1} C_{\ell_{n1}}^q)z + \ldots + (\sum_{\ell_{N-r}\in L_{N-r}} U_{\ell_{N-r}} C_{\ell_{N-r}}^q)z^{N-r}]$.

Notice that the product of the eigenvalues $E_i$, $i = 1, \ldots, N$ can be considered as the determinant of certain $N \times N-$matrix $H = (a_{i,j})$, so $\prod_{i=1}^{N} E_i = \sum_{\sigma\in P_n} a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}$, where $E_i = E_i(q)$, $a_{i,j} = a_{i,j}(q)$ and $P_n$ is the group of permutations of $n-$elements. Besides $\sum_{i=1}^{N} E_i = Tr(H) = \sum_i a_{i,i}$. Since $H$ is typical:

$$a_{i_1,j_1}^{n_1} a_{i_2,j_2}^{n_2}, \ldots, a_{i_k,j_k}^{n_k} \neq 1 \text{ for any } (i_1, i_2, \ldots, i_k)\,(j_1, j_2, \ldots, j_k) \text{ and } (n_1, n_2, \ldots, n_k) \in \mathbf{Z}^k. \tag{21}$$

The coefficient of $z^r$ in the expansion of $D(z,q)$ is of the form

$$\frac{a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}}{a_{i_1,i_1} \ldots a_{i_r,i_r}}, \tag{22}$$

where $\sigma \in P_n$ fixes $(i_1, \ldots, i_r)$, and the coefficient of $z^{N-r}$ is of the form

$$\frac{a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}}{a_{i_1,i_1} \ldots a_{i_{N-r},i_{N-r}}} \tag{23}$$

with $\sigma \in P_n$ fixing $(i_1, \ldots, i_{N-r})$.

Then, we have $\sum_{\sigma\in P_n} a_{1,\sigma(1)} \ldots a_{N,\sigma(N)} = \sum_{j_r,\ell_{N-r}} N_{j_r} U_{\ell_{N-r}} B_{j_r}^q C_{\ell_{N-r}}^q$, so that there is a correspondence between $a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}$ and the coefficients $B_{i_r}^q C_{\ell_{N-r}}^q$. Thus comparing the coefficients of $z^r$ we have $B_{j_r}^q C_{\ell_0}^q = \frac{a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}}{a_{i_1,i_1} \ldots a_{i_r,i_r}}$ and also a similar expression for $z^{N-r}$. If $\sigma \in P_n$ does not have fixed points then $a_{1,\sigma(1)} \ldots a_{N,\sigma(N)}$ appears in the constant term of the development of the $D(z,q)$, but it is not possible to write it as a product of the coefficients $B_{j_r}^q C_{\ell_{N-r}}^q$. To illustrate this, consider the cyclic permutation $\overline{\sigma} = (1,2,3)$ and the sum $\sum_{\sigma\in P_3} a_{1,\sigma(1)} a_{2,\sigma(2)} a_{3,\sigma(3)}$, which of course includes $\overline{\sigma}$. The coefficient of $z^2$ is a sum of terms $a_{i.j} a_{j,i}$ and $a_{i.i} a_{j,j}$. Now $a_{1,2} a_{2,3} a_{3,4}$ must be of the form $a_{i.j} a_{j,i} a_{m,n}$, which could not be possible by (21). $\square$

As we have pointed out Pollicott and Weiss provided examples of locally constant potentials in which the rigidity phenomenon is not verified. More specifically they found finite range potentials $\varphi_1$ ,$\varphi_2$ with the same free energy but non-equivalent in the sense that $\varphi_1$ is not cohomologous to $\varphi_2 \circ \tau$, where $\tau$ is a homeomorphism which commutes with the Bernoulli shift. To ensure the rigidity is imposed the condition of genericity (see [25] for the definition) on the matrix $\mathbf{L}_\varphi$ (defined at the beginning of this section) associated to the potential $\varphi$.

When the transfer operator $\mathcal{L}_q$ is restricted to the set of locally constant potentials it is reduced to a matrix $\mathbf{L}_\varphi$. Now if the genericity condition is imposed on the matrix $H$, originated by the truncation of the Fredholm determinant, then $D_N(z, q)$ determines the matrix and the potential. The genericity condition allows to recover the coefficients of $D_n(q)$ in the expansion of $D_N(z, q)$.

*Conclusion* The local multifractal rigidity was proved on weaker conditions than those of [25], say expansiveness and specification for the dynamics and potentials belonging the bounded distortion class, instead of the Hölder continuous maps which are included in our wider class. On the other hand was proved a rigidity phenomenon for long range potentials, so extending the results of [25], valid for generic finite range potentials.

## 5 A Case with Infinite Alphabet

We consider now a lattice system with countable spins, i.e. a system modelled by a Markov subshift $\Sigma_A^+ = \left\{x = (x_i)_{i\in\mathbf{N}} : x_i \in I, \forall i \in \mathbf{N}, A_{x_i,x_{i+1}} = 1\right\}$, $I$ infinite countable. Let $f : [0, 1] \to [0, 1]$ be the Gauss map, i.e $f(t) = \frac{1}{t} \bmod 1$. If any $t \in [0, 1]$ is represented by its continued fraction $t = \frac{1}{i_1 + \frac{1}{i_2 + \frac{1}{i_3 + \cdots}}}$, then the assignation $t \rightleftharpoons \iota = (i_n)_{n\in\mathbf{N}}$ gives a symbolic representation of the dynamical system $(I, f)$. More generally if $f$ is an analytic expanding map a symbolic representation is obtained via Markov partitions. We consider the potential $\varphi(t) = \log\left|f'(t)\right|$ and the spin system induced the Gauss map in the way described above.

For every positive integer $n$ holds: $f^n(t) = t$ if and only if $i_n = i_{n+k}$, for every $k$, where $\iota = (i_n)$ is the infinite sequence associated to $t$. Hence the following notation can be introduced: for any number $t$ with period $n$, with respect to $f$, the associated sequence will be denoted by $[i_1...i_n]$.

The partition function for the system $(I, f)$ reads:

$$Z_n(q) = \sum_{x\in P_n(f)} \prod_{j=0}^{n-1} \left|\left(f^{'}\right)^j(t)\right|^q, \tag{24}$$

setting $\phi_q := \exp(q\varphi)$ and replacing $t$ by its symbolic representation from the continued fraction, the partition function for the associated spin system can be expressed as:

$$Z_n(q) = \sum_{(i_1,\ldots,i_n)} \prod_{j=0}^{n-1} \phi_q\left(\left[i_{1+k}\ldots i_n, i_1, .., i_k\right]\right). \tag{25}$$

the convergence, i.e. the existence of the "thermodynamic limit", is ensured if$\left|\phi_q(t)\right| \sim |t|^{\gamma}$, as $t \to 0$, for some $\gamma = \gamma(q) > 1$. This condition is satisfied with $\gamma = 2q$.

To define the transfer operators let us consider the Markov partition $\mathcal{P} = \{I_n = [\frac{1}{n+1}, \frac{1}{n})\}_{n\in\mathbf{N}}$, we have $f\mid_{I_n}(t) = \frac{1}{t} - n$, so $f\mid_{I_n}$ is analytic if $t \neq 0$. and $|(f^2)'| \geq 4$. For the special case where we have a Markov partition $\mathcal{P} = \{I_n\}$ for expanding analytic maps the charts $\psi_n$ can be defined as the branches of $f\mid_{I_n}$, in our case it is $\psi_n(t) = \frac{1}{t+n}$, being $\phi_q \circ \psi_n$ analytic in a complex neighborhood of each $I_n$. Now the transfer operator becomes:

$$\mathcal{L}_q(\varkappa)(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n}\right)^{2q} \chi\left(\frac{1}{z+n}\right), \tag{26}$$

where $q$ must be $> \frac{1}{2}$ by convergence reasons already mentioned. These operators are proved to be nuclear in some adequate functional space, indeed for this can be taken $\mathcal{A}_\infty(U)$ such that $\psi_n(\overline{U}) \subset U$ and $\phi_q \circ \psi_n \in \mathcal{A}_\infty(U)$. The open complex set $U$ can be choose as the disc $|z-1| < \frac{3}{2}$.

Therefore in this particular case and for the temperature parameter $q > \frac{1}{2}$ the results about multifractal rigidity valid for the finite alphabet case can be extended to infinite spin systems following the scheme of the earlier section.

More general cases are found by considering the so called *boundary hyperbolic maps*, which are functions originated by the action of Kleinian finitely generated groups $\Gamma$ on the hyperbolic disc $H^2$ such that to any point in the limit set $\Lambda$ of this action can be assigned a sequence in the generators of the group. These maps $f : \Lambda \to \Lambda$ were introduced, to codify hyperbolic geodesics, by Series [27, 28] who proved that the system $(\Lambda, f)$ has a Markov partition which leads to symbolic representation by a subshift with an alphabet which in general does not agree with the generator set of the group. The alphabet is infinite if and only if $\Gamma$ contains parabolic elements. The Gauss map is a special case of such a map, corresponding to the action of $\Gamma = SL_2(\mathbf{Z})$. For the connection of the boundary hyperbolic maps with multifractal analysis one can see [18, 23].

## Appendix

### Topological Entropy for Non-Compact Nor Invariant Sets

Let $f : X \to X$ be a continuous map and $(X, d)$ a compact metrisable space. Let $\mathcal{U} = \{U_1, ..., U_N\}$ be a finite covering of $X$ . A string is defined as a sequence $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ such that $\left\{U_{\ell_0}, U_{\ell_1}, ..., U_{\ell_{n-1}}\right\} \subset \mathcal{U}, \ell_i \in \{1, 2, ..., N\}$. The length of the string $L = (\ell_0, \ell_1, ..., \ell_{n-1})$ is denoted by $n(L) = n$. Let call $W_n(\mathcal{U})$ the set of the strings $L$ with length $n$ for the covering $\mathcal{U}$.

Let

$$X(L) = U_{\ell_0} \cap f^{-1}\left(U_{\ell_1}\right) \cap ... \cap f^{-n+1}\left(U_{\ell_{n-1}}\right),$$

if $Z \subset X$ we say that $\Pi = \{L = (\ell_0, \ell_1, ..., \ell_{n-1})\}$ covers $Z$ if

$$Z \subset \bigcup_{L \in \Pi} X(L).$$

For any real number $s$:

$$M(Z, \mathcal{U}, s, n) = \inf_{\Gamma} \sum_{L \in \Pi} \exp(-sn(L)),$$

where the infimum is taken over all collections of strings $\Pi \subset W_n(\mathcal{U})$ which cover $Z$.

Let

$$M(\mathcal{U}, Z, s) = \lim_{n \to \infty} M(\mathcal{U}, Z, s, n).$$

There is a unique number $\bar{s}$ such that $M(\mathcal{U}, Z, .)$ jumps from $+\infty$ to 0, now let

$$h_{\text{top}}(f, Z, \mathcal{U}) = \bar{s} = \sup\{s : M(\mathcal{U}, Z, s) = +\infty\} = \inf\{s : M(\mathcal{U}, Z, s) = 0\}. \tag{27}$$

Finally the number

$$h_{\text{top}}(f, Z) = \lim_{\Delta(\mathcal{U}) \to 0} h_{\text{top}}(f, Z, \mathcal{U}), \ \Delta(\mathcal{U}) = \text{diameter of } \mathcal{U} \tag{28}$$

is the *topological entropy* of $f$ restricted to $Z$.

### Gibbs Measures in Lattice Spin Systems

We present here a formulation of the notion of Gibbs states in lattice spin models: Let $X$ be "the configuration space" which, as we already pointed out, is mathematically described as the set $\Sigma_A = \left\{x = (x_i)_{i \in \mathbf{Z}} : x_i \in \Omega, \forall i \in \mathbf{Z}, A_{x_i, x_{i+1}} = 1\right\}$, where $A$ is a $k \times k$ matrix with 0, 1 entries and $\Omega = \{0, 1, 2, ..., k-1\}$. The integers $i$ are called the *sites* and the corresponding coordinate $x_i$ the *spin* at the site $i$. The matrix $A$ indicates which configurations are allowed. From $A$ is defined a probability vector $p=(p_0,p_1,...,p_{k-1})$ and a stochastic matrix $E=\left(E_{i,j}\right)_{i,j=0,...,k-1}$ $(\sum_j E_{i,j}=1)$ such that $\sum_i p_i E_{i,j}=p_j$. (see e.g. [32]). This space is equipped with the $\sigma$-álgebra $\mathcal{B}$ generated by the semi-algebra of elementary cylinders: $C^{-m,...,m}_{\alpha_{-m},...,\alpha_m}=\{x\in\Sigma_A : x_i=\alpha_i,\ i=-m,...,m\}$. The Gibbs states will be probability measures defined on $(\Sigma_A,\mathcal{B})$. In this space are considered as dynamics the shift $\sigma:\Sigma_A\to\Sigma_A, \sigma x=x^{'}$, where $x^{'}_i=x_{i+1}$. A Gibbs state in the space of symbolic dynamics is the product measure defined on cylinders by

$$\mu\left(C^{-m,...,m}_{\alpha_{-m},...,\alpha_m}\right)=p_{\alpha_{-m}}E_{\alpha_{-m},\alpha_{-m+1}}...E_{\alpha_{m-1},\alpha_m}. \tag{29}$$

For a potential $\varphi\in C(\Sigma_A)$, which physically can be interpreted as a description of the interaction energy between one spin and the remaining, the statistical sum $S_n(\varphi)(x)=\sum_{i=0}^{n-1}\varphi\left(\sigma^i(x)\right)$ can be decomposed as $S_n(\varphi)(x)=H(x_0,x_1,...,x_{n-1})+W(x_0,x_1,...,x_{n-1}\mid x_n,x_{n+1},...)$ [26], where $H$ describes the energy of the spins $x_0,x_1,...,x_{n-1}$ and $W$ the interaction of $x_0,x_1,...,x_{n-1}$ and the spins $x_n,x_{n+1},....$ For a configuration $x$, let us denote by $x^{(n)}$ any member of $\Sigma_A$ with $x^{(n)}_i=x_i$, this is the election of a *boundary condition* for the system. The partition function is defined now as: $Z_n(\varphi)=\sum_{i_{0,...,}i_{n-1\in\Omega}}\exp\left(S_n(\varphi)\left(x^{(n)}\right)\right)$. Finally Gibbs states are defined as measures which satisfy the equation:

$$d\mu\left(x_{0,...,}x_{n-1}\right)=Z_n(\varphi)^{-1}\exp\left(S_n(\varphi)\left(x^{(n)}\right)\right), \tag{30}$$

for any configuration $x$ and every $n\in\mathbf{N}$.

The parallelism between shift dynamical systems and statistical mechanics of spin systems by interpreting the symbolic sequences as spin configurations over the lattice $\mathbf{Z}$ was done by Sinai [29]. The above analysis is rooted in his ideas.

The definition of Gibbs states used in a "probabilistic context" is usually given as follows: let $\Lambda$ be a finite subset of $\mathbf{Z}$ and let us denote $x_\Lambda=(x_i)_{i\in\Lambda}$, the projection of the sequence $x$ on $\Lambda$. For prescribed conditional probabilities $P(x_\Lambda\mid x_{\Lambda^C})$ let $H_\Lambda(x)$ be the Hamiltonian describing the energy excess of $x$ over the energy of $x_{\Lambda^C}$, which will be done by

$$P(x_\Lambda\mid x_{\Lambda^C})=\frac{1}{Z_\Lambda}\exp\left(-H_\Lambda(x)\right), \tag{31}$$

where $Z_\Lambda$ is the partition function. Here the inverse of the temperature $\beta$ is summed into $H_\Lambda$, or alternatively can be taken units in such a way that $\beta=1$. Thus a probability measure $\mu$ is a Gibbs state for a family of conditional probabilities $P(x_\Lambda\mid x_{\Lambda^C})_{\Lambda\text{ finite}\subset\mathbf{Z}}$ if $\mu\left(x_\Lambda\text{ occurs in }\Lambda\mid x_{\Lambda^C}\text{ occurs in }\Lambda^C\right)=P(x_\Lambda\mid x_{\Lambda^C})$, for every $x$ $\mu$-a.e.

To compare this definition with the earlier one notice these analogies: the Hamiltonian $H_\Lambda$ has its correlate in the statistical sum for the potential $\varphi$, the finite set $\Lambda$ indicates the sites for a finite set of spins and $\Lambda^C$ the remaining. The partition function $Z_\Lambda$ in (31) is obtained by summing over all the configurations $x^{'}$ which agree with $x$ in $\Lambda^C$, while in the definition as function of the potential $\varphi$ the summation is over the sites whose spins agree with a configuration $x$. This establishes a correlation between both expressions, the summation indexes in $Z_\Lambda$ correspond to the boundary conditions in $Z_n(\varphi)$.

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