Physical peculiarities of divergences emerging in q-deformed statistics

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Abstract

It was found in [Europhysics Letters **104**, (2013), 60003] that classical Tsallis theory exhibits poles in the partition function $\mathcal{Z}$ and the mean energy $<\mathcal{U}>$. These occur at a countably set of the q-line. We give here, via a simple procedure, a mathematical account of them. Further, by focusing attention upon the pole-physics, we encounter interesting effects. In particular, for the specific heat, we uncover hidden gravitational effects.

K-WORDS: Tsallis entropy, divergences, partition function, specific heat.

1 Introduction

Generalized or q-statistical mechanics à la Tsallis has generated manifold applications in the last 25 years [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. It has been shown (see for instance, [12, 13]) that the Tsallis q-statistics is of great importance for dealing with some astrophysical issues involving self-gravitating systems [14]. Moreover, this statistics has proved its utility in variegated scientific fields, with several thousands publications and authors [2], so that studying its structural features is an important issue for physics, astronomy, biology, neurology, economics, etc. [1]. The success of the q-statistics reaffirms the well grounded notion asserting that there is much physics whose origin is of purely statistical nature (not mechanical). As a spectacular example, me mention the application of q-ideas to high energy experimental physics, where the q-statistics appears to adequately describe the transverse momentum distributions of different hadrons [15, 16, 17].

In this work we show that as yet unexplored gravitational effects characterize this q-theory on account of *divergences* that, in some circumstances, emerge, within the q-statistical framework, in both the mean energy and the partition function.

Divergences are an important topic in theoretical physics. Indeed, the study and elimination of divergences of a physical theory is perhaps one of the most important aspects of theoretical physics. The quintessential typical example is the attempt to quantify the gravitational field, which so far has not been achieved. Some examples of elimination of divergences can be seen in references

[18, 19, 20, 21, 22].

We will use here an extremely simplified version of the ideas of [18, 19, 20, 21, 22] in connection with Tsallis q-statistics [1, 2], with emphasis in its applicability to gravitational issues [12, 13], in particular self-gravitating systems [14]. We will see that the removal of the above mentioned divergences leads to illuminating insights.

2 The divergences of q-statistics

As we have shown in [23], the q-partition function of the *classical* Harmonic Oscillator (HO) in ν dimensions can be written in the form

$$\mathcal{Z} = \frac{\pi^\nu}{\Gamma(\nu)} \int\limits_0^\infty \frac{u^{\nu-1}}{[1+\beta(q-1)u]^{\frac{1}{q-1}}} du, \tag{2.1}$$

where u refers to the phase space energy and β is the inverse temperature. The result of integral (2.1) is, according to [24],

$$\mathcal{Z} = \frac{\pi^\nu}{[\beta(q-1)]^\nu} \frac{\Gamma\left(\frac{1}{q-1}-\nu\right)}{\Gamma\left(\frac{1}{q-1}\right)} \tag{2.2}$$

This result is valid for $q \neq 1$ and we have selected $1 \leq q < 2$. Of course, $q = 1$ is the Boltzmann statistics instance, for which the q-exponential transforms itself into the ordinary exponential function (and the integral (2.1) is convergent). According to (2.2), the singularities (divergences) of (2.1) are given by the poles of the Γ function that appears in the numerator of (2.2), i.e., for

$$\frac{1}{q-1} - \nu = -p \quad \text{for} \quad p = 0, 1, 2, 3, ...,$$

or, equivalently, for

$$q = \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, ..., \frac{\nu}{\nu-1}, \frac{\nu+1}{\nu}$$

In a similar way, we have for the q-mean energy of the HO,

$$<\mathcal{U}>= \frac{\pi^\nu}{\Gamma(\nu)\mathcal{Z}} \int\limits_0^\infty \frac{u^\nu}{[1+\beta(q-1)u]^{\frac{1}{q-1}}} du \tag{2.3}$$

The result of (2.3) is, using [24] once again,

$$<\mathcal{U}>= \frac{\nu\pi^\nu}{\mathcal{Z}[\beta(q-1)]^{\nu+1}} \frac{\Gamma\left(\frac{1}{q-1}-\nu-1\right)}{\Gamma\left(\frac{1}{q-1}\right)}, \tag{2.4}$$

where we assume that $\mathcal{Z}$ is the physical partition function, which has no singularities. In this case, the singularities of (2.4) are given by:

$$\frac{1}{q-1} - \nu - 1 = -p \quad \text{for } p = 0, 1, 2, 3, ...,$$

or, equivalently,

$$q = \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, ..., \frac{\nu+1}{\nu}, \frac{\nu+2}{\nu+1}.$$

As usual [25], in terms of the so-called q-logarithms [1] $\ln_q(x) = \frac{x^{1-q}-1}{1-q}$, the entropy is cast in the fashion

$$S = \ln_q \mathcal{Z} + \mathcal{Z}^{1-q}\beta <\mathcal{U}> \tag{2.5}$$

and it is finite if $\mathcal{Z}$ and $<\mathcal{U}>$ are also finite.

Our purpose here is then to derive, for the classical HO, physical thermo-statistical variables $\mathcal{Z}$, $<\mathcal{U}>$, and $\mathcal{S}$, by appropriately treating (regularizing) the above singularities. As an illustration, we specify things for the cases of dimensions one, two, three, and N.

3 The one-dimensional case

In one dimension $\mathcal{Z}$ is regular and $<\mathcal{U}>$ has a singularity at $q = \frac{3}{2}$. For $q \neq \frac{3}{2}$, $\mathcal{Z}$ and $<\mathcal{U}>$ can be easily evaluated. The result is straightforward

$$\mathcal{Z} = \frac{\pi}{\beta(2-q)}, \tag{3.1}$$

$$<\mathcal{U}>= \frac{1}{\beta(3-2q)}. \tag{3.2}$$

According to (3.2), in the regular case, as $<\mathcal{U}>\geq 0$, one should have $q < \frac{3}{2}$. At $q = 3/2$ we have a pole in the mean energy, that we wish to investigate. Instead, when $q = \frac{3}{2}$, we have for $\mathcal{Z}$

$$\mathcal{Z} = \frac{2\pi}{\beta}, \tag{3.3}$$

a regular value. Regularization is needed then only for $<\mathcal{U}>$.

3.1 Dealing with the divergence

In order to proceed with such regularizing procedure, the main idea is to write $<\mathcal{U}>$ as a function of the dimension ν, in the fashion

$$<\mathcal{U}>= \frac{2^{\nu+1}\nu\pi^{\nu}}{\mathcal{Z}\beta^{\nu+1}}\Gamma(1-\nu), \tag{3.4}$$

and carefully dissect this expression. First we recast things as

$$<\mathcal{U}>= \frac{2^{\nu+1}[\nu-1+1]\pi^{\nu}}{\mathcal{Z}\beta^{\nu+1}}\Gamma(1-\nu), \tag{3.5}$$

and remember that $(\nu-1)\Gamma(1-\nu) = -\Gamma(2-\nu)$ to obtain

$$< \mathcal{U} >= -\frac{1}{\pi \mathcal{Z}} \left(\frac{2\pi}{\beta}\right)^{\nu+1} \Gamma(2-\nu) + \frac{1}{\pi \mathcal{Z}} \left(\frac{2\pi}{\beta}\right)^2 \left(\frac{2\pi}{\beta}\right)^{\nu-1} \Gamma(1-\nu). \quad (3.6)$$

We realize that the first term of (3.6) is finite, while the second one is singular for $\nu = 1$ (the physical dimension in this instance is unity). The trick here is to appeal to a Taylor's expansion, around $\nu = 1$, *of the third factor in the second term*, i.e., $\frac{2\pi}{\beta}^{\nu-1} = \exp[(\nu-1)\ln\frac{2\pi}{\beta}]$. Notice also that, from (3.3), $\mathcal{Z} = \frac{2\pi}{\beta}$. Accordingly, we have

$$< \mathcal{U} >= -\frac{1}{(2\pi^2/\beta)} \left(\frac{2\pi}{\beta}\right)^{\nu+1} \Gamma(2-\nu) + (2/\beta)\times$$
$$\left[1 + (\nu-1)\ln\left(\frac{2\pi}{\beta}\right) + \frac{(\nu-1)^2}{2}\ln^2\left(\frac{2\pi}{\beta}\right) + \cdots\right]\Gamma(1-\nu). \quad (3.7)$$

We use now once again the fact that $(\nu-1)\Gamma(1-\nu) = -\Gamma(2-\nu)$ to write

$$< \mathcal{U} >= -\frac{1}{(2\pi^2/\beta)} \left(\frac{2\pi}{\beta}\right)^{\nu+1} \Gamma(2-\nu) + (2/\beta)\times$$
$$\left[1 - \ln\left(\frac{2\pi}{\beta}\right) - \frac{(\nu-1)}{2}\ln^2\left(\frac{2\pi}{\beta}\right) + \cdots\right]\Gamma(2-\nu), \quad (3.8)$$

and then, in the limit $\nu \to 1$, after cancellations and series' terms that vanish, we are left with

$$< \mathcal{U} >= -\frac{2}{\beta}\left[1 + \ln\left(\frac{2\pi}{\beta}\right)\right], \quad (3.9)$$

that is to be regarded as the physical value of $< \mathcal{U} >$ [18, 19, 20, 21, 22, 26]. Using now (2.5) we immediately get for $\mathcal{S}$

$$\mathcal{S} = \ln_{\frac{3}{2}}\left(\frac{2\pi}{\beta}\right) - \sqrt{\frac{2\pi}{\beta}}\left[1 + \ln\left(\frac{2\pi}{\beta}\right)\right] \quad (3.10)$$

3.2 Direct proof of the existence of an upper bound to the canonical bath' temperature

Since the mean energy must be positive, according to (3.9) the possible values of β are restricted by the constraint $\beta > 2\pi e$, entailing $T < 1/2\pi e k_B$, with k_B Boltzmann's constant. There is an upper bound to the physical temperature, which cannot be infinite. This agrees with the considerations made in [27]: *q-statistics refers to systems in thermal contact with a* **finite** *bath.*

3.3 A fancier conjecture

On a more conjectural fashion, one is also reminded here of the Hagedorn temperature. This is the temperature at which ordinary matter is no longer stable and would evaporate, transforming itself into quark matter, a sort of boiling point of hadronic matter. This temperature would exist on account of the fact that the accessible energy would be so high that quark-antiquark pairs would be be spontaneously extracted from the vacuum. A putative system at such a high temperature is able to accommodate any amount of energy because the newly emerging quarks would provide additional degrees of freedom. The Hagedorn temperature would thus be unsurmountable [28].

4 The two-dimensional case

For two dimensions, $\mathcal{Z}$ has a singularity at $q = \frac{3}{2}$ and $<\mathcal{U}>$ has singularities at $q = \frac{3}{2}$ and $q = \frac{4}{3}$. Save for the case of these singularities, we can evaluate their values of the main statistical quantities without the use of dimensional regularization. Thus, we obtain

$$\mathcal{Z} = \frac{\pi^2}{\beta^2(2-q)(3-2q)}, \tag{4.1}$$

$$<\mathcal{U}>= \frac{2}{\beta(4-3q)}, \tag{4.2}$$

$$\mathcal{S} = \ln_q\left[\frac{\pi^2}{\beta^2(2-q)(3-2q)}\right] + \left[\frac{\pi^2}{\beta^2(2-q)(3-2q)}\right]^{1-q}\frac{2}{4-3q} \tag{4.3}$$

According to (4.2), in the regular case $q < \frac{4}{3}$.

4.1 The $q = 3/2$ pole

For $q = \frac{3}{2}$ we must employ the treatment of the preceding Section, i.e., regularize, both $\mathcal{Z}$ and $\mathcal{U}$. We start with $\mathcal{Z}$. From (2.2) we have

$$\mathcal{Z} = \left(\frac{2\pi}{\beta}\right)^{\nu}\Gamma(2-\nu), \tag{4.4}$$

which can be rewritten as

$$\mathcal{Z} = \left(\frac{2\pi}{\beta}\right)^{2}\left(\frac{2\pi}{\beta}\right)^{\nu-2}\Gamma(2-\nu). \tag{4.5}$$

With this form for $\mathcal{Z}$, we can expand in Taylor's series, around $\nu = 2$, the factor $\left(\frac{2\pi}{\beta}\right)^{\nu-2} = \exp\left[(\nu-2)\ln\frac{2\pi}{\beta}\right]$, noting also that $(\nu-2)\Gamma(2-\nu) = -\Gamma(3-\nu)$, i.e.,

$$\mathcal{Z} = \left(\frac{2\pi}{\beta}\right)^{2}\Gamma(2-\nu)\left[1+(\nu-2)\ln\left(\frac{2\pi}{\beta}\right)\cdots\right], \tag{4.6}$$

and thus we obtain the physical value of $\mathcal{Z}$ as

$$\mathcal{Z} = -\frac{4\pi^2}{\beta^2}\ln\left(\frac{2\pi}{\beta}\right). \tag{4.7}$$

For $\mathcal{U}$ the situation is similar. From (2.4) we have

$$<\mathcal{U}>=\frac{\nu}{\mathcal{Z}\pi}\left(\frac{2\pi}{\beta}\right)^{\nu+1}\Gamma(1-\nu), \tag{4.8}$$

where $\mathcal{Z}$ is given by (4.7). Proceeding in the same way as we did in the one dimensional case, and omitting here from intermediate steps, we rewrite $<\mathcal{U}>$ in the fashion

$$<\mathcal{U}>=\frac{\Gamma(3-\nu)}{\mathcal{Z}\pi(\nu-1)}\left(\frac{2\pi}{\beta}\right)^{\nu+1}+\frac{2}{\mathcal{Z}\pi}\left(\frac{2\pi}{\beta}\right)^{3}\left(\frac{2\pi}{\beta}\right)^{\nu-2}\frac{\Gamma(2-\nu)}{1-\nu}, \tag{4.9}$$

and we obtain the physical value of $<\mathcal{U}>$:

$$<\mathcal{U}>=\frac{8\pi^2}{\mathcal{Z}\beta^3}+\frac{16\pi^2}{\mathcal{Z}\beta^3}\ln\left(\frac{2\pi}{\beta}\right), \tag{4.10}$$

so that replacing Z by the value given in (4.7) we have

$$<\mathcal{U}>=\frac{2}{\beta(\ln\beta-\ln 2\pi)}+\frac{4}{\beta}. \tag{4.11}$$

From (4.11) we see that the possible values of β are given by $\beta>2\pi$. *Again, a temperature's upper bound is detected.*

Now, from the physical values of $\mathcal{Z}$ and $<\mathcal{U}>$, as given by (4.7) and (4.11), respectively, and from (2.5), we find the physical value of $\mathcal{S}$ as

$$\mathcal{S}=\ln_{\frac{3}{2}}\left[\frac{4\pi^2}{\beta^2}\ln\left(\frac{\beta}{2\pi}\right)\right]+\left[\frac{4\pi^2}{\beta^2}\ln\left(\frac{\beta}{2\pi}\right)\right]^{-\frac{1}{2}}\left[\frac{2}{(\ln\beta-\ln 2\pi)}+4\right] \tag{4.12}$$

4.2 The $q=4/3$ pole

For $q=\frac{4}{3}$, $\mathcal{Z}$ is finite and $<\mathcal{U}>$ has a pole. The procedure for finding their physical values is similar to that for the case $q=\frac{3}{2}$. For this reason, we only indicate the results obtained for $\mathcal{Z}$, $<\mathcal{U}>$, and $\mathcal{S}$. One finds

$$\mathcal{Z}=\frac{9\pi^2}{2\beta^2}, \tag{4.13}$$

$$< \mathcal{U} >= \frac{6}{\beta}\left[\ln\left(\frac{\beta}{3\pi}\right) - \frac{1}{2}\right], \tag{4.14}$$

$$\mathcal{S} = \ln_{\frac{4}{3}}\left(\frac{9\pi^2}{2\beta^2}\right) + \left(\frac{9\pi^2}{2\beta^2}\right)^{-\frac{1}{3}}\left[6\ln\left(\frac{\beta}{3\pi}\right) - 3\right] \tag{4.15}$$

From (4.14) we see that the possible values of β are given by the constraint $\beta > 3\pi\sqrt{e}$.

5 The three-dimensional case

In three dimensions, $\mathcal{Z}$ has poles at $q = \frac{3}{2}$ and $q = \frac{4}{3}$ while $< \mathcal{U} >$ exhibits them at $q = \frac{3}{2}$, $q = \frac{4}{3}$, and $q = \frac{5}{4}$. Consequently, after regularization, we have

$$\mathcal{Z} = \frac{\pi^3}{\beta^3(2-q)(3-2q)(4-3q)}, \tag{5.1}$$

$$< \mathcal{U} >= \frac{3}{\beta(5-4q)}. \tag{5.2}$$

From (5.1) and (5.2) we obtain for the entropy

$$\mathcal{S} = \ln_q\left[\frac{\pi^3}{\beta^3(2-q)(3-2q)(4-3q)}\right] + \left[\frac{\pi^3}{\beta^3(2-q)(3-2q)(4-3q)}\right]^{q-1}\frac{3}{5-4q} \tag{5.3}$$

In this case q should satisfy the condition $q < \frac{5}{4}$ for the mean energy to be a positive quantity.

5.1 The $q = 3/2$ pole

For $q = \frac{3}{2}$ we have

$$\mathcal{Z} = \left(\frac{2\pi}{\beta}\right)^{\nu}\Gamma(2-\nu). \tag{5.4}$$

Proceeding as in the previous cases and making now the Taylor's expansion around $\nu = 3$, $\mathcal{Z}$ acquires the appearance

$$\mathcal{Z} = \left(\frac{2\pi}{\beta}\right)^3 \frac{\Gamma(3-\nu)}{2-\nu}\left[1 + (\nu - 3)\ln\left(\frac{2\pi}{\beta}\right) + \cdots\right]. \tag{5.5}$$

From (5.5) it is easy to obtain the physical value of $\mathcal{Z}$ as

$$\mathcal{Z} = \frac{8\pi^3}{\beta^3}\ln\left(\frac{2\pi}{\beta}\right). \tag{5.6}$$

In a similar vein have for $<\mathcal{U}>$

$$<\mathcal{U}>= \frac{1}{\beta(\ln\beta - \ln 2\pi)} - \frac{3}{\beta}, \tag{5.7}$$

and from (5.6) and (5.7)

$$\mathcal{S} = \ln_{\frac{3}{2}}\left[\frac{8\pi^3}{\beta^3}\ln\left(\frac{2\pi}{\beta}\right)\right] + \left[\frac{8\pi^3}{\beta^3}\ln\left(\frac{2\pi}{\beta}\right)\right]^{-\frac{1}{2}}\left(\frac{1}{\ln\beta - \ln 2\pi} - 3\right) \tag{5.8}$$

with $2\pi < \beta < 2\pi e^{\frac{1}{3}}$. This entails that the system exhibits positive entropy only for a small range of very high temperatures.

5.2 The $q = 4/3$ and $q = 5/4$ poles

For $q = \frac{4}{3}$ and $q = \frac{5}{4}$ we give only the corresponding results, since the calculations are entirely similar to those for the case $q = \frac{3}{2}$. Thus, for $q = \frac{4}{3}$ we have

$$\mathcal{Z} = \frac{27\pi^3}{2\beta^3}\ln\left(\frac{\beta}{3\pi}\right), \tag{5.9}$$

$$<\mathcal{U}>= \frac{3}{\beta(\ln\beta - \ln 3\pi)} - \frac{9}{\beta}, \tag{5.10}$$

$$\mathcal{S} = \ln_{\frac{4}{3}}\left[\frac{27\pi^3}{2\beta^3}\ln\left(\frac{\beta}{3\pi}\right)\right] + \left[\frac{27\pi^3}{2\beta^3}\ln\left(\frac{\beta}{3\pi}\right)\right]^{-\frac{1}{3}}\left(\frac{3}{\ln\beta - \ln 3\pi} - 9\right) \tag{5.11}$$

with $3\pi < \beta < 3\pi e^{\frac{1}{3}}$. This entails, again, that the system exhibits positive entropy only for a small range of very high temperatures.

For $q = \frac{5}{4}$:

$$\mathcal{Z} = \frac{32\pi^3}{3\beta^3}, \tag{5.12}$$

$$< \mathcal{U} >= \frac{12}{\beta} \ln\left(\frac{\beta}{4\pi}\right) - \frac{4}{\beta}, \tag{5.13}$$

$$\mathcal{S} = \ln_{\frac{5}{4}}\left(\frac{32\pi^3}{3\beta^3}\right) + \left(\frac{32\pi^3}{3\beta^3}\right)^{-\frac{1}{4}} \left[12 \ln\left(\frac{\beta}{4\pi}\right) - 4\right], \tag{5.14}$$

with $\beta > 4\pi e^{\frac{1}{3}}$.

6 The N-Dimensional Case

Repeating the calculation made for 2, 3 and 4 dimensions, with more algebraic work we get for $\mathcal{Z}$ the expression:

$$\mathcal{Z}_{\frac{\nu-k+1}{\nu-k}} = \frac{(-1)^{k+1}}{k!\Gamma(\nu-k)} \left[\frac{(\nu-k)\pi}{\beta}\right]^{\nu} \ln\left[\frac{(\nu-k)\pi}{\beta}\right] \tag{6.1}$$

Here $k = 0, 1, 2, 3........, \nu - 2$,where ν is the dimension of the space. And for $< U >$:

$$< U >_{\frac{\nu-k+2}{\nu-k+1}} = \frac{(-1)^{k+1}}{k!\Gamma(\nu-k)\beta Z} \left[\frac{(\nu+1-k)\pi}{\beta}\right]^{\nu} +$$
$$\frac{(-1)^{k+1}\nu}{k!\Gamma(\nu-k)\beta Z} \left[\frac{(\nu+1-k)\pi}{\beta}\right]^{\nu} \ln\left[\frac{(\nu+1-k)\pi}{\beta}\right] \tag{6.2}$$

where $k = 0, 1, 2, 3........, \nu - 1$.

7 Specific Heats

We set $k \equiv k_B$. For $\nu = 1$, in the regular case we have for the specific heat C:

$$\mathcal{C} = \frac{k}{3-2q}, \tag{7.1}$$

with $q < \frac{3}{2}$.

For $\nu = 2$ one has

$$\mathcal{C} = \frac{2k}{4-3q}, \tag{7.2}$$

with $q < \frac{4}{3}$.

Finally, for $\nu = 3$ one ascertains that

$$\mathcal{C} = \frac{3k}{5-4q}, \tag{7.3}$$

with $q < \frac{5}{4}$.

7.1 Specific heats at the poles

For $\nu = 1$; $q = \frac{3}{2}$

$$\mathcal{C} = -2k(\ln kT + \ln 2\pi + 2). \tag{7.4}$$

with $kT < \frac{1}{2\pi e}$.

For $\nu = 2$; $q = \frac{3}{2}$

$$\mathcal{C} = \frac{2k}{(\ln kT + \ln 2\pi)^2} - \frac{2k}{(\ln kT + \ln 2\pi)} + 4k, \tag{7.5}$$

with $kT < \frac{1}{2\pi}$.

For $\nu = 2$ and $q = \frac{4}{3}$ things become:

$$\mathcal{C} = -6k\left(\ln kT + \ln 3\pi + \frac{3}{2}\right), \tag{7.6}$$

with $kT < \frac{1}{3\pi\sqrt{e}}$.

For $\nu = 3$; $q = \frac{3}{2}$,

$$\mathcal{C} = \frac{k}{(\ln kT + \ln 2\pi)^2} - \frac{k}{(\ln kT + \ln 2\pi)} - 3k, \tag{7.7}$$

with $\frac{1}{2\pi e^{\frac{1}{3}}} < kT < \frac{1}{2\pi}$.

For $\nu = 3$ and $q = \frac{4}{3}$ one has

$$\mathcal{C} = \frac{3k}{(\ln kT + \ln 3\pi)^2} - \frac{3k}{(\ln kT + \ln 3\pi)} - 9k, \tag{7.8}$$

with $\frac{1}{3\pi e^{\frac{1}{3}}} < kT < \frac{1}{3\pi}$

Finally, for $\nu = 3$ and $q = \frac{5}{4}$

$$\mathcal{C} = -12k\left(\ln kT + \ln 4\pi + \frac{4}{3}\right) \tag{7.9}$$

with $kT < \frac{1}{4\pi e^{\frac{1}{3}}}$.

Figs, 1, 2, and 3 plot the pole-specific heats within their allowed temperature ranges, for one, two, and three dimensions, respectively. The most distinguished feature emerges in the cases in which we deal with $< U > -$poles for which Z is regular. We see in such a case that negative specific heats arise. Such an occurrence has been associated to self-gravitational systems [14, 29]. In turn, Verlinde has associated this type of systems to an entropic force [30]. It is natural to conjecture then that such a force may appear at the energy poles.

Notice also that temperature ranges are restricted. There is an $T-$upper bound that one may wish to link to the Hagedorn temperature (see above) [28]. In two and three dimensions there is also a lower bound, so that the system (at the poles) would be stable only in a limited $T-$range.

8 Discussion

In this work we have appealed to an an elementary regularization procedure to study the poles in the partition function and the mean energy that appear, for specific, discrete q-values, in Tsallis' statistics. We studied the thermodynamic behavior at the poles and found interesting peculiarities. The analysis was made in one, two, three, and N dimensions. Amongst pole-traits we emphasize:

- We have proved that there is an upper bound to the temperature at the poles, confirming the findings of Ref. [27].
- In some cases, Tsallis' entropies are positive only for a restricted temperature-range.
- Negative specific heats, characteristic trait of self-gravitating systems, are encountered.

Our physical results derive only from statistics, not from mechanical effects. This fact reminds us of a similar occurrence in the case of the entropic force conjectured by Verlinde [30].

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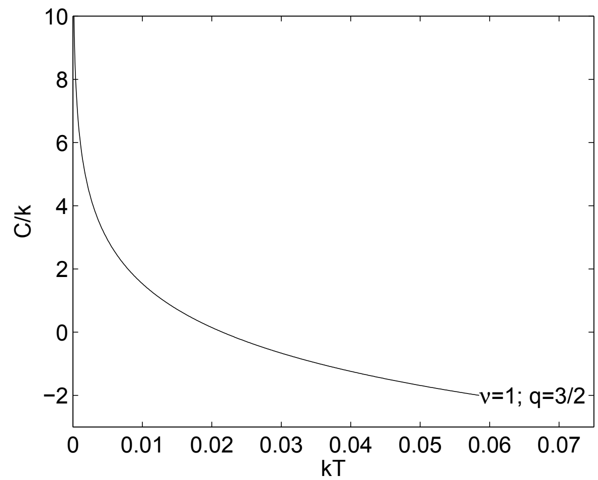

Figure 1: One dimension: specific heats at the pole versus temperature T, plotted within the allowed temperature range.

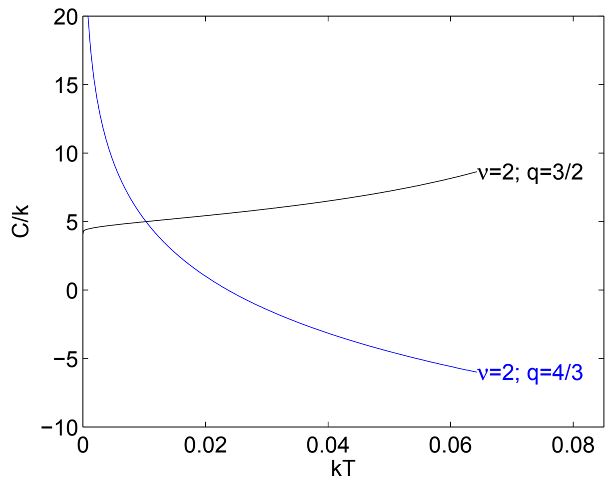

Figure 2: Two dimensions: specific heats at the two poles versus temperature T, plotted within the allowed temperature ranges in the two cases.

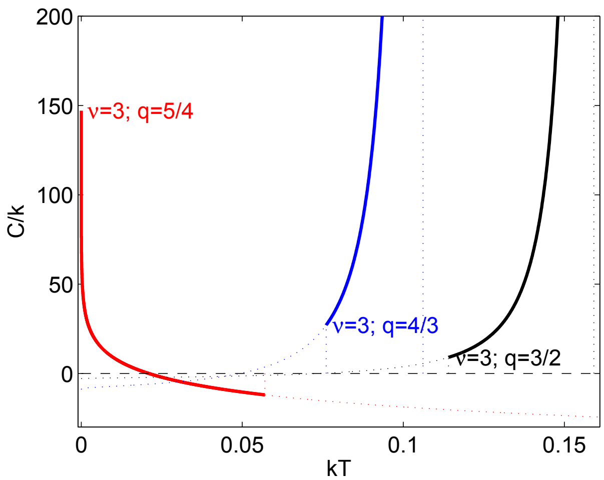

Figure 3: Three dimensions: specific heats at the three poles versus temperature T. The vertical lines demarcate the allowed temperature ranges in the three cases. Dashed lines are continuations of the $C-$values outside the domains of validity