

# On the high-density expansion for Euclidean random matrices

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**Abstract.** Diagrammatic techniques to compute perturbatively the spectral properties of Euclidean random matrices (ERM) in the high-density regime are introduced and discussed in detail. Such techniques are developed in two alternative and very different formulations of the mathematical problem and are shown to give identical results up to second order in the perturbative expansion. One method, based on writing the so-called resolvent function as a Taylor series, allows us to group the diagrams into a small number of topological classes, providing a simple way to determine the infrared (small momenta) behaviour of the theory up to third order, which is of interest for the comparison with

experiments. The other method, which reformulates the problem as a field theory, can instead be used to study the infrared behaviour at any perturbative order.

**Keywords:** random matrix theory and extensions

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## 1. Introduction

Random matrices [1] are  $N \times N$  matrices whose entries are random numbers drawn from a certain probability distribution. Their statistical spectral properties in the large  $N$  limit describe a wide range of physical phenomena: nuclear spectra [2], quantum chaos [3], localization in electronic systems [4], diffusion in random graphs [5], liquid dynamics [6]

and the glass transition [7], complex networks [8] and superstrings [9]. Random matrices may be grouped in a few universality classes according to their statistical properties [1]. For most of these classes, the density of eigenvalues follows Wigner's semicircle law. It has thus become of interest to identify ensembles where the semicircle law is modified in a non-trivial way. One such ensemble results when the corresponding physical problem has a conserved quantity (e.g. momentum in the case of propagating excitations, or number density in diffusion problems). Under such circumstances, the random matrix that best describes the problem is typically a Laplacian matrix [5], which has the property

$$\sum_j \mathcal{M}_{ij} = 0. \quad (1)$$

This encodes the property that a vector whose components are identical is an eigenvector with eigenvalue zero.

A kind of random matrix of particular relevance in the study of off-lattice systems is the so-called Euclidean random matrices (ERM) [10, 21, 23]. Place  $N$  particles in positions  $\mathbf{x}_i$ ,  $i = 1, 2, \dots, N$ , belonging to some region of  $D$ -dimensional Euclidean space, of volume  $V$ . The positions are drawn randomly from some probability distribution function  $P(\{\mathbf{x}_i\})$ . The entries of an ERM are a deterministic function of these random positions,  $M_{ij} = f(\mathbf{x}_i - \mathbf{x}_j)$ . If a conservation law is relevant for the problem at hand, we will rather have a Laplacian ERM:

$$\mathcal{M}_{ij} = \delta_{ij} \sum_{k=1}^N f(\mathbf{x}_j - \mathbf{x}_k) - f(\mathbf{x}_i - \mathbf{x}_j). \quad (2)$$

Note that we never find the same particle label twice in the argument of the function  $f(\mathbf{x}_i - \mathbf{x}_j)$ , since the term  $f(\mathbf{x}_i - \mathbf{x}_i)$  cancels. In a diagonal term,  $\delta_{ij} f(\mathbf{x}_i - \mathbf{x}_k)$ , the  $k$ th particle shall be called a *medium* particle, while the  $i$ th particle will be the *chain* particle.

The function  $f$  in equation (2) is quite general: only rotational invariance and the existence of the Fourier transform  $\hat{f}(p)$  are assumed ( $p = \sqrt{\mathbf{p} \cdot \mathbf{p}}$ ). Furthermore, even if in this work  $f$  will be a scalar function, for some applications it should rather be a matrix-valued function. It must be so, for instance, to account for the vector nature (longitudinal or transversal) of vibrational dynamics [18]. Most of our results extend as such to this more general case.

ERMs describe topologically disordered systems, at variance with problems where the  $N$  positions  $\{\mathbf{x}_i\}$  are placed on a crystalline lattice [30]. We will be considering an extreme case, in which the  $N$  positions are placed with uniform probability on the volume  $V$ . The particle-number density,  $\rho = N/V$ , will be held fixed while we take the large  $N$  limit. Note that there are two sources of statistical correlation among the entries of matrix (2), even if the positions  $\{\mathbf{x}_i\}$  are totally uncorrelated. First, it is a Laplacian matrix; recall equation (1). Second, due to the triangular inequality of Euclidean geometry, the distances from two neighbouring particles to a third one are necessarily similar.

Specific applications of ERMs include disordered d-wave superconductors [11], disordered magnetic semiconductors [12] (very similar to a spin-glass model [13]), Instantaneous Normal Modes in liquids [6, 14], vibrations in glasses [7], [15]–[18], the gelation transition in polymers [19] and vibrations in DNA [20]. ERMs have been studied analytically and numerically both in the low particle-number density regime [14], [22]–[24] and for high densities [10], [15]–[18], [31].

In order to compute the basic spectral properties of ERM it turns out to be convenient to introduce the *resolvent*:

$$G(\mathbf{p}, z) = \lim_{N \rightarrow \infty} \frac{1}{N} \overline{\sum_{i,j=1}^N e^{i\mathbf{p} \cdot (\mathbf{x}_j - \mathbf{x}_i)} \left[ \frac{1}{z - \mathcal{M}} \right]_{ij}}, \quad (3)$$

where the complex number  $z \equiv \lambda + i\eta$  has a tiny imaginary part  $\eta$  and the overbar stands for an average over the  $\{\mathbf{x}_i\}$ . If the ERM describes physical excitations (phonons, electrons, etc) in topologically disordered systems, the resolvent (3) corresponds to the single-particle Green function, or propagator, for such excitations. If the system is isotropic, the resolvent depends only on  $p$ .

The density of eigenvalues  $g(\lambda)$ , or density of states (DOS), is given by

$$g(\lambda) = -\frac{1}{\pi} \lim_{p \rightarrow \infty} \text{Im}[G(\mathbf{p}, \lambda + i0^+)]. \quad (4)$$

This limiting behaviour is characteristic of topologically disordered systems. It does not hold for lattice systems. We note as well that the constraint (1) implies that a plane wave  $e^{i\mathbf{p} \cdot \mathbf{x}_i}$  is an eigenvector of the matrix (2) if  $p = 0$ :

$$G(\mathbf{0}, z) = \frac{1}{z}. \quad (5)$$

As we shall discuss below, the resolvent takes a very simple form in the high-density limit (it is actually the *bare* propagator of the theory):

$$G_0(\mathbf{p}, z) = \frac{1}{z - \epsilon(\mathbf{p})}, \quad \epsilon(\mathbf{p}) = \rho[\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})]. \quad (6)$$

The physical interpretation is quite appealing [10]. The system behaves as an elastic continuum medium. In the large  $\rho$  limit, the plane waves  $e^{i\mathbf{p} \cdot \mathbf{x}_i}$  become exact eigenvectors of the matrix (2), with eigenvalues given by the dispersion relation  $\epsilon(\mathbf{p}) = \rho[\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})]$ . In particular, for small  $p$ ,  $\epsilon(\mathbf{p}) = c^2 p^2 + \mathcal{O}(p^4)$ , where  $c$  is the speed of sound. This neat physical picture motivates the introduction of a high-density expansion.

At large, but finite  $\rho$ , the resolvent can be written

$$G(\mathbf{p}, z) = \frac{1}{z - \epsilon(\mathbf{p}) - \Sigma(\mathbf{p}, z)}. \quad (7)$$

The self-energy  $\Sigma(\mathbf{p}, z)$ , which is introduced to encode all the information about the *interactions* (a standard practice in the Green function formalism), vanishes when  $\rho$  tends to  $\infty$ . In our case, the interaction involved is that between the propagating excitations and the topological disorder. An important theoretical challenge is to compute the self-energy at finite densities  $\rho$ . In fact, in this case an eigenvector can be thought of as a packet of plane waves (see section 2). The width of such a packet is related to the imaginary part of  $\Sigma$ .

Some of us have argued that in the limit of small  $p, z$  the leading term at  $1/\rho^2$  order has the form [15]–[17]

$$\text{Im} \Sigma(\mathbf{p}, z + i0^+) = \mathcal{A} z^{(D-2)/2} p^2 + \mathcal{O}(z^{(D-2)/2} p^4, z^D p^2), \quad (8)$$

( $D$  is the space dimension, while  $\mathcal{A}$  is an amplitude). This has been disputed recently by an independent computation to order  $1/\rho^2$ , claiming that the actual analytic structure of the self-energy for small  $z$  and  $p$  is [31]

$$\text{Im } \Sigma(\mathbf{p}, z + i0^+) \propto z^{D/2} p^2 + \dots, \quad (9)$$

analogous to what one finds in the Rayleigh theory of scattering and in lattice models where disordered spring constants mimic the effect of topological disorder [30].

By reconsidering in detail the perturbative expansion, in this work we show that the prefactor  $\mathcal{A}$  in equation (8) is actually null, due to algebraic cancellations, and that this cancellation arises at all orders in the perturbative expansion in  $1/\rho$ . This is not related to any known symmetry of the problem, but rather reflects the mathematical structure of the perturbative contributions. On the other hand, we will also show that the result in [31], recall equation (9), is incomplete, since the imaginary part admits a formal expansion for small  $z$ :

$$\text{Im } \Sigma(\mathbf{p}, z + i0^+) = z^{(D-2)/2} \sum_{n=0}^{\infty} g_n(p^2) z^n. \quad (10)$$

The constraint (5) implies that  $g_n(0) = 0$  for all  $n$ , so that in general  $g_n(p) = \mathcal{A}_n p^2 + \mathcal{O}(p^4)$ . However, we find that, for all functions  $f$  and all  $\rho$ ,  $\mathcal{A}_0 = 0$ , so that  $g_0(p) \sim p^4$  while  $g_1(p) \sim p^2$ . In this respect, we confirm that the interaction between free excitations and disorder in topologically disordered systems (as long as ERMs describe them) has a peculiar mathematical structure that is different from disordered lattice systems (for lattice systems  $g_0$  vanishes identically). To show this we shall compute the self-energy perturbatively within two unrelated approaches: (a) an improved form of the combinatorial formalism introduced in [16] and (b) a field-theoretic formulation. The field theory introduced here is quite different from standard formalisms in the theory of random matrices (see, e.g., [10, 15]). It probably deserves an in-depth study, which is left for future work. We remark that our combinatorial formalism is simpler than the field theory, and is probably the method of choice to carry out higher-order computations in the  $1/\rho$  expansion. However, it has the drawback that the asymptotic  $g_0(p^2) \sim p^4$  appears at order  $1/\rho^2$  from an exact cancellation of two contributions of order  $p^2$  (at order  $1/\rho^3$  we find an exact cancellation of *ten* contributions of order  $p^2$ ). The field-theoretic framework clarifies that these cancellations are not accidental and thus not restricted to low orders in the  $1/\rho$  expansion.

The layout of the remaining part of this work is as follows: in section 2 we discuss a particular phenomenon (phonons in topologically disordered systems) where a theory based on ERMs has been proposed in recent years. In section 3 we anticipate our main result, namely the leading order of  $\text{Im } \Sigma(\mathbf{p}, z + i0^+)$ . In section 4 we discuss in detail the combinatorial formalism up to order  $1/\rho^2$ . We describe the rules to group all the diagrams that arise at this order in a very small number of diagrams, according to their topological structure, and show that up to second order in the function  $g_0(p^2)$  the prefactor of the term  $\propto p^2$  cancels out. We also see that this cancellation appears in a given class of diagrams at  $1/\rho^3$ . In order to shed a light onto the mathematical origin of such a cancellation, in section 5 we introduce a field-theoretical formulation that, despite producing a much larger number of diagrams, allows us to give an argument explaining the origin of the cancellation at any perturbative order.

## 2. A case study for ERM: phonons in topologically disordered systems

Although ERMs have a wide range of applications, in this paper we are mainly interested in the study of phonons in amorphous systems, such as glasses or supercooled liquids [25], since the large amount of experimental evidence may provide fundamental insights about the correctness of the theory. Of particular interest is the case where the frequencies  $\omega(p)$  of the phonons with wavevector  $p$  lie in the GHz to the THz region (high-frequency sound). This is, in fact, the range explored by neutron and x-ray inelastic scattering experiments. These give the inelastic contribution to the dynamic structure factor, i.e. a Brillouin-like peak with position  $\omega(p)$  and width  $\Gamma(p)$ . Summarizing the experimental findings, for  $p < p_0$  ( $p_0$  is the first maximum of the static structure factor, typically a few  $\text{nm}^{-1}$  [26]) one finds a linear dispersion relation  $\omega(p) \sim cp$ , where the speed of sound  $c$  is quite close to that obtained by acoustic measurements. The dispersion relation typically saturates at  $p \sim p_0$ . Moreover, the  $p$  dependence of the peak width is often described by  $\Gamma(p) \propto p^\alpha$ . Interestingly enough,  $\Gamma(p)$  also saturates as the momentum becomes  $p \sim p_0$ . There has been a hot debate among different experimental groups about the value of the exponent  $\alpha$  [27, 28], some claiming  $\alpha \sim 2$  and some  $\alpha \sim 4$ . There is now some consensus that, in the region where  $\Gamma$  is independent of temperature (i.e.  $\omega(p) \geq 1$  THz), one has  $\alpha = 4$ , while at lower frequencies (the GHz region), where  $\Gamma$  has a strong temperature dependence, the experimental value is  $\alpha = 2$  [29].

A simple model of the high-frequency sound is afforded by scalar harmonic vibrations around a topologically disordered structure made of  $N$  oscillation centres  $\mathbf{x}_i$ , placed with uniform probability on a volume  $V^9$ . Particle displacements  $\varphi_i$  have an elastic energy

$$U(\{\varphi_i\}) = \frac{1}{2} \sum_{i,j=1}^N f(\mathbf{x}_i - \mathbf{x}_j)(\varphi_i - \varphi_j)^2 = \sum_{i,j=1}^N \mathcal{M}_{i,j} \varphi_i \varphi_j, \quad (11)$$

where the matrix  $\mathcal{M}$  has the form equation (2) and  $f(\mathbf{x})$  is the spring constant connecting particles separated by the vector  $\mathbf{x}$ . We assume that  $f(\mathbf{x})$  is spherically symmetric, so that  $\hat{f}(\mathbf{p}) = g(p^2)$ , where  $g$  is a smooth function. In the framework of the one-phonon approximation, the inelastic dynamic structure factor is related to the resolvent via

$$S(\mathbf{p}, \omega) = -\frac{p^2}{\omega\pi} \text{Im} G(\mathbf{p}, \omega^2 + i0^+). \quad (12)$$

As a consequence, the width of the Brillouin peak is related to the imaginary part of  $\Sigma$  by

$$\text{Im} \Sigma(\mathbf{p}, \omega(p)) = \omega(p)\Gamma(p). \quad (13)$$

Then equation (10) implies that  $\Gamma(p) \sim p^4$  for very small  $p$  (for  $p \sim p_0$  the width saturates and a mixed, more complex, scaling should be expected). In that regime the phonon-disorder interaction can be thought of as a scattering phenomenon of the Rayleigh type.

Since ERMs describe the dynamics of vibrating particles within the context of the harmonic approximation, the theoretical predictions based on ERM theory must be compared with experiments in the region where  $\Gamma$  is independent of temperature; in fact,

<sup>9</sup> One may just as easily introduce a model for vector harmonic vibrations [18]. In fact, all our results within the combinatorial formalism translate directly to the vector case.

the temperature dependence is an indication that the width of the peak is rather due to thermal processes, such as anharmonicities or relaxations, which require more refined theoretical approaches.

We finally mention that vibrational frequencies  $\omega$  are related to ERM eigenvalues  $\lambda$  ( $z = \lambda + i0^+$ ) by the relation  $\lambda = \omega^2$ , see equation (12). Hence, the width of spectral peaks in  $\lambda$  space and in  $\omega$  space are related by equation (13). Furthermore, equations (4) and (10) imply that the DOS in  $\lambda$  space behaves for small  $\lambda$  as  $g_\lambda(\lambda) \propto \lambda^{(D-2)/2}$ , which translates to frequency space as a Debye spectrum  $g_\omega(\omega) \propto \omega^{D-1}$  (because of the Jacobian in the change of variable:  $d\lambda = 2\omega d\omega$ ). At this point, the reader may object that lattice systems have a Debye spectrum even if  $g_0$  in equation (10) vanishes for them. In fact, their Debye spectrum is possible because equation (4) does not hold in the lattice case.

### 3. The main result

The main result of this work is the following. Expanding the self-energy in powers of  $1/\rho$ , i.e.

$$\Sigma(\mathbf{p}, z) = \Sigma^{(1)}(\mathbf{p}, z) + \Sigma^{(2)}(\mathbf{p}, z) + \dots, \quad (14)$$

where  $\Sigma^{(k)}$  is of order  $1/\rho^k$ , one has only one first-order contribution:

$$\Sigma^{(1)}(\mathbf{p}, z) = \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q}, \mathbf{p}), \quad (15)$$

while at second order

$$\Sigma^{(2)}(\mathbf{p}, z) = \Sigma_A^{(2)}(\mathbf{p}, z) + \Sigma_B^{(2)}(\mathbf{p}, z) + \Sigma_C^{(2)}(\mathbf{p}, z), \quad (16)$$

where the three topologically different pieces are

$$\Sigma_A^{(2)}(\mathbf{p}, z) = \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) \Sigma^{(1)}(\mathbf{q}, z) G_0(\mathbf{q}, z) V(\mathbf{q}, \mathbf{p}), \quad (17a)$$

$$\begin{aligned} \Sigma_B^{(2)}(\mathbf{p}, z) &= \frac{1}{\rho^2} \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{d^D \mathbf{k}}{(2\pi)^D} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) \\ &\quad \times V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{k}) G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p}), \end{aligned} \quad (17b)$$

$$\begin{aligned} \Sigma_C^{(2)}(\mathbf{p}, z) &= \frac{1}{\rho^2} \int \frac{d^D \mathbf{q}}{(2\pi)^D} \frac{d^D \mathbf{k}}{(2\pi)^D} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, 2\mathbf{q} - \mathbf{p}) \\ &\quad \times G_0(\mathbf{k}, z) V(\mathbf{p} - \mathbf{k}, \mathbf{p}). \end{aligned} \quad (17c)$$

In equations (15)–(17) we have used

$$V(\mathbf{q}, \mathbf{p}) = \rho[\hat{f}(\mathbf{q}) - \hat{f}(\mathbf{p} - \mathbf{q})], \quad (18)$$

which, as we will see below, plays the role of the interaction vertex. The bare propagator  $G_0$  was defined in equation (6). Note that  $V(\mathbf{q}, \mathbf{p}) \neq V(\mathbf{p}, \mathbf{q})$ . Other useful identities are

$$V(\mathbf{q}, \mathbf{p}) = V(-\mathbf{q}, -\mathbf{p}), \quad V(\mathbf{q}, \mathbf{p}) = -V(\mathbf{p} - \mathbf{q}, \mathbf{p}). \quad (19)$$

Note that, since  $V(\mathbf{q}, \mathbf{0}) = 0$ , we have

$$\Sigma_A^{(2)}(\mathbf{0}, z) = \Sigma_B^{(2)}(\mathbf{0}, z) = \Sigma_C^{(2)}(\mathbf{0}, z) = 0. \quad (20)$$



The high-density expansion for Laplacian ERM was introduced in [15, 16]. Equation (15) was already reported there but, instead of equation (16), one had 39 diagrams of order  $1/\rho^2$ . Even if the final expressions were cumbersome, a numerical evaluation of the amplitude  $\mathcal{A}$  in equation (8) was attempted for a simple choice of the function  $f$ . Presumably because of a numerical mistake, it was wrongly concluded that  $\mathcal{A} \neq 0$ . Afterwards, it was announced (without supporting technical details) that the 39 diagrams previously found at order  $1/\rho^2$  could be grouped as in equation (16) [18]. Unfortunately, a numerical re-evaluation of the amplitude  $\mathcal{A}$  was not attempted from these simpler expressions.

We remark as well that an independent computation of  $\Sigma$  to order  $1/\rho^2$  has appeared recently [31]. We have checked that their results are consistent with ours, letting aside contact terms (in fact, these authors explicitly state that some contact terms are lacking from their final expressions). Thus, their failure in identifying the  $g_0$  term in equation (10) is not due to discrepancies in the final expressions. The underlying reason is rather more mundane, as we explain below.

At first order the theory has the following behaviour. For small  $\lambda$ ,  $z = \lambda + i0^+$ , we approximate the imaginary part of the propagator  $G_0$  by

$$\text{Im } G_0(\mathbf{q}, \lambda + i0^+) = -\frac{\pi}{2\sqrt{\lambda}} \delta\left(q - \frac{\sqrt{\lambda}}{c}\right) \quad (21)$$

(assuming a linear dispersion relation  $\epsilon(\mathbf{p}) \approx c^2 p^2$ ). Then the only contribution to the imaginary part comes from  $q = \sqrt{\lambda}/c$ . To evaluate the vertex  $V(\mathbf{q}, \mathbf{p})$  at small  $q$ , and small  $p$  we just need to recall that  $\hat{f}(\mathbf{p}) = g(p^2)$ . It is important to avoid any assumptions about the ratio  $p/q$ , which can be either very large or very small when both  $p$  and  $q$  are small (at the Brillouin peak  $p/q \sim 1$ , but in [31] it was unjustifiedly assumed that  $p \ll q$ ). Then

$$\begin{aligned} V(\mathbf{q}, \mathbf{p}) &= g(q^2) - g(q^2 + p^2 - 2\mathbf{p} \cdot \mathbf{q}) \\ &\approx g(0) + g'(0)q^2 - g(0) - g'(0)[q^2 + p^2 - 2\mathbf{p} \cdot \mathbf{q}] \\ &= -g'(0)[p^2 - 2\mathbf{p} \cdot \mathbf{q}]. \end{aligned} \quad (22)$$

If we now square the vertex function and perform the angular integral, we obtain ( $S_D$  is the surface of the sphere in  $D$  dimensions)

$$[g'(0)]^2 S_D \left( p^4 + \frac{1}{D} q^2 p^2 \right). \quad (23)$$

The integral over  $q$  is now straightforward, thanks to Dirac's  $\delta$  function in equation (15). We get

$$\text{Im } \Sigma^{(1)}(\mathbf{p}, \lambda + i0^+) \propto - \left[ \lambda^{(D-2)/2} p^4 + \lambda^{D/2} \frac{p^2}{Dc^2} \right]. \quad (24)$$

Hence, already at order  $1/\rho$ ,  $g_0(p^2)$  in equation (10) is of order  $p^4$ . Had we neglected the  $p^4$  term in favour of the  $q^2 p^2$  term (as done in [31]), we would have failed in identifying the  $g_0$  term. The physical reason for which the presence of such a term is mandatory (namely the existence of a Debye spectrum) was discussed in the concluding paragraph of section 2.



Let us now check that the amplitude  $\mathcal{A}$  in equation (8) vanishes. We merely need to compute the imaginary part of the self-energy at its lowest order in  $\lambda$ , namely  $\lambda^{(D-2)/2}$ . A general term of the diagrammatic expansion involves the factor

$$\int d^D \mathbf{q} G_0(\mathbf{q}, z) = \int dq d\Omega_D q^{D-1} G_0(\mathbf{q}, z), \quad (25)$$

where we have expressed the measure in terms of polar coordinates in  $D$  dimensions. Every bare propagator is associated to one or more vertices that are smooth functions of the involved momenta. In fact, we can expand the product of such vertices in a Taylor series. Now, the point is that, if we want the lowest order in  $z$ , we have to exclude all the terms that are proportional to  $q$  and we have to take only the zeroth-order term of the Taylor expansion. We can obtain this term simply making the following substitution:

$$\tilde{\text{Im}} \int d^D \mathbf{q} G_0(\mathbf{q}, \lambda + i0^+) = -\frac{\pi}{2} \lambda^{(D-2)/2} \int dq d\Omega_D \delta(q) \quad (26)$$

where  $\tilde{\text{Im}}$  stands for the imaginary part proportional to  $\lambda^{(D-2)/2}$ . Then

$$\tilde{\text{Im}} \Sigma_A^{(2)}(\mathbf{p}, \lambda + i0^+) = -\frac{\pi}{2} \lambda^{(D-2)/2} \int dq \int d\Omega_D V^2(q, p) \propto \lambda^{(D-2)/2} p^2 \quad (27)$$

and

$$\tilde{\text{Im}} \Sigma_A^{(2)}(\mathbf{p}, \lambda + i0^+) = -\tilde{\text{Im}} \Sigma_B^{(2)}(\mathbf{p}, \lambda + i0^+). \quad (28)$$

It follows that the amplitude  $\mathcal{A}$  vanishes, because the  $\Sigma_C^{(2)}$  contribution is already of order  $p^4$ :

$$\tilde{\text{Im}} \Sigma_C^{(2)}(\mathbf{p}, \lambda + i0^+) \propto -2\lambda^{(D-2)/2} V(\mathbf{p}, \mathbf{p}) \Sigma^{(1)}(\mathbf{p}, z) \propto \lambda^{(D-2)/2} p^4. \quad (29)$$

In the following, we will show explicitly that the cancellation of the  $\lambda^{(D-2)/2} p^2$  term arises even for a given topological class (quite large) of diagrams at  $1/\rho^3$  order, and we will provide an argument that predicts such a cancellation at any perturbative order.

#### 4. The combinatorial computation

The first approach to the computation of the resolvent is based on the expansion of equation (3) as a power series:

$$G(\mathbf{p}, z) = \sum_{R=1}^N \frac{1}{z^{R+1}} \left( \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i,j=1}^N \overline{e^{i\mathbf{p} \cdot (\mathbf{x}_j - \mathbf{x}_i)} [\mathcal{M}^R]_{ij}} \right). \quad (30)$$

Although the final results will only depend on  $p$ , in order to develop the formalism it is convenient to reintroduce the dependence on  $\mathbf{p}$ .

#### 4.1. Organizing the calculation. The bare propagator

4.1.1. *Momentum shift: choosing wisely the integration order.* The  $R$ th term of the expansion equation (30) is

$$\overline{\sum_{i_0, i_1, \dots, i_R} \frac{1}{N} \sum_{i, j=1}^N e^{i\mathbf{p} \cdot (\mathbf{x}_{i_R} - \mathbf{x}_{i_0})} [\mathcal{M}_{i_0, i_1} \mathcal{M}_{i_1, i_2} \cdots \mathcal{M}_{i_{R-1}, i_R}]}, \quad (31)$$

where the average over the vibrational centres take the form of a multi-dimensional integral with measure

$$\frac{1}{V^N} \int \prod_{i=1}^N d\mathbf{x}_i.$$

As for all such integrals, although the final result is independent of the order in which the individual integrals are performed, the difficulties encountered in a real computation are dramatically smaller if one finds a wise ordering for iterated integrations.

Now consider the expression

$$e^{-i\mathbf{p} \cdot \mathbf{x}_{i_l}} \left[ \delta_{i_l, i_{l+1}} \sum_{k_l \neq i_l} f(\mathbf{x}_{k_l} - \mathbf{x}_{i_l}) - [1 - \delta_{i_l, i_{l+1}}] f(\mathbf{x}_{i_l} - \mathbf{x}_{i_{l+1}}) \right], \quad (32)$$

which arises as a factor when introducing the explicit form (equation (2)) of  $\mathcal{M}$  into equation (31). When dealing with a diagonal term, we shall integrate over the position of the *medium* particle,  $\mathbf{x}_{k_l}$ ; when dealing with an off-diagonal term, we shall integrate over  $\mathbf{x}_{i_l}$ . For a diagonal term, the integral over the position of the medium particle is easy, if the particle index  $k_l$  does not appear elsewhere in the chain (even if the index  $i_l$  is *sure* to appear at least once more along the chain). For the non-diagonal term, the integral over  $\mathbf{x}_{i_l}$  is very simple if it does not appear later in the chain (even if  $i_{l+1}$  appears twice or more times in the chain, to the right). The two integrals yield

$$\frac{1}{V} [\hat{f}(\mathbf{0}) \delta_{i_l, i_{l+1}} - [1 - \delta_{i_l, i_{l+1}}] \hat{f}(\mathbf{p})] e^{-i\mathbf{p} \cdot \mathbf{x}_{i_{l+1}}}.$$

Since a term of order  $R$  has  $R$  such factors, the number of values the index  $k_l$  (or  $i_l$ ) can take without violating the non-repetition condition is between  $N$  and  $N - R$ . But both  $N/V$  and  $(N - R)/V$  tend to  $\rho$  in the thermodynamic limit, hence momentum can shift through non-index-repeating elements from left to right:

$$e^{-i\mathbf{p} \cdot \mathbf{x}_{i_l}} \left[ \delta_{i_l, i_{l+1}} \sum_{k_l \neq i_l} f(\mathbf{x}_{k_l} - \mathbf{x}_{i_l}) - [1 - \delta_{i_l, i_{l+1}}] f(\mathbf{x}_{i_l} - \mathbf{x}_{i_{l+1}}) \right] \longrightarrow \rho [\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})] e^{-i\mathbf{p} \cdot \mathbf{x}_{i_{l+1}}}. \quad (33)$$

Similarly, momentum can shift through non-repeating elements from right to left. In that case, one would integrate over  $\mathbf{x}_{k_l}$  (diagonal term) or over  $\mathbf{x}_{i_{l+1}}$  (non-diagonal):

$$\left[ \delta_{i_l, i_{l+1}} \sum_{k_l \neq i_l} f(\mathbf{x}_{k_l} - \mathbf{x}_{i_l}) - [1 - \delta_{i_l, i_{l+1}}] f(\mathbf{x}_{i_l} - \mathbf{x}_{i_{l+1}}) \right] e^{i\mathbf{p} \cdot \mathbf{x}_{i_{l+1}}} \longrightarrow e^{i\mathbf{p} \cdot \mathbf{x}_{i_l}} \rho [\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})]. \quad (34)$$

Note that a given matrix element might be considered as non-repeating for momentum shift from right to left, but it could not be suitable for the left-to-right momentum shift.

At this point, the computation of the leading order is straightforward. If there are no obstacles for momentum shift, we just push the left exponential  $e^{-i\mathbf{p}\cdot\mathbf{x}_{i_0}}$  to the right until it cancels out with  $e^{i\mathbf{p}\cdot\mathbf{x}_{i_R}}$ , leaving us with (since there are precisely  $R$  matrix elements)

$$\rho^R[\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})]^R. \tag{35}$$

Then the high-density limit of the sum in equation (30) is

$$G_0(\mathbf{p}, z) = \frac{1}{z - \epsilon(\mathbf{p})}, \tag{36}$$

which is then the bare propagator of the theory, as announced in equation (6).

*4.1.2. Repeated indices.* Now consider a situation where we can shift the external momentum  $\mathbf{p}$  from left to right until a particular particle index (say  $i_l = 1$  or  $k_l = 1$ ) is repeated in the chain somewhere to the right, so that we must stop. At this point, we shift the external momentum from right to left, until a particle label repetition  $i_{r+1} = 2$  or  $k_r = 2$  stop us. We depict this situation as

$$\dots 1 [\text{stuff}] 2 \dots \tag{37}$$

At this point we will have

$$\rho^S[\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})]^S e^{-i\mathbf{p}\cdot\mathbf{x}_{i_l}} 1 [\text{stuff}] 2 e^{i\mathbf{p}\cdot\mathbf{x}_{i_{r+1}}} \rho^L[\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})]^L.$$

Since the very same scheme of particle label repetitions  $1[\text{stuff}]2$  can be found for all values  $L, S = 0, 1, 2, \dots$ , we can sum all those terms to find a contribution

$$G_0(\mathbf{p}, z) e^{-i\mathbf{p}\cdot\mathbf{x}_{i_l}} 1 [\text{stuff}] 2 e^{i\mathbf{p}\cdot\mathbf{x}_{i_{r+1}}} G_0(\mathbf{p}, z).$$

We interpret the two factors  $G_0(\mathbf{p}, z)$  as the external legs for a Dyson resummation of the self-energy.

Clearly particle label repetitions are going to be very important in what follows, so some terminology will be useful. A generic factor

$$\left[ \delta_{i_l, i_{l+1}} \sum_{k_l \neq i_l} f(\mathbf{x}_{k_l} - \mathbf{x}_{i_l}) - [1 - \delta_{i_l, i_{l+1}}] f(\mathbf{x}_{i_l} - \mathbf{x}_{i_{l+1}}) \right] \tag{38}$$

will be called an L-stop if particles  $k_l$  or  $i_l$  are repeated somewhere to the right (so that momentum cannot be shifted from left to right through index  $i_l$ ). Similarly, we shall call it an R-stop if  $k_l$  or  $i_{l+1}$  are repeated somewhere to the left. We note that a matrix element can be both an L-stop and an R-stop (if  $k_l$  is repeated both to the right and to the left, or if  $i_l$  is repeated to the right while  $i_{l+1}$  is repeated to the left).

To make momentum flow through an L-stop or R-stops we resort to the so-called fake-particle trick. Consider a particle label, say 1, that appears twice (for instance, in an L-stop and in an R-stop to its right). Before carrying out the average over  $\{\mathbf{x}_i\}$ , we multiply the term by 1 written as

$$1 = \int d^D \mathbf{y}_{\bar{1}} \delta(\mathbf{x}_1 - \mathbf{y}_{\bar{1}}) = \frac{1}{(2\pi)^D} \int d^D \mathbf{y}_{\bar{1}} d^D \mathbf{q} e^{i\mathbf{q}(\mathbf{x}_1 - \mathbf{y}_{\bar{1}})}. \tag{39}$$

Then we can pretend that particle 1 takes two identities, 1 and  $\bar{1}$ , so that there is no repetition. The price we pay for this simplification is that:

- we have an extra integration over the internal momentum  $\mathbf{q}$ ,
- we have to deal with an extra factor  $e^{i\mathbf{q}\mathbf{x}_1}$  at the L-stop and an extra  $e^{-i\mathbf{q}\mathbf{y}_1}$  at the R-stop, and
- the fake particle  $\mathbf{y}_1$  does not bring a combinatorial  $N$  factor or an  $1/V$  from the normalization of the  $\mathbf{y}_1$  integral, so that there is a lacking factor of  $\rho$  (this we can ignore if we add compensating  $1/\rho$  to the final expression).

However, the modified momentum-shift formulae are simple enough to justify these inconveniences. Integrating over  $\mathbf{x}_1$  we obtain

$$e^{-i\mathbf{p}\mathbf{x}_{i_l}} e^{i\mathbf{q}\mathbf{x}_1} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] \longrightarrow V(\mathbf{q}, \mathbf{p}) e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}_{i_{l+1}}}. \quad (40a)$$

Similarly, integrating over  $\mathbf{y}_1$  at the R-stop, we have

$$[\delta_{i_r, i_{r+1}} f(\mathbf{y}_1 - \mathbf{x}_{i_r}) - (1 - \delta_{1, i_r}) \delta_{1, i_{r+1}} f(\mathbf{x}_{i_r} - \mathbf{y}_1)] e^{i\mathbf{p}\mathbf{x}_{i_{r+1}}} e^{-i\mathbf{q}\mathbf{y}_1} \longrightarrow e^{i(\mathbf{p}-\mathbf{q})\mathbf{x}_{i_r}} V(\mathbf{q}, \mathbf{p}). \quad (40b)$$

As a warning on momentum shift, note that one may shift momentum from left to right as long as there is nothing to the left still needing integration (and similarly for right-to-left shifts). Momentum shift can be visualized as a zip with two heads: one pulls both heads until they meet (and then there are no integrals left to be done).

4.1.3. *The reduction formula.* Imagine we face the situation

$$\dots 1[\text{stuff}]1 \dots,$$

i.e. the leftmost stop is an L-stop at index position  $l$ , the rightmost stop is an R-stop at index position  $r + 1$  and the particle that prevents the two momentum shifts is the same at both ends, say  $i_l = i_{r+1} = 1$  or any other possible combination ( $k_l = i_{r+1} = 1$ ,  $i_l = k_r = 1$  or  $k_l = k_r$ ). If the particle label 1 does not appear again inside the brackets, a nice reduction formula follows:

$$\begin{aligned} e^{-i\mathbf{p}\mathbf{x}_{i_l}} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - [1 - \delta_{1, i_{l+1}}] \delta_{i_l, 1} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] \times [\text{stuff}] \times [\delta_{i_r, i_{r+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_r}) \\ - [1 - \delta_{i_r, 1}] \delta_{i_{r+1}, 1} f(\mathbf{x}_{i_r} - \mathbf{x}_1)] e^{i\mathbf{p}\mathbf{x}_{i_{r+1}}} \longrightarrow \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V^2(\mathbf{q}, \mathbf{p}) \\ \times e^{-i\mathbf{q}\mathbf{x}_{i_{l+1}}} [\text{stuff}] e^{+i\mathbf{q}\mathbf{x}_{i_r}}. \end{aligned} \quad (41)$$

This can be proved by averaging over  $\mathbf{x}_1$ . To adjust the power of  $\rho$ , just recall that there were order  $N$  choices for (say) the index coincidence  $k_l = i_{r+1} = 1$ . Note that the proof of equation (41) involves doing four different integrals. Let us see how the fake-particle formulae (equations (40)) yield the same result effortlessly. The introduction of the fake particle transforms the left-hand side of equation (41) to

$$\begin{aligned} e^{-i\mathbf{p}\mathbf{x}_{i_l}} e^{i\mathbf{q}\mathbf{x}_1} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] \times [\text{stuff}] \times [\delta_{i_r, i_{r+1}} f(\mathbf{y}_1 - \mathbf{x}_{i_r}) \\ - (1 - \delta_{1, i_r}) \delta_{1, i_{r+1}} f(\mathbf{x}_{i_r} - \mathbf{y}_1)] e^{i\mathbf{p}\mathbf{x}_{i_{r+1}}} e^{-i\mathbf{q}\mathbf{y}_1}. \end{aligned} \quad (42)$$

We now merely shift momentum to the right using equation (40a) and to the left using equation (40b) to obtain

$$\frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V(\mathbf{q}, \mathbf{p})^2 e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}_{i_{l+1}}} \times [\text{stuff}] \times e^{+i(\mathbf{p}-\mathbf{q})\mathbf{x}_{i_r}}.$$

Now a change of integration variable  $\mathbf{q} \rightarrow \mathbf{p} - \mathbf{q}$  and the second of the identities (19) yield equation (41).

Both  $\Sigma^{(1)}$  and  $\Sigma_A^{(2)}$  follow directly from equation (41). Also, more general expressions can be found easily from it, as we shall see below.

#### 4.2. Order $1/\rho$

If no further particle label repetition arise, the momentum  $e^{-i\mathbf{q}\cdot\mathbf{x}_{i+1}}$  in equation (41) can be shifted to the right until it is killed by the second exponential  $e^{i\mathbf{q}\cdot\mathbf{x}_i}$ . We have then a set of contributions of the form

$$\sum_{a+b+c+2=R} \frac{[\rho(\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p}))]^a}{z^{a+1}} \times \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V^2(\mathbf{q}, \mathbf{p}) \frac{[\rho(\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{q}))]^b}{z^{b+1}} \times \frac{[\rho(\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p}))]^c}{z^{c+1}}, \quad (43)$$

composed of the product of three harmonic series that are easily seen to add up to

$$G_0(\mathbf{p}, z) \times \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V^2(\mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) \times G_0(\mathbf{p}, z). \quad (44)$$

If we interpret the two factors  $G_0(\mathbf{p}, z)$  as external legs of a Dyson resummation, we get

$$\Sigma^{(1)}(\mathbf{p}, z) = \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V^2(\mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z), \quad (45)$$

which is the first-order result anticipated in section 3.

#### 4.3. Order $1/\rho^2$

The cases with two pairs of repeated indices, or one index occurring three times, contribute to the second-order corrections. The contributions separate naturally into three kinds, according to the arrangement of the repeated indices.

*4.3.1. The nested case:*  $\Sigma_A^{(2)}$ . Take now the scheme of particle repetitions giving rise to equation (43) and place it in between an external pair of particle repetitions:

$$\dots 2 \dots 1 \dots 1 \dots 2 \dots$$

Assume that the index 2 happens twice and only twice in the chain. We are thus entitled to use the reduction formula, equation (41), for particle 2. The inner momentum  $\mathbf{q}$  can then be shifted (from either side) until it hits particle 1, where it produces a contribution such as equation (43). The only difference is in that the role previously played by the external momentum  $\mathbf{p}$  is now played by the internal momentum  $\mathbf{q}$ . We get, without need for further computation:

$$\Sigma_A^{(2)}(\mathbf{p}, z) = \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^3} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) \Sigma^{(1)}(\mathbf{q}, z) G_0(\mathbf{q}, z) V(\mathbf{q}, \mathbf{p}). \quad (46)$$

4.3.2. *The interleaved case:*  $\Sigma_B^{(2)}$ . The  $\Sigma_B^{(2)}$  piece in equation (16) arises from the pattern

$$\dots 1 \dots 2 \dots 1 \dots 2 \dots$$

A moment's thought indicates that the leftmost 1 must belong to an L-stop, while the rightmost 2 must be an R-stop. Furthermore, the internal 2 should be an L-stop (otherwise, one would use a fake particle to shift momentum from left to right over it trivially). For the same reason, the internal 1 should belong to an R-stop.

Our previous success with the reduction formula, equation (41), suggests that we try to deal with all such terms at once, by performing the integral

$$\begin{aligned} & \int \frac{d^D \mathbf{x}_1 d^D \mathbf{x}_2}{V^2} e^{-i\mathbf{p}\mathbf{x}_{i_l}} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] [\dots] \\ & \quad \times [\delta_{i_r, i_{r+1}} f(\mathbf{x}_2 - \mathbf{x}_{i_r}) - (1 - \delta_{2, i_{r+1}}) \delta_{2, i_r} f(\mathbf{x}_2 - \mathbf{x}_{i_{r+1}})] [\dots] \\ & \quad \times [\delta_{i_s, i_{s+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_s}) - (1 - \delta_{1, i_s}) \delta_{1, i_{s+1}} f(\mathbf{x}_{i_s} - \mathbf{x}_1)] [\dots] \\ & \quad \times [\delta_{i_z, i_{z+1}} f(\mathbf{x}_2 - \mathbf{x}_{i_z}) - (1 - \delta_{2, i_z}) \delta_{2, i_{z+1}} f(\mathbf{x}_{i_z} - \mathbf{x}_2)] e^{i\mathbf{p}\mathbf{x}_{i_{z+1}}}. \end{aligned} \quad (47)$$

Here the several  $[\dots]$  stand for arbitrary numbers of matrix elements without momentum stops arising. Note that not all the  $\dots 1 \dots 2 \dots 1 \dots 2 \dots$  terms have the form equation (47): the central 2, 1 particles may also collapse onto a single matrix element (necessarily non-diagonal) which is both an R-stop and an L-stop. Such terms will be considered in section 4.3.3.

We now introduce two fake particles,  $\tilde{1}$  and  $\tilde{2}$ , to transform the above integral into

$$\begin{aligned} & \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} \int \frac{d^D \mathbf{x}_1 d^D \mathbf{y}_{\tilde{1}} d^D \mathbf{x}_2 d^D \mathbf{y}_{\tilde{2}}}{V^2} e^{-i\mathbf{p}\mathbf{x}_{i_l}} \\ & \quad \times [\delta_{i_l, i_{l+1}} e^{i\mathbf{q}\mathbf{x}_1} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} e^{i\mathbf{q}\mathbf{x}_1} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] [\dots] \\ & \quad \times [\delta_{i_r, i_{r+1}} e^{i\mathbf{k}\mathbf{x}_2} f(\mathbf{x}_2 - \mathbf{x}_{i_r}) - (1 - \delta_{2, i_{r+1}}) \delta_{2, i_r} e^{i\mathbf{k}\mathbf{x}_2} f(\mathbf{x}_2 - \mathbf{x}_{i_{r+1}})] [\dots] \\ & \quad \times [\delta_{i_s, i_{s+1}} e^{-i\mathbf{q}\mathbf{y}_{\tilde{1}}} f(\mathbf{y}_{\tilde{1}} - \mathbf{x}_{i_s}) - (1 - \delta_{\tilde{1}, i_s}) \delta_{\tilde{1}, i_{s+1}} e^{-i\mathbf{q}\mathbf{y}_{\tilde{1}}} f(\mathbf{x}_{i_s} - \mathbf{y}_{\tilde{1}})] [\dots] \\ & \quad \times [\delta_{i_z, i_{z+1}} e^{-i\mathbf{k}\mathbf{y}_{\tilde{2}}} f(\mathbf{y}_{\tilde{2}} - \mathbf{x}_{i_z}) - (1 - \delta_{\tilde{2}, i_z}) \delta_{\tilde{2}, i_{z+1}} e^{-i\mathbf{k}\mathbf{y}_{\tilde{2}}} f(\mathbf{x}_{i_z} - \mathbf{y}_{\tilde{2}})] e^{i\mathbf{p}\mathbf{x}_{i_{z+1}}}. \end{aligned} \quad (48)$$

One then shifts momentum from left to right up to  $i_s$  and from right to left again up to  $i_s$  to find

$$\begin{aligned} & \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{p} - \mathbf{q}, z) V(\mathbf{k}, \mathbf{p} - \mathbf{q}) G_0(\mathbf{p} - \mathbf{q} - \mathbf{k}, z) \\ & \quad \times V(\mathbf{q}, \mathbf{p} - \mathbf{k}) G_0(\mathbf{p} - \mathbf{k}, z) V(\mathbf{k}, \mathbf{p}). \end{aligned} \quad (49)$$

Equation (17b) follows after changing integration variables according to

$$\mathbf{p} - \mathbf{q} \rightarrow \mathbf{q}, \quad \text{and} \quad \mathbf{q} - \mathbf{k} \rightarrow \mathbf{k}. \quad (50)$$

4.3.3. *The collapse of an L-stop and an R-stop:*  $\Sigma_C^{(2)}$ . As we have remarked, it can happen that the L-stop and R-stop of equation (47) actually belong to the same matrix element, necessarily non-diagonal. However, any non-diagonal term should be paired with a diagonal one. As we mentioned in section 4.1.1, a diagonal term can be both an L-stop and an R-stop if the medium particle is repeated both to the left and to the right.



Hence we will be considering here this kind of term ( $O$ : of diagonal matrix element,  $D$ : diagonal matrix element):

$$1 \dots O(21) \dots 2 \quad + \quad 1 \dots D(1) \dots 1. \quad (51)$$

The terms with an off-diagonal index appearing three times ( $1 \dots O(1?) \dots 1$ ) do not belong to  $\Sigma_C^{(2)}$  and are considered in section 4.3.4.

Let us start with the case  $1 \dots D(1) \dots 1$ :

$$\begin{aligned} & \int \frac{d^D \mathbf{x}_1}{V} e^{-i\mathbf{p}\mathbf{x}_{i_l}} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] [\dots] \\ & \quad \times \delta_{i_r, i_{r+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_r}) [\dots] [\delta_{i_s, i_{s+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_s}) - (1 - \delta_{1, i_s}) \\ & \quad \times \delta_{1, i_{s+1}} f(\mathbf{x}_{i_s} - \mathbf{x}_1)] e^{i\mathbf{p}\mathbf{x}_{i_{s+1}}}. \end{aligned} \quad (52)$$

We now introduce two extra fake particles to substitute for particle 1, namely  $\tilde{1}$  and  $\hat{1}$  via the identity

$$\begin{aligned} 1 &= \int d^D \mathbf{y}_{\tilde{1}} \int d^D \mathbf{z}_{\hat{1}} \delta(\mathbf{x}_1 - \mathbf{y}_{\tilde{1}}) \delta(\mathbf{y}_{\tilde{1}} - \mathbf{z}_{\hat{1}}) \\ &= \frac{1}{(2\pi)^{2D}} \int d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{y}_{\tilde{1}} d^D \mathbf{z}_{\hat{1}} e^{i\mathbf{q}(\mathbf{x}_1 - \mathbf{y}_{\tilde{1}})} e^{i\mathbf{k}(\mathbf{y}_{\tilde{1}} - \mathbf{z}_{\hat{1}})}, \end{aligned} \quad (53)$$

to find

$$\begin{aligned} & \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} \int \frac{d^D \mathbf{x}_1 d^D \mathbf{y}_{\tilde{1}} d^D \mathbf{z}_{\hat{1}}}{V} e^{-i\mathbf{p}\mathbf{x}_{i_l}} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] \\ & \quad \times e^{i\mathbf{q}\mathbf{x}_1} [\dots] \delta_{i_r, i_{r+1}} f(\mathbf{y}_{\tilde{1}} - \mathbf{x}_{i_r}) e^{i(\mathbf{k} - \mathbf{q})\mathbf{x}_{\tilde{1}}} [\dots] e^{-i\mathbf{k}\mathbf{z}_{\hat{1}}} \\ & \quad \times [\delta_{i_s, i_{s+1}} f(\mathbf{z}_{\hat{1}} - \mathbf{x}_{i_s}) - (1 - \delta_{\hat{1}, i_{s+1}}) \delta_{\hat{1}, i_{s+1}} f(\mathbf{x}_{i_s} - \mathbf{z}_{\hat{1}})] e^{-i\mathbf{p}\mathbf{x}_{i_{s+1}}}. \end{aligned} \quad (54)$$

Finally we shift momentum from left to right up to  $i_r$ , from right to left up to  $i_{r+1}$  and integrate over  $\tilde{\mathbf{x}}_1$ . We obtain

$$1 \dots D(1) \dots 1 = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{p} - \mathbf{q}, z) [\rho \hat{f}(\mathbf{k} - \mathbf{q})] G_0(\mathbf{p} - \mathbf{k}) V(\mathbf{k}, \mathbf{p}). \quad (55)$$

Consider now  $1 \dots O(21) \dots 2$ :

$$\begin{aligned} & \int \frac{d^D \mathbf{x}_1 d^D \mathbf{x}_2}{V^2} e^{-i\mathbf{p}\mathbf{x}_{i_l}} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] [\dots] \\ & \quad \times [-\delta_{i_r, 2}] \delta_{i_{r+1}, 1} f(\mathbf{x}_2 - \mathbf{x}_1) [\dots] [\delta_{i_s, i_{s+1}} f(\mathbf{x}_2 - \mathbf{x}_{i_s}) - (1 - \delta_{2, i_s}) \delta_{2, i_{s+1}} \\ & \quad \times f(\mathbf{x}_{i_s} - \mathbf{x}_2)] e^{i\mathbf{p}\mathbf{x}_{i_{s+1}}}. \end{aligned} \quad (56)$$

Introducing two fake particles,  $\tilde{1}$  and  $\tilde{2}$ , we can rewrite it as

$$\begin{aligned} & \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{x}_1 d^D \mathbf{x}_2}{(2\pi)^{2D} V^2} e^{-i\mathbf{p}\mathbf{x}_{i_l}} e^{i\mathbf{q}\mathbf{x}_1} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] [\dots] \\ & \quad \times [-e^{-i\mathbf{q}\mathbf{y}_{\tilde{1}}} \delta_{i_r, 2} \delta_{i_{r+1}, \tilde{1}} f(\mathbf{x}_2 - \mathbf{y}_{\tilde{1}}) e^{i\mathbf{k}\mathbf{x}_2} [\dots] [\delta_{i_s, i_{s+1}} f(\mathbf{x}_2 - \mathbf{x}_{i_s}) \\ & \quad - (1 - \delta_{\tilde{2}, i_s}) \delta_{\tilde{2}, i_{s+1}} f(\mathbf{x}_{i_s} - \mathbf{y}_{\tilde{2}})] e^{-i\mathbf{k}\mathbf{y}_{\tilde{2}}} e^{i\mathbf{p}\mathbf{x}_{i_{s+1}}}. \end{aligned} \quad (57)$$

Shifting momentum left to right up to  $i_r$  and from right to left up to  $i_{r+1}$  we are left with

$$-\frac{1}{V} \int d^D \mathbf{x}_2 d^D \mathbf{y}_{\tilde{1}} f(\mathbf{x}_2 - \mathbf{y}_{\tilde{1}}) e^{-i(\mathbf{p} - \mathbf{q} - \mathbf{k})(\mathbf{x}_2 - \mathbf{x}_1)} = -\hat{f}(\mathbf{p} - \mathbf{q} - \mathbf{k}), \quad (58)$$

so that, adjusting the power of  $\rho$ , we get

$$1 \dots O(21) \dots 2 = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{p} - \mathbf{q}, z) [-\rho \hat{f}(\mathbf{p} - \mathbf{q} - \mathbf{k})] \\ \times G_0(\mathbf{p} - \mathbf{k}, z) V(\mathbf{k}, \mathbf{p}). \quad (59)$$

Adding together the two pieces, equations (59) and (55), we finally find

$$\Sigma_C^{(2)} = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{p} - \mathbf{q}, z) V(\mathbf{k} - \mathbf{q}, \mathbf{p} - 2\mathbf{q}) G_0(\mathbf{p} - \mathbf{k}, z) V(\mathbf{k}, \mathbf{p}), \quad (60)$$

which after the change of variables

$$\mathbf{q} \longrightarrow \mathbf{p} - \mathbf{q}, \quad \mathbf{k} \longrightarrow \mathbf{p} - \mathbf{k}, \quad (61)$$

and use of identities equation (19) yield equation (17c).

4.3.4. *The Dyson resummation to order  $1/\rho^2$ .* Recalling equation (14), we notice that we have still not identified the pattern of particle label repetitions that gives rise to the second-order terms appearing in the Dyson resummation of the first-order self-energy:

$$\Sigma^{(1)}(\mathbf{p}, z) G_0(\mathbf{p}, z) \Sigma^{(1)}(\mathbf{p}, z), \quad (62)$$

(we have not written the irrelevant external legs). The natural candidate is

$$1 \dots 1 \dots 2 \dots 2, \quad (63)$$

where the sequence is L-stop, R-stop, L-stop, R-stop. This expectation is correct, but it will turn out that the constraint imposed by the matrix-product structure needs extra terms to build equation (62). These missing terms will be provided by the pattern  $1 \dots O(1?) \dots 1$ .

Let us first compute blindly the term  $1 \dots 1 \dots 2 \dots 2$ , incurring a quite instructive mistake. We introduce only one fake particle,  $\tilde{1}$ :

$$e^{-i\mathbf{p}\mathbf{x}_{i_l}} e^{i\mathbf{q}\mathbf{x}_1} [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] [\dots] \\ \times e^{-i\mathbf{q}\mathbf{y}_{\tilde{1}}} [\delta_{i_r, i_{r+1}} f(\mathbf{y}_{\tilde{1}} - \mathbf{x}_{i_r}) - (1 - \delta_{\tilde{1}, i_r}) \delta_{\tilde{1}, i_{r+1}} f(\mathbf{x}_{i_r} - \mathbf{y}_{\tilde{1}})] [\dots] \\ \times [\delta_{i_s, i_{s+1}} f(\mathbf{x}_2 - \mathbf{x}_{i_s}) - (1 - \delta_{2, i_{s+1}}) \delta_{2, i_s} f(\mathbf{x}_2 - \mathbf{x}_{i_{s+1}})] [\dots] \\ \times [\delta_{i_z, i_{z+1}} f(\mathbf{x}_2 - \mathbf{x}_{i_r}) - (1 - \delta_{2, i_z}) \delta_{2, i_{z+1}} f(\mathbf{x}_{i_r} - \mathbf{x}_2)] e^{i\mathbf{p}\mathbf{x}_{i_{z+1}}}. \quad (64)$$

We shift momentum from left to right up to  $i_r$  as usual. At this point, we still need to push the momentum to the right (*this* is unusual). We need to perform two integrals:

$$\int d^D \mathbf{y}_{\tilde{1}} e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}_{i_r}} e^{-i\mathbf{q}\mathbf{y}_{\tilde{1}}} [\delta_{i_r, i_{r+1}} f(\mathbf{y}_{\tilde{1}} - \mathbf{x}_{i_r})] = \hat{f}(\mathbf{q}) e^{i\mathbf{p}\mathbf{x}_{i_{s+1}}}, \quad (65)$$

$$\int d^D \mathbf{x}_{i_r} e^{-i(\mathbf{p}-\mathbf{q})\mathbf{x}_{i_r}} e^{-i\mathbf{q}\mathbf{y}_{\tilde{1}}} [-(1 - \delta_{\tilde{1}, i_r}) \delta_{\tilde{1}, i_{r+1}} f(\mathbf{x}_{i_r} - \mathbf{y}_{\tilde{1}})] = -\hat{f}(\mathbf{p} - \mathbf{q}) e^{i\mathbf{p}\mathbf{x}_{i_{s+1}}}. \quad (66)$$

Hence the integrations up to this point yield

$$\frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{p} - \mathbf{q}, z) V(\mathbf{q}, \mathbf{p}) e^{i\mathbf{p}\mathbf{x}_{i_{s+1}}} = \Sigma^{(1)}(\mathbf{p}, z) e^{i\mathbf{p}\mathbf{x}_{i_{s+1}}}. \quad (67)$$

It seems to be an easy matter to complete the computation: one pushes momentum  $p$  to the right up to  $i_s$ , seemingly yielding a bare propagator  $G_0(\mathbf{p}, z)$ , and we would be left with  $2 \dots 2$  (a standard diagram for the self-energy at order  $1/\rho$ ). However, after some reflection it is clear that an R-stop and an L-stop such as  $\dots 1 \dots 2 \dots$ , where both particle 1 and particle 2 appear in off-diagonal matrix elements, must be separated by at least one off-diagonal matrix element. Hence if there are  $S$  matrix elements between the R-stop and the L-stop, when shifting momentum  $p$  we will encounter a factor

$$\rho^S [\hat{f}(\mathbf{0}) - \hat{f}(\mathbf{p})]^S - [\rho \hat{f}(\mathbf{0})]^S$$

which, adding the geometric series, means

$$G_0(\mathbf{p}, z) - \frac{1}{z - \rho \hat{f}(\mathbf{0})}.$$

Hence the correct result is

$$\begin{aligned} 1 \dots 1 \dots 2 \dots 2 &= \Sigma^{(1)}(\mathbf{p}, z) G_0(\mathbf{p}, z) \Sigma^{(1)}(\mathbf{p}, z) \\ &- \frac{1}{z - \rho \hat{f}(\mathbf{0})} \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{p} - \mathbf{q}, z) [\rho \hat{f}(\mathbf{p} - \mathbf{q})] \\ &\times [\rho \hat{f}(\mathbf{p} - \mathbf{k})] G_0(\mathbf{p} - \mathbf{k}, z) V(\mathbf{k}, \mathbf{p}). \end{aligned} \tag{68}$$

We will now show that the second term in equation (68) is cancelled by the contribution from

$$1 \dots O(1?) \dots 1.$$

In this pattern, the leftmost 1 belongs to an L-stop and the rightmost one to an R-stop. The first observation is that the central 1 in the  $O(1?)$  must necessarily appear in an R-stop (because we never find the same particle in any matrix element  $f(\mathbf{x}_i - \mathbf{x}_j)$ , and due to the constraint imposed by the matrix product). The second observation is that there must be at least one off-diagonal matrix element between the two R-stops sharing the common particle 1. Introducing fake particles 1 and  $\tilde{1}$ , we are left with

$$\begin{aligned} &\int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} \int \frac{d^D \mathbf{x}_1 d^D \mathbf{y}_{\tilde{1}} d^D \mathbf{z}_{\tilde{1}}}{V} e^{-i\mathbf{p}\mathbf{x}_{i_l}} \\ &\times [\delta_{i_l, i_{l+1}} f(\mathbf{x}_1 - \mathbf{x}_{i_l}) - (1 - \delta_{1, i_{l+1}}) \delta_{1, i_l} f(\mathbf{x}_1 - \mathbf{x}_{i_{l+1}})] e^{i\mathbf{q}\mathbf{x}_1} [\dots] \\ &\times [-(1 - \delta_{\tilde{1}, i_{r+1}}) \delta_{\tilde{1}, i_r} f(\mathbf{x}_{i_r} - \mathbf{y}_{\tilde{1}})] e^{i(\mathbf{k}-\mathbf{q})\mathbf{y}_{\tilde{1}}} [\dots]' \\ &\times e^{-i\mathbf{k}\mathbf{z}_{\tilde{1}}} [\delta_{i_s, i_{s+1}} f(\mathbf{z}_{\tilde{1}} - \mathbf{x}_{i_s}) - (1 - \delta_{\tilde{1}, i_s}) \delta_{\tilde{1}, i_{s+1}} f(\mathbf{x}_{i_s} - \mathbf{z}_{\tilde{1}})] e^{-i\mathbf{p}\mathbf{x}_{i_{s+1}}}. \end{aligned} \tag{69}$$

All that remains is a simple momentum shift, keeping in mind that, when going over the factor  $[\dots]'$ , it will give

$$G_0(\mathbf{p} - \mathbf{k}, z) - \frac{1}{z - \rho \hat{f}(\mathbf{0})} = -\rho \hat{f}(\mathbf{p} - \mathbf{k}) \frac{G_0(\mathbf{p} - \mathbf{k}, z)}{z - \rho \hat{f}(\mathbf{0})}. \tag{70}$$

Thus one finally finds

$$\begin{aligned} 1 \dots O(1?) \dots 1 &= \frac{1}{z - \rho \hat{f}(\mathbf{0})} \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{p} - \mathbf{q}, z) [\rho \hat{f}(\mathbf{p} - \mathbf{q})] \\ &\times [\rho \hat{f}(\mathbf{p} - \mathbf{k})] G_0(\mathbf{p} - \mathbf{k}, z) V(\mathbf{k}, \mathbf{p}). \end{aligned} \tag{71}$$

#### 4.4. Higher orders

The argument of section 4.3.1 is fully general. Consider the contribution of order  $1/\rho^n$  to the *propagator*, rather than the self-energy (i.e. let us include both the connected and disconnected pieces). We can write this as  $G_0(\mathbf{p}, z)H^{(n)}(\mathbf{p}, z)G_0(\mathbf{p}, z)$ . Let us emphasize that  $H^{(n)}(\mathbf{p}, z)$  refers to the full contribution to the propagator at order  $1/\rho^n$ , not to a particular topological subset (such as the cactus [17]).

We may enclose the scheme of particle label repetitions that generates  $H^{(n)}(\mathbf{p}, z)$  within an L-stop and an R-stop with equal particle labels that do not appear again along the chain. Under such circumstances, we are entitled to use the reduction formula, equation (41), which yields

$$\Sigma_A^{(n+1)}(\mathbf{p}, z) = \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} V(\mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) H^{(n)}(\mathbf{q}, z) G_0(\mathbf{q}, z) V(\mathbf{q}, \mathbf{p}). \quad (72)$$

Clearly this is not the full self-energy at order  $1/\rho^{n+1}$ , but it is a genuine part of it that automatically verifies

$$\Sigma_A^{(n+1)}(\mathbf{p}, z) = 0. \quad (73)$$

In particular if  $n = 1$  this gives the  $1/\rho^2$  term  $\Sigma_A^2(\mathbf{p}, z)$  discussed above. It is interesting to note that

$$\text{Im} \Sigma_A^{(n+1)}(\mathbf{p}, \lambda + i0^+) \sim p^2 \lambda^{(D-2)/2}, \quad (74)$$

since for  $q \sim 1$  and  $z = \lambda + i0^+$ , for small  $\lambda$  it is expected that (for the Debye spectrum see section 2)

$$\text{Im} H^{(n)}(\mathbf{q}, z) \propto \lambda^{(D-2)/2}. \quad (75)$$

Thus, the vanishing of the amplitude  $\mathcal{A}$  in equation (8) implies that non-trivial cancellations occur at all orders in perturbation theory. Since we are presenting arguments for such a cancellation, we agree with [31] in that the  $\Sigma_A^{(n+1)}$  terms alone do not reproduce the correct analytic structure of the theory.

*4.4.1. Towards the self-energy at third order.* Using the combinatorial rules described above, it is possible to push the perturbative computation to order  $1/\rho^3$ , which has never been attempted before. Here we will limit ourselves to the terms without collapse of an R-stop with an L-stop (i.e. we will retain only the terms with 6 vertex functions). The reason is that the combinatorial computation suggests very simple Feynman rules that can be used to obtain the diagrams, without lengthy computations. The purpose is to check that, at least within this subclass of diagrams, the cancellation of the prefactor of the  $p^2 \omega^{D-2}$  term still occurs.

Let us describe the Feynman rules. Take for instance a term such as

$$L1 \dots L3 \dots L2 \dots R3 \dots R1 \dots R2.$$

The rules are as follows:

- (1) Draw an horizontal full line and mark on it the stops (preserving the ordering).
- (2) Join the corresponding L- and R-stops with a dashed line.

- (3) The diagram has an incoming momentum  $p$ , from left to right.
- (4) Attach a momentum to every line (full or dashed), applying momentum conservation at each stop.
- (5) Associate a bare propagator,  $G_0$ , to each full line.
- (6) Associate a vertex function to every stop, such that its first argument is *always* the momentum running over the dotted line.
- (7) For an L-stop, the second argument of the vertex will be the momentum running over the full line to its left.
- (8) For an R-stop, the second argument of the vertex will be the momentum running over the full line to its right.
- (9) Multiply by  $1/\rho^3$  and integrate over the internal momenta.

Applying these rules to the patterns without stop collapse, we obtain the following contributions.

(a) *Terms* L1...L2...L3...R3...R2...R1.

$$\begin{aligned}
 I_1 = \text{[Diagram: A semi-circular chain of three nested dashed arcs on a solid line]} &= \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) \\
 &\times G_0(\mathbf{l}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) G_0(\mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{q}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}). \tag{76}
 \end{aligned}$$

Now we wish to compute ( $S_D$  : surface of the unit sphere in  $D$  dimensions) the limit

$$J_1 = -\frac{(2\pi)^D}{\pi S_D} \lim_{z \rightarrow 0^+} \frac{\text{Im } I_1(p, z)}{z^{(D-2)/2}}, \tag{77}$$

and in general,  $J_k$ , defined from  $I_k(p, z)$  as the same limiting procedure.

The rules to obtain the limit painlessly are simple:

- (1) Locate a propagator,  $G(q)$ , whose running momentum is never a second argument of a vertex function  $V(\cdot, q)$ .
- (2) Substitute that propagator by  $-(\pi S_D / (2\pi) D) z^{(D-2)/2} \delta(q)$  and perform the  $q$  integral.
- (3) Apply the simplification

$$G_0(\mathbf{q}, z) V(\mathbf{q}, \mathbf{q}) = \frac{V(\mathbf{q}, \mathbf{q})}{z + V(\mathbf{q}, \mathbf{q})} = 1 + \mathcal{O}(z). \tag{78}$$

For  $I_1$  only  $l = 0$  gives a contribution to  $J_1$ , hence

$$\begin{aligned}
 J_1 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} &V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) \\
 &\times G_0(\mathbf{q}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}). \tag{79}
 \end{aligned}$$

(b) *Terms* L1...L2...L3...R2...R3...R1

$$\begin{aligned}
 I_2 = \text{Diagram} &= \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) \\
 &\times G_0(\mathbf{l}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q} - \mathbf{k} + \mathbf{l}) \\
 &\times G_0(\mathbf{q} - \mathbf{k} + \mathbf{l}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{q}) G_0(\mathbf{q}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}).
 \end{aligned} \tag{80}$$

For  $I_2$  one easily realizes that only  $l = 0$  contributes to  $J_2$ :

$$J_2 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) V(\mathbf{k}, \mathbf{q}) G_0(\mathbf{q}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}). \tag{81}$$

Since  $V(\mathbf{q} - \mathbf{k}, \mathbf{q}) = -V(\mathbf{k}, \mathbf{q})$ , one has  $J_2 = -J_1$ .

(c) *Terms* L1...L2...R2...L3...R3...R1

$$\begin{aligned}
 I_3 = \text{Diagram} &= \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) \\
 &\times G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{l}, \mathbf{q}) G_0(\mathbf{l}, z) V(\mathbf{q} - \mathbf{l}, \mathbf{q}) G_0(\mathbf{q}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}).
 \end{aligned} \tag{82}$$

Both  $\mathbf{k} = 0$  and  $\mathbf{l} = 0$  contribute to  $J_3$  (for  $\mathbf{k} = 0$  we changed the dummy variable  $\mathbf{l}$  to  $\mathbf{k}$ ):

$$J_3 = \frac{2}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{q}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}). \tag{83}$$

(d) *Terms* L1...L3...R3...L2...R1...R2

$$\begin{aligned}
 I_4 = \text{Diagram} &= \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) \\
 &\times G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{l}, \mathbf{q}) G_0(\mathbf{l}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{l}) \\
 &\times G_0(\mathbf{p} - \mathbf{q} + \mathbf{l}, z) V(\mathbf{q} - \mathbf{l}, \mathbf{p}).
 \end{aligned} \tag{84}$$

For  $J_4$  both  $\mathbf{k} = 0$  and  $\mathbf{l} = 0$  are relevant:

$$\begin{aligned}
 J_4 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} &V(\mathbf{p} - \mathbf{q}, \mathbf{p}) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{k}) \\
 &\times G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p}) + \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) \\
 &\times G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) V(\mathbf{q}, \mathbf{p}).
 \end{aligned} \tag{85}$$



(e) Terms L1...L2...L3...R3...R1...R2

$$\begin{aligned}
 I_5 = \text{Diagram} &= \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) \\
 &\times G_0(\mathbf{l}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) G_0(\mathbf{k}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{k}) \\
 &\times G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p}).
 \end{aligned} \tag{86}$$

The only relevant contribution is now  $\mathbf{l} = 0$ :

$$\begin{aligned}
 J_5 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} &V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{k}) \\
 &\times G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p}).
 \end{aligned} \tag{87}$$

(f) Terms L1...L2...R1...L3...R3...R2

$$\begin{aligned}
 I_6 = \text{Diagram} &= \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{k}) \\
 &\times G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{p} - \mathbf{q} + \mathbf{k} - \mathbf{l}, \mathbf{p} - \mathbf{q} + \mathbf{k}) G_0(\mathbf{l}, z) \\
 &\times V(\mathbf{p} - \mathbf{q} + \mathbf{k} - \mathbf{l}, \mathbf{p} - \mathbf{q} + \mathbf{k}) G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p}).
 \end{aligned} \tag{88}$$

$J_6$  stems both from  $\mathbf{k} = 0$  and from  $\mathbf{l} = 0$ . For the  $\mathbf{k} = 0$  contribution, we make the change of variable  $\mathbf{q} \rightarrow \mathbf{p} - \mathbf{q}$  to identify the cancellation with  $J_3$ :

$$\begin{aligned}
 J_6 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} &V(\mathbf{q}, \mathbf{p}) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{q}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}) \\
 &+ \frac{1}{\rho^3} \int \frac{d^D q d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{k}) V(\mathbf{q} - \mathbf{k}, \mathbf{p}).
 \end{aligned} \tag{89}$$

(g) Terms L1...L2...L3...R1...R3...R2

$$\begin{aligned}
 I_7 = \text{Diagram} &= \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \\
 &\times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) \\
 &\times G_0(\mathbf{l}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{l}) G_0(\mathbf{p} - \mathbf{q} + \mathbf{l}, z) \\
 &\times V(\mathbf{k} - \mathbf{l}, \mathbf{p} - \mathbf{q} + \mathbf{k}) G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p}).
 \end{aligned} \tag{90}$$

Only  $\mathbf{l} = 0$  contributes to  $J_7$ :

$$J_7 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) V(\mathbf{k}, \mathbf{p} - \mathbf{q} + \mathbf{k}) \times G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p}). \quad (91)$$

(h) Terms L1...L3...L2...R1...R3...R2

$$I_8 = \text{Diagram} = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) \times G_0(\mathbf{l}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{l}) G_0(\mathbf{p} - \mathbf{q} + \mathbf{l}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{p} + \mathbf{l} - \mathbf{k}) \times G_0(\mathbf{p} + \mathbf{l} - \mathbf{k}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{p}). \quad (92)$$

Again, only  $\mathbf{l} = 0$  matters:

$$J_8 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) V(\mathbf{q} - \mathbf{k}, \mathbf{p} - \mathbf{k}) G_0(\mathbf{p} - \mathbf{k}, z) V(\mathbf{k}, \mathbf{p}). \quad (93)$$

(i) Terms L1...L3...L2...R3...R1...R2

$$I_9 = \text{Diagram} = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{k}) \times G_0(\mathbf{l}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q} - \mathbf{k} + \mathbf{l}) G_0(\mathbf{q} - \mathbf{k} + \mathbf{l}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{k} + \mathbf{l}) \times G_0(\mathbf{p} - \mathbf{k} + \mathbf{l}, z) V(\mathbf{k} - \mathbf{l}, \mathbf{p}). \quad (94)$$

And, once again, only  $\mathbf{l} = 0$  contributes:

$$J_9 = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{k}) G_0(\mathbf{p} - \mathbf{k}, z) V(\mathbf{k}, \mathbf{p}). \quad (95)$$

(j) Terms L1...L3...R1...L2...R3...R2

$$I_{10} = \text{Diagram} = \frac{1}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k} d^D \mathbf{l}}{(2\pi)^{3D}} \times V(\mathbf{p} - \mathbf{q}, \mathbf{p}) G_0(\mathbf{q}, z) V(\mathbf{q} - \mathbf{k}, \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} + \mathbf{k}) \times G_0(\mathbf{p} - \mathbf{q} + \mathbf{k}, z) V(\mathbf{p} - \mathbf{q} + \mathbf{k} - \mathbf{l}, \mathbf{p} - \mathbf{q} + \mathbf{k}) G_0(\mathbf{l}, z) \times V(\mathbf{q} - \mathbf{k}, \mathbf{q} - \mathbf{k} + \mathbf{l}) G_0(\mathbf{q} - \mathbf{k} + \mathbf{l}, z) V(\mathbf{p} - \mathbf{q} + \mathbf{k} - \mathbf{l}, \mathbf{p}). \quad (96)$$

Here we have a contribution from  $\mathbf{k} = 0$  as well as from  $\mathbf{l} = 0$ :

$$J_{10} = \frac{2}{\rho^3} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} V(\mathbf{p} - \mathbf{q}, \mathbf{p}) V(\mathbf{p} - \mathbf{q} - \mathbf{k}, \mathbf{p} - \mathbf{q}) G_0(\mathbf{k}, z) V(\mathbf{q}, \mathbf{q} + \mathbf{k}) \times G_0(\mathbf{q} + \mathbf{k}, z) V(\mathbf{p} - \mathbf{q} - \mathbf{k}, \mathbf{p}). \quad (97)$$

4.4.2. *Resummation of the imaginary parts.* The resummation of the imaginary parts of the previous diagrams is simple. Using the properties of the vertex  $V(\mathbf{p}, \mathbf{q})$  and changing carefully the integration variables when necessary we can show that

$$J_1 + J_2 = 0 \quad (98)$$

$$J_3 + J_4 + J_6 + J_{10} = 0 \quad (99)$$

$$J_5 + J_7 = 0 \quad (100)$$

$$J_8 + J_9 = 0 \quad (101)$$

so that

$$\sum_{i=1}^{10} J_i = 0, \quad (102)$$

and the total contribution to the imaginary part proportional to  $z^{(D-2)/2}$  vanishes.

## 5. A field theory approach

In this section we will introduce a field-theoretical representation for the resolvent  $G(\mathbf{p}, z)$ . Within this formalism one is able to obtain the perturbative computation for the self-energy in a more straightforward way than with previous formulations [15]. Interestingly enough, due to the ultraviolet behaviour of the bare propagator of the field involved, such a perturbative expansion yields some divergent terms that can be summed up to zero. The starting point is the following representation for the resolvent:

$$G(\mathbf{p}, z) = \frac{1}{N} \sum_{ij} e^{i\mathbf{p} \cdot (\mathbf{x}_i - \mathbf{x}_j)} \frac{\int (\prod_i^N d\phi_i) \phi_i \phi_j \exp\{-(1/2) \sum_{lm} \phi_l [(z - \sum_k f(\mathbf{x}_l - \mathbf{x}_k)) \delta_{lm} + f(\mathbf{x}_l - \mathbf{x}_m)] \phi_m\}}{\int (\prod_i^N d\phi_i) \exp\{-(1/2) \sum_{lm} \phi_l [(z - \sum_k f(\mathbf{x}_l - \mathbf{x}_k)) \delta_{lm} + f(\mathbf{x}_l - \mathbf{x}_m)] \phi_m\}}. \quad (103)$$

Introducing the fields

$$\phi(\mathbf{x}) \equiv \begin{cases} \phi_i & \mathbf{x} = \mathbf{x}_i, \\ \text{arbitrary} & \mathbf{x} \neq \mathbf{x}_i, \end{cases} \quad (104)$$

$$\rho(\mathbf{x}) \equiv \frac{1}{\rho} \sum_k \delta(\mathbf{x} - \mathbf{x}_k), \quad (105)$$

one has

$$G(\mathbf{p}, z) = \frac{\rho^2}{N} \int d^D \mathbf{x} d^D \mathbf{y} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \frac{\rho(\mathbf{x})\rho(\mathbf{y})}{Z_\rho} \int \left( \prod_i^N d\phi(\mathbf{x}_i) \phi(\mathbf{x}) \phi(\mathbf{y}) \exp\{S_\rho[\phi]\} \right), \quad (106)$$

where we have introduced the action and the partition function at a fixed realization of the disorder, given respectively by

$$S_\rho[\phi] = -\frac{\rho}{2} \int d^D \mathbf{x} d^D \mathbf{y} \phi(\mathbf{x}) \left[ z\rho(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) - \rho\rho(\mathbf{x})\delta(\mathbf{x} - \mathbf{y}) \int d^D \boldsymbol{\sigma} f(\mathbf{x} - \boldsymbol{\sigma})\rho(\boldsymbol{\sigma}) + \rho\rho(\mathbf{x})f(\mathbf{x} - \mathbf{y})\rho(\mathbf{y}) \right] \phi(\mathbf{y}), \quad (107)$$

$$Z_\rho = \int \left( \prod_i^N d\phi(\mathbf{x}_i) \right) \exp\{S_\rho[\phi]\}. \quad (108)$$

Now we note that the action equation (107) depends on the field  $\phi$  only through the values that it assumes on the random positions  $\{\mathbf{x}_i\}$ . In fact, in the action, the field  $\phi$  is always multiplied by the random field  $\rho$ , which selects the random points of the lattice  $\{\mathbf{x}_i\}$ . So, we can substitute the discretized functional measure with the continuous one: this is a crucial step. The continuous version of the functional integral is invariant under the following transformation of the field  $\phi$ , which we shall call a gauge transformation:

$$\phi'(\mathbf{x}) = \phi(\mathbf{x}) + h(\mathbf{x}), \quad \text{with } h(\mathbf{x}_i) = 0, \quad i = 1, \dots, N. \quad (109)$$

This is a local transformation, but we can see that its global version is trivial because the condition equation (109) implies that a global transformation can be possible only for  $h = 0$ . This local symmetry is not present in other field-theoretic formulations [15].

We now look at the resolvent: it can be written in the form

$$G(\mathbf{p}, z) = \int d^D \mathbf{x} d^D \mathbf{y} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \frac{\rho^2}{N} \rho(\mathbf{x})\rho(\mathbf{y}) \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle, \quad (110)$$

where  $\langle \cdot \rangle$  stands for the average over the action  $S_\rho[\phi]$ . We immediately see that  $\rho(\mathbf{x})\rho(\mathbf{y}) \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle$  is gauge-invariant. With the change of variables

$$\rho(\mathbf{x}) = 1 + \delta\rho(\mathbf{x}) \quad (111)$$

the resolvent can be written

$$G(\mathbf{p}, z) = \int d^D \mathbf{x} d^D \mathbf{y} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \left\{ \frac{\rho^2}{N} \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle + 2\frac{\rho^2}{N} \delta\rho(\mathbf{x}) \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle + \frac{\rho^2}{N} \delta\rho(\mathbf{x})\delta\rho(\mathbf{y}) \langle \phi(\mathbf{x})\phi(\mathbf{y}) \rangle \right\}, \quad (112)$$

with the action

$$S_\rho[\phi] = -\frac{\rho}{2} \int d^D \mathbf{x} d^D \mathbf{y} \phi(\mathbf{x}) [(z - \rho\tilde{f}(0))\delta(\mathbf{x} - \mathbf{y}) + \rho f(\mathbf{x} - \mathbf{y})] \phi(\mathbf{y}) - \frac{\rho}{2} \int d^D \mathbf{x} d^D \mathbf{y} d^D \boldsymbol{\sigma} \phi(\mathbf{x})\phi(\mathbf{y})\delta\rho(\boldsymbol{\sigma})V_3(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma}) - \frac{\rho}{2} \int d^D \mathbf{x} d^D \mathbf{y} d^D \boldsymbol{\sigma} d^D \boldsymbol{\gamma} \phi(\mathbf{x})\phi(\mathbf{y})\delta\rho(\boldsymbol{\sigma})\delta\rho(\boldsymbol{\gamma})V_4(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma}, \boldsymbol{\gamma}), \quad (113)$$

where

$$V_3(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma}) = [(z - \rho \hat{f}(\mathbf{0}))\delta(\mathbf{x} - \mathbf{y})\delta(\boldsymbol{\sigma} - \mathbf{x}) - \rho\delta(\mathbf{x} - \mathbf{y})f(\mathbf{x} - \boldsymbol{\sigma}) - \rho f(\mathbf{x} - \mathbf{y})(\delta(\mathbf{x} - \boldsymbol{\sigma}) + \delta(\boldsymbol{\sigma} - \mathbf{y}))], \quad (114)$$

$$V_4(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma}, \boldsymbol{\gamma}) = \rho\delta(\boldsymbol{\gamma} - \mathbf{x})f(\mathbf{x} - \mathbf{y})\delta(\mathbf{y} - \boldsymbol{\sigma}) - \rho\delta(\boldsymbol{\gamma} - \mathbf{x})\delta(\mathbf{x} - \mathbf{y})f(\mathbf{x} - \boldsymbol{\sigma}). \quad (115)$$

Note that the first term of equation (112), when computed in the limit  $\delta\rho = 0$  of the action (equation (113)), yields the bare propagator  $G_0(\mathbf{p}, z)$ . The first term then corresponds to the free (Gaussian) part of the field theory and the terms involving three and four fields are the interacting part. The latter can be treated perturbatively using standard diagrammatic techniques of field theory. One can easily see from the form of the interacting terms that in such diagrams no loops involving the  $\delta\rho$  field may arise because, at fixed disorder, it acts as an external field while a generic  $n$ -loop diagram comes from the average over the disorder and yields a  $1/\rho^n$  contribution to the resolvent.

### 5.1. The correlation functions for the density field

In order to perform the loop expansion one needs the  $n$ -point correlation functions of the  $\delta\rho$  field. Since

$$\overline{\rho(\mathbf{x})} = 1 + \overline{\delta\rho(\mathbf{x})} = \int \left( \prod_i^N \frac{d^D \mathbf{x}_i}{V} \right) \frac{1}{\rho} \sum_k^N \delta(\mathbf{x} - \mathbf{x}_k) = \frac{1}{\rho} \frac{N}{V} = 1, \quad (116)$$

one has

$$\overline{\delta\rho(\mathbf{x})} = 0. \quad (117)$$

Similarly, the fact that

$$\begin{aligned} \overline{\rho(\mathbf{x})\rho(\mathbf{y})} &= 1 + \overline{\delta\rho(\mathbf{x})} + \overline{\delta\rho(\mathbf{y})} + \overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})} = 1 + \overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})} \\ &= \frac{1}{\rho^2} \sum_k^N \sum_j^N \int \left( \prod_i^N \frac{d^D \mathbf{x}_i}{V} \right) \delta(\mathbf{x} - \mathbf{x}_k)\delta(\mathbf{y} - \mathbf{x}_j) \\ &= \frac{1}{\rho^2} \sum_{k \neq j}^N \int \left( \prod_i^N \frac{d^D \mathbf{x}_i}{V} \right) \delta(\mathbf{x} - \mathbf{x}_k)\delta(\mathbf{y} - \mathbf{x}_j) \\ &\quad + \frac{1}{\rho^2} \sum_k^N \int \left( \prod_i^N \frac{d^D \mathbf{x}_i}{V} \right) \delta(\mathbf{x} - \mathbf{x}_k)\delta(\mathbf{y} - \mathbf{x}_k) \\ &= \frac{N(N-1)}{\rho^2 V^2} + \frac{N}{V\rho^2} \delta(\mathbf{x} - \mathbf{y}) \longrightarrow 1 + \frac{1}{\rho} \delta(\mathbf{x} - \mathbf{y}) \end{aligned} \quad (118)$$

implies that

$$\overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})} = \frac{1}{\rho} \delta(\mathbf{x} - \mathbf{y}). \quad (119)$$

To carry our the perturbative expansion up to order  $1/\rho^2$ , the three- and four-point correlations are needed. These can be derived according to the lines described above,

giving

$$\overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})\delta\rho(\mathbf{z})} = \frac{1}{\rho^2}\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{y}-\mathbf{z}) = \overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})} \cdot \overline{\delta\rho(\mathbf{y})\delta\rho(\mathbf{z})}, \quad (120)$$

$$\begin{aligned} \overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})\delta\rho(\mathbf{z})\delta\rho(\mathbf{t})} &= \frac{1}{\rho^3}\delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{y}-\mathbf{z})\delta(\mathbf{z}-\mathbf{t}) + \overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})} \cdot \overline{\delta\rho(\mathbf{z})\delta\rho(\mathbf{t})} \\ &+ \overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{z})} \cdot \overline{\delta\rho(\mathbf{y})\delta\rho(\mathbf{t})} + \overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{t})} \cdot \overline{\delta\rho(\mathbf{y})\delta\rho(\mathbf{z})}. \end{aligned} \quad (121)$$

*5.1.1. The general expression.* We may write as well the expression for the arbitrary  $n$ -point correlation  $\overline{\delta\rho(\mathbf{y}_1)\delta\rho(\mathbf{y}_2)\cdots\delta\rho(\mathbf{y}_k)}$ , needed to compute the self-energy to order  $1/\rho^3$  or higher in the field theory. To give our result, we shall need some notations.

Let  $\omega$  be an arbitrary partition of the set  $\{1, 2, \dots, k\}$  into subsets. For instance, for  $k = 4$ ,  $\omega$  could be a partition into two subsets, such as  $\omega = \{\{1, 2\}, \{3, 4\}\}$  or  $\{\{1, 3\}, \{2, 4\}\}$ , or a partition into four subsets such as  $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ , etc. Let  $\|\alpha\|$  be the cardinality of the set  $\alpha$ , for instance, if  $\omega = \{\{1, 2\}, \{3, 4\}\}$ , then  $\|\omega\| = 2$ .

We also define  $\mathcal{P}^{(k)}$ , the set of all possible partitions of  $\{1, 2, \dots, k\}$ . Given a partition  $\omega$ , the subsets associated with it will be  $\Omega_{l,\omega}$ , with  $l = 1, 2, \dots, \|\omega\|$ . We shall need to consider  $\mathcal{H}^{(k)}$ , a subset of the set of all partitions  $\mathcal{P}^{(k)}$ .  $\mathcal{H}^{(k)}$  is made of all partitions  $\omega$  such that  $\|\Omega_{l,\omega}\| > 1$  for all  $l = 1, 2, \dots, \|\omega\|$ , i.e. partitions in which none of the subsets contains less than *two* integers. Then the general result is

$$\overline{\delta\rho(\mathbf{y}_1)\delta\rho(\mathbf{y}_2)\cdots\delta\rho(\mathbf{y}_k)} = \sum_{\omega \in \mathcal{H}^{(k)}} \frac{1}{\rho^{k-\|\omega\|}} \left[ \prod_{l=1}^{\|\omega\|} \left( \prod_{r=1}^{\|\Omega_{l,\omega}\|-1} \delta(\mathbf{y}_{\alpha_r^{(l,\omega)}} - \mathbf{y}_{\alpha_{r+1}^{(l,\omega)}}) \right) \right]. \quad (122)$$

The proof is given in the appendix. To recover equation (120) from this formula, note that the set  $\mathcal{H}^{(3)}$  of allowed partitions of  $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$  contains a single partition, with just one subset ( $\|\omega\| = 1$ ), namely  $\omega = \{\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}\}$ . On the other hand, to obtain equation (121) we need the set  $\mathcal{H}^{(4)}$  of allowed partitions for  $\{\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}\}\}$ . There are four such partitions, namely  $\omega_1 = \{\{\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}\}\}$ ,  $\omega_2 = \{\{\mathbf{x}, \mathbf{y}\}, \{\mathbf{z}, \mathbf{t}\}\}$ ,  $\omega_3 = \{\{\mathbf{x}, \mathbf{z}\}, \{\mathbf{y}, \mathbf{t}\}\}$  and  $\omega_4 = \{\{\mathbf{x}, \mathbf{t}\}, \{\mathbf{y}, \mathbf{z}\}\}$ . Clearly,  $\|\omega_1\| = 1$ , while  $\|\omega_2\| = \|\omega_3\| = \|\omega_4\| = 2$ .

## 5.2. Diagrammatic expansion: one loop

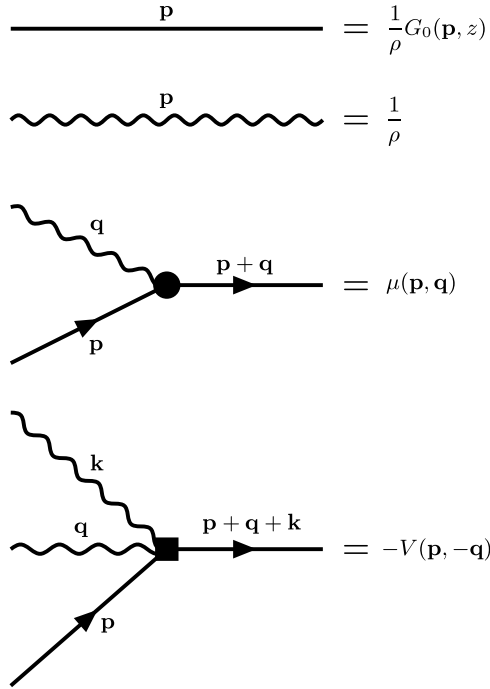
In order to write down the one-loop term it turns out to be convenient to write  $V_3$  and  $V_4$  in terms of the interaction vertex (equation (18)):

$$\begin{aligned} \int d^D\mathbf{x} d^D\boldsymbol{\sigma} e^{i\mathbf{p}_1 \cdot \mathbf{x} + i\mathbf{p}_2 \cdot \boldsymbol{\sigma}} V_3(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma}) &= [G_0^{-1}(\mathbf{p}_1) - V(\mathbf{p}_2, -\mathbf{p}_1)] e^{i(\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{y}} \\ &\equiv \mu(\mathbf{p}_1, \mathbf{p}_2) e^{i(\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{y}} \end{aligned} \quad (123)$$

$$\int d^D\mathbf{x} d^D\boldsymbol{\sigma} d^D\boldsymbol{\gamma} e^{i\mathbf{p}_1 \cdot \mathbf{x} + i\mathbf{p}_2 \cdot \boldsymbol{\sigma} + i\mathbf{p}_3 \cdot \boldsymbol{\gamma}} V_4(\mathbf{x}, \mathbf{y}, \boldsymbol{\sigma}, \boldsymbol{\gamma}) = -V(\mathbf{p}_2, -\mathbf{p}_1) e^{i(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \cdot \mathbf{y}}. \quad (124)$$

The latter expression depends only on two momenta. Thus when the vertex  $V_4$  is involved, one has to make its expression symmetric by joining the  $\delta\rho$  propagators with the two possible external links offered by this vertex.





**Figure 1.** Diagrammatic notation.

Figure 1 defines our diagrammatic notation. Note that the vertex  $V_4$  is not symmetric under the interchange of the two  $\delta\rho$  lines. Now we are able to write down the one-loop diagrams.

Recalling that the resolvent  $G(\mathbf{p}, z)$  is given by equation (112), we compute the one-loop contribution to  $\overline{\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle}$ :

$$L_1^{(1)} \equiv \text{---} \bullet \text{---} \text{---} \bullet \text{---} = \frac{G_0^2(\mathbf{p}, z)}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} G_0(\mathbf{p} - \mathbf{q}, z) \mu^2(\mathbf{p}, -\mathbf{q}) \quad (125)$$

$$\text{---} \blacksquare \text{---} = 0 \quad (126)$$

The last diagram gives a general result: every tadpole made with a vertex with four fields gives a vanishing contribution due to the form of the vertex. The term with one external  $\delta\rho$  insertion, arising from  $\overline{\delta\rho(\mathbf{x})\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle}$  is given by

$$L_2^{(1)} \equiv \text{---} \bullet \text{---} \text{---} \text{---} = -\frac{2G_0^2(\mathbf{p}, z)}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} G_0(\mathbf{p} - \mathbf{q}, z) \mu(\mathbf{p}, -\mathbf{q}) \quad (127)$$

The last contribution to the self-energy at one loop comes from  $\overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle}$  and is

$$L_3^{(1)} \equiv \text{---} \text{---} \text{---} = \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} G_0(\mathbf{p} - \mathbf{q}, z). \quad (128)$$

Note that this last contribution has an ultraviolet divergence since the propagator goes to a finite constant when the internal momentum goes to infinity. Nevertheless, by adding the four diagrams

$$\begin{aligned} L_1^{(1)} + L_2^{(1)} + L_1^{(3)} &= \frac{1}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} G_0(\mathbf{p} - \mathbf{q}, z) [\mu(\mathbf{p}, -\mathbf{q}) G_0(\mathbf{p}, z) - 1]^2 \\ &= \frac{G_0^2(\mathbf{p}, z)}{\rho} \int \frac{d^D \mathbf{q}}{(2\pi)^D} G_0(\mathbf{q}, z) V^2(\mathbf{q}, \mathbf{p}) \equiv G_0^2(\mathbf{p}, z) \Sigma^{(1)}(\mathbf{p}, z), \end{aligned} \quad (129)$$

the divergence disappears and one recovers the combinatorial result for the one-loop self-energy.

### 5.3. Two loops

Let us first consider the two-loop diagrams arising from  $\overline{\langle \phi(\mathbf{x}) \phi(\mathbf{y}) \rangle}$ :

$$\begin{aligned} L_1^{(2)} &\equiv \text{Diagram 1} = \frac{G_0^2(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0^2(\mathbf{q}, z) G_0(\mathbf{q} - \mathbf{k}, z) \\ &\quad \times \mu^2(\mathbf{p}, \mathbf{q} - \mathbf{p}) \mu^2(\mathbf{q}, -\mathbf{k}) \end{aligned} \quad (130)$$

$$\begin{aligned} L_2^{(2)} &\equiv \text{Diagram 2} = \frac{G_0^2(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) G_0(\mathbf{p} - \mathbf{q} - \mathbf{k}, z) \\ &\quad \times \mu(\mathbf{p}, \mathbf{q} - \mathbf{p}) \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}) \mu(\mathbf{q}, \mathbf{k} - \mathbf{p}) \mu(\mathbf{k}, \mathbf{q} - \mathbf{p}) \end{aligned} \quad (131)$$

$$\begin{aligned} L_3^{(2)} &\equiv \text{Diagram 3} = -\frac{G_0^2(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \mu(\mathbf{p}, \mathbf{q} - \mathbf{p}) \\ &\quad \times \mu(\mathbf{p}, \mathbf{k} - \mathbf{q}) \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}) \end{aligned} \quad (132)$$

$$\begin{aligned} L_4^{(2)} &\equiv \text{Diagram 4} = \frac{G_0^2(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \mu(\mathbf{p}, \mathbf{q} - \mathbf{p}) \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}) \\ &\quad \times [V(\mathbf{p} - \mathbf{q}, -\mathbf{q}) - V(\mathbf{k} - \mathbf{p}, -\mathbf{q})] \end{aligned} \quad (133)$$

$$\begin{aligned} L_5^{(2)} &\equiv \text{Diagram 5} = -\frac{2G_0^2(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{q} - \mathbf{k}, z) \mu(\mathbf{q}, -\mathbf{k}) \\ &\quad \times [-V(\mathbf{p} - \mathbf{q}, \mathbf{k} - \mathbf{q}) - V(\mathbf{k}, \mathbf{k} - \mathbf{q})] \end{aligned} \quad (134)$$

$$\begin{aligned} L_6^{(2)} &\equiv \text{Diagram 6} = \frac{G_0^2(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{p} - \mathbf{q} - \mathbf{k}, z) V(\mathbf{q}, \mathbf{p}) \\ &\quad \times [V(\mathbf{q}, \mathbf{k} + \mathbf{q} - \mathbf{p}) + V(\mathbf{k}, \mathbf{k} + \mathbf{q} - \mathbf{p})] \end{aligned} \quad (135)$$

$$\begin{aligned} L_7^{(2)} &\equiv \text{Diagram 7} = \frac{G_0^3(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{k}, z) G_0(\mathbf{q}, z) \mu^2(\mathbf{p}, \mathbf{q} - \mathbf{p}) \\ &\quad \times \mu^2(\mathbf{p}, \mathbf{k} - \mathbf{p}). \end{aligned} \quad (136)$$

The diagram  $L_7^{(2)}$  seems to be already included in the Dyson resummation of the one-loop result. However, since diagrams with one and zero external legs have to be included in the diagrammatic expansion, it also provides a genuine contribution to the two-loop result. Note that in order to obtain  $L_3^{(2)}$  one uses equation (120) for the three-point correlation of  $\delta\rho$ . The other diagrams involve only the disconnected part of the four-point correlation function, while the connected one would only matter at three loops.

Next we must consider the contribution arising from  $\overline{\delta\rho(\mathbf{x})\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle}$ :

$$L_8^{(2)} \equiv \text{Diagram} = -\frac{2G_0(\mathbf{p}, z)}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0^2(\mathbf{q}, z) G_0(\mathbf{q} - \mathbf{k}, z) \mu(\mathbf{p}, \mathbf{q} - \mathbf{p}) \times \mu^2(\mathbf{q}, -\mathbf{k}) \quad (137)$$

$$L_9^{(2)} \equiv \text{Diagram} = -\frac{2G_0(\mathbf{p}, z)}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) G_0(\mathbf{p} - \mathbf{q} - \mathbf{k}, z) \times \mu^2(\mathbf{k}, \mathbf{q} - \mathbf{p}) \mu(\mathbf{q}, \mathbf{k} - \mathbf{p}) \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}) \quad (138)$$

$$L_{10}^{(2)} \equiv \text{Diagram} = \frac{2G_0(\mathbf{p}, z)}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \mu(\mathbf{q}, \mathbf{k} - \mathbf{q}) \times \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}) \quad (139)$$

$$L_{11}^{(2)} \equiv \text{Diagram} = -\frac{2G_0(\mathbf{p}, z)}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \mu(\mathbf{q}, \mathbf{k} - \mathbf{q}) \times [V(\mathbf{p} - \mathbf{q}, -\mathbf{k}) + V(\mathbf{q} - \mathbf{k}, -\mathbf{k})] \quad (140)$$

$$L_{12}^{(2)} \equiv \text{Diagram} = -\frac{2G_0(\mathbf{p}, z)}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}) \times [V(\mathbf{p} - \mathbf{q}, -\mathbf{q}) + V(\mathbf{k} - \mathbf{p}, -\mathbf{q})] \quad (141)$$

$$L_{13}^{(2)} \equiv \text{Diagram} = -\frac{2G_0^2(\mathbf{p}, z)}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \mu^2(\mathbf{p}, \mathbf{k} - \mathbf{p}) \times \mu(\mathbf{p}, \mathbf{q} - \mathbf{p}). \quad (142)$$

As before, we have used the disconnected part of the four-point function, apart from  $L_{10}^{(2)}$  where we have used the three-point function. Note also that  $L_{13}^{(2)}$  arises both in the Dyson resummation and in the two-loop expansion.

Finally, we consider the diagrams arising from  $\overline{\delta\rho(\mathbf{x})\delta\rho(\mathbf{y})\langle\phi(\mathbf{x})\phi(\mathbf{y})\rangle}$ :

$$L_{14}^{(2)} \equiv \text{Diagram} = \frac{1}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0^2(\mathbf{q}, z) G_0(\mathbf{q} - \mathbf{k}, z) \mu^2(\mathbf{q}, -\mathbf{k}) \quad (143)$$

$$L_{15}^{(2)} \equiv \text{Diagram} = \frac{1}{\rho^2} \int \frac{d^D\mathbf{q} d^D\mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) G_0(\mathbf{p} - \mathbf{q} - \mathbf{k}, z) \times \mu^2(\mathbf{q}, \mathbf{k} - \mathbf{p}) \mu(\mathbf{k}, \mathbf{q} - \mathbf{p}) \quad (144)$$

$$L_{16}^{(2)} \equiv \text{diagram} = -\frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \mu(\mathbf{q}, \mathbf{k} - \mathbf{q}) \quad (145)$$

$$L_{17}^{(2)} \equiv \text{diagram} = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \times [V(\mathbf{p} - \mathbf{q}, -\mathbf{q}) + V(\mathbf{k} - \mathbf{p}, -\mathbf{q})] \quad (146)$$

$$L_{18}^{(2)} \equiv \text{diagram} = \frac{G_0(\mathbf{p}, z)}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) \times \mu(\mathbf{p}, \mathbf{q} - \mathbf{p}) \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}). \quad (147)$$

We now show how the diagrams can be summed up to give the combinatorial expressions for the self-energy. Consider the diagram  $L_1^{(2)}$ ; it has the same topology (in the sense of momenta flow and vertex positions) of  $\Sigma_A^{(2)}$ . In fact, it can be combined with  $L_8^{(2)}$  and  $L_{14}^{(2)}$  to give

$$L_1^{(2)} + L_8^{(2)} + L_{14}^{(2)} = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0^2(\mathbf{p}, z) G_0^2(\mathbf{q}, z) G_0(\mathbf{q} - \mathbf{k}, z) \times V^2(\mathbf{p} - \mathbf{q}, \mathbf{p}) \mu^2(\mathbf{q}, -\mathbf{k}) \quad (148)$$

$$= \Sigma_A^{(2)}(\mathbf{p}, z) + \Omega_1(\mathbf{p}, z), \quad (149)$$

where we have defined

$$\Omega_1(\mathbf{p}, z) = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} [G_0^2(\mathbf{p}, z) G_0(\mathbf{k}, z) V^2(\mathbf{p} - \mathbf{q}, \mathbf{p}) - 2G_0^2(\mathbf{p}, z) G_0(\mathbf{q}, z) \times G_0(\mathbf{k}, z) V^2(\mathbf{p} - \mathbf{q}, \mathbf{p}) V(\mathbf{q} - \mathbf{k}, q)]. \quad (150)$$

In the same way we can combine  $L_2^{(2)}$ ,  $L_9^{(2)}$  and  $L_{15}^{(2)}$ ; they have the same topology of  $\Sigma_B(\mathbf{p}, z)$ :

$$L_2^{(2)} + L_9^{(2)} + L_{15}^{(2)} = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) G_0(\mathbf{p} - \mathbf{q} - \mathbf{k}, z) \mu(\mathbf{q}, \mathbf{k} - \mathbf{p}) \times \mu(\mathbf{k}, \mathbf{q} - \mathbf{p}) V(\mathbf{p} - \mathbf{q}, \mathbf{p}) V(\mathbf{p} - \mathbf{k}, \mathbf{p}) = \Sigma_B^{(2)}(\mathbf{p}, z) + \Omega_2(\mathbf{p}, z) \quad (151)$$

where

$$\Omega_2(\mathbf{p}, z) = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} [G_0^2(\mathbf{p}, z) G_0(\mathbf{p} - \mathbf{q} - \mathbf{k}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}) V(\mathbf{p} - \mathbf{k}, \mathbf{p}) - 2G_0^2(\mathbf{p}, z) G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) V(\mathbf{q} + \mathbf{k}, \mathbf{p}) V(\mathbf{p} - \mathbf{k}, \mathbf{p}) V(\mathbf{q} + \mathbf{k}, \mathbf{k})]. \quad (152)$$

We now add the diagrams  $L_7^{(2)}$ ,  $L_{13}^{(2)}$  and  $L_{18}^{(2)}$  because they produce the Dyson resummation of the self-energy at one loop that we want to isolate from the other contributions that have to be included in the self-energy at two loops. They give

$$L_7^{(2)} + L_{13}^{(2)} + L_{18}^{(2)} = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} G_0^3(\mathbf{p}, z) G_0(\mathbf{q}, z) G_0(\mathbf{k}, z) V(\mathbf{p} - \mathbf{q}, \mathbf{p}) \times V(\mathbf{p} - \mathbf{k}, \mathbf{p}) \mu(\mathbf{p}, \mathbf{q} - \mathbf{p}) \mu(\mathbf{p}, \mathbf{k} - \mathbf{p}) = G_0(\mathbf{p}, z) \Sigma^{(1)}(\mathbf{p}, z) \times G_0(\mathbf{p}, z) \Sigma^{(1)}(\mathbf{p}, z) G_0(\mathbf{p}, z) + \Omega_3(\mathbf{p}, z), \quad (153)$$

where

$$\Omega_3(\mathbf{p}, z) = \frac{1}{\rho^2} \int \frac{d^D \mathbf{q} d^D \mathbf{k}}{(2\pi)^{2D}} [G_0(\mathbf{p}, z)G_0(\mathbf{q}, z)G_0(\mathbf{k}, z)V(\mathbf{p} - \mathbf{q}, \mathbf{p})V(\mathbf{p} - \mathbf{k}, \mathbf{p}) \times (1 - 2G_0(\mathbf{p}, z)V(\mathbf{p} - \mathbf{q}, \mathbf{p}))]. \quad (154)$$

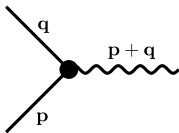
At this point one can check that

$$\Omega_1(\mathbf{p}, z) + \Omega_2(\mathbf{p}, z) + \Omega_3(\mathbf{p}, z) + L_3^{(2)} + L_4^{(2)} + L_5^{(2)} + L_6^{(2)} + L_{10}^{(2)} + L_{11}^{(2)} + L_{12}^{(2)} + L_{16}^{(2)} + L_{17}^{(2)} = \Sigma_C^{(2)}(\mathbf{p}, z) \quad (155)$$

and the combinatorial result is recovered.

### 5.4. The small $p$ behaviour

We will now prove that the prefactor of the term  $p^2 \lambda^{(D-2)/2}$  is zero to all orders in perturbation theory. For this purpose, the field theory approach turns out to be very convenient. Consider the vertex  $V_3$  with three fields:



$$V_3 = G_0^{-1}(\mathbf{p}, z) + V(\mathbf{q}, \mathbf{p}) = z - \rho \hat{f}(\mathbf{0}) + \rho \hat{f}(\mathbf{p}) + \rho \hat{f}(\mathbf{q}) - \rho \hat{f}(\mathbf{p} - \mathbf{q}). \quad (156)$$

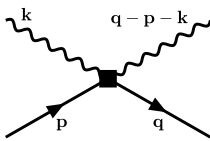
It is easy to see that this vertex is symmetric and can be written as

$$V_3 = z + S(\mathbf{p}, \mathbf{q}), \quad \text{where } S(\mathbf{p}, \mathbf{q}) = S(\mathbf{q}, \mathbf{p}). \quad (157)$$

Moreover one can check directly that

$$S(\mathbf{p}, \mathbf{0}) = S(\mathbf{0}, \mathbf{p}) = 0. \quad (158)$$

Consider now the vertex with four fields. We see that the Wick contractions between the fields  $\delta\rho$  symmetrize the vertex. In fact in every diagram this vertex appears in the form



$$= -(V(\mathbf{k}, -\mathbf{p}) + V(\mathbf{q} - \mathbf{p} - \mathbf{k}, -\mathbf{p})) = -\rho(\hat{f}(\mathbf{k}) - \hat{f}(\mathbf{p} + \mathbf{k}) + \hat{f}(\mathbf{q} - \mathbf{p} - \mathbf{k}) - \hat{f}(\mathbf{q} - \mathbf{k})). \quad (159)$$

The important thing is that this vertex vanishes when one of the two  $G_0$  bare propagators carries a null momentum. Consider now a diagram that arises from the expansion of the resolvent  $G(p, z)$ . At the lowest order in  $z$  when the diagram contains some three-field vertices one has to consider only the symmetric part of these vertices. Let us apply the method explained above in order to extract the contribution to the self-energy proportional to  $z^{(D-2)/2}$ . Apparently, if one sets to zero the momentum of a bare propagator that enters into a vertex then its contribution to the imaginary part vanishes. This seems very strange because from this argument it follows that only  $L_3^{(1)}$  contributes to the imaginary part. Moreover if we consider the two-loop contributions we see that there are no contributions

to the imaginary part of the self-energy because the diagrammatic expansion  $L_1^{(2)}-L_{18}^{(2)}$  contains at least one vertex that vanishes when we set to zero one of the momenta brought by a  $\phi$  propagator. However, the argument is not complete. Actually, the diagrammatic expansion  $L_1^{(2)}-L_{18}^{(2)}$  does not contain only the contribution

$$G_0(\mathbf{p}, z)\Sigma^{(2)}(\mathbf{p}, z)G_0(\mathbf{p}, z) \tag{160}$$

since it contains also the Dyson resummation of the one-loop self-energy. This is the fact that completes the argument and will lead us to prove that a contribution proportional to  $z^{(D-2)/2}p^2$  cannot appear at any order in perturbation theory.

We start checking the argument just given at the one-loop level. Let us introduce the notation

$$\text{-----} = G_0^{-1}(\mathbf{p}, z), \tag{161}$$

so that the self-energy at one loop can be written diagrammatically as

$$\Sigma^{(1)}(\mathbf{p}, z) = \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} + \text{---}\text{---}\text{---} \tag{162}$$

From this expansion and from the above argument one sees that the imaginary part of the self-energy (we will refer always to the imaginary part proportional to  $z^{(D-2)/2}$ ) may come from the last diagram and is correctly given by

$$\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0^+} \Sigma^{(1)}(\mathbf{p}, z + i\epsilon) \propto z^{(D-2)/2}V^2(\mathbf{p}, \mathbf{p}), \tag{163}$$

which is also the contribution that can be easily calculated from the combinatorial expression. Now consider the expansion at two loops. From the combinatorial expressions of the self-energy we immediately see that the imaginary part comes from only  $\Sigma_C^{(2)}$  and can be rewritten in the form

$$\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \Sigma^{(2)}(\mathbf{p}, z + i\epsilon) \propto -2z^{(D-2)/2}V(\mathbf{p}, \mathbf{p})\Sigma^{(1)}(\mathbf{p}, z). \tag{164}$$

Consider now the diagrammatic expansion for the two-loop self-energy  $L_1^{(2)}-L_{18}^{(2)}$ . We have to extract from this expansion the term

$$\Sigma^{(1)}(\mathbf{p}, z)G_0(\mathbf{p}, z)\Sigma^{(1)}(\mathbf{p}, z). \tag{165}$$

Now we will do this in a diagrammatic way. Consider the diagrammatic expansion for the above term (figure 2). If we want the imaginary part of the self-energy proportional to  $z^{(D-2)/2}$  we have to consider the term

$$\Lambda(\mathbf{p}, z) \equiv [G_0(\mathbf{p}, z)]^{-1} \sum_{i=1}^{18} L_i^{(2)}[G_0(\mathbf{p}, z)]^{-1}$$

and the diagrams in figure 2. When we calculate this contribution we have to set to zero the momentum carried by one internal propagator  $G_0$  so that the contribution coming from  $\Lambda(\mathbf{p}, z)$  does not matter. We have to calculate only the term coming from the Dyson

$$\begin{aligned}
 & \Sigma^{(1)}(\mathbf{p}, z)G_0(\mathbf{p}, z)\Sigma^{(1)}(\mathbf{p}, z) = \\
 & = \left\{ \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} + \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} + \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \right\} \\
 & \cdot \text{---} \cdot \left\{ \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} + \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} + \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \right\} = \\
 & = 2\Sigma^{(1)}(\mathbf{p}, z) \cdot \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} + \\
 & + \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---}
 \end{aligned}$$

Figure 2. Diagrammatic expression for equation (165).

resummation so that the imaginary part of the self-energy at two loops is given by

$$\begin{aligned}
 \tilde{\text{Im}} \lim_{\epsilon \rightarrow 0^+} \Sigma^{(2)}(\mathbf{p}, z + i\epsilon) &= -2 \left[ \tilde{\text{Im}} \lim_{\epsilon \rightarrow 0^+} \Sigma^{(1)}(\mathbf{p}, z + i\epsilon) \right] \\
 &\times \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} - 2\Sigma^{(1)}(\mathbf{p}, z) \tilde{\text{Im}} \lim_{\epsilon \rightarrow 0^+} \left[ \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \right] \\
 &+ \tilde{\text{Im}} \lim_{\epsilon \rightarrow 0^+} \left[ \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \right] \tag{166}
 \end{aligned}$$

$$= -2\Sigma^{(1)}(\mathbf{p}, z) \tilde{\text{Im}} \lim_{\epsilon \rightarrow 0^+} \left[ \text{---} \overset{\text{cloud}}{\curvearrowright} \text{---} \right] \propto -2z^{(D-2)/2} V(\mathbf{p}, \mathbf{p}) \Sigma^{(1)}(\mathbf{p}, z). \tag{167}$$

At this point we can give also the analytical argument

$$\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \Sigma^{(2)}(\mathbf{p}, z + i\epsilon) = -\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \left[ \Sigma^{(1)}(\mathbf{p}, z + i\epsilon) G_0(\mathbf{p}, z + i\epsilon) \Sigma^{(1)}(\mathbf{p}, z + i\epsilon) \right] \tag{168}$$

$$\propto -2V(\mathbf{p}, \mathbf{p}) \Sigma^{(1)}(\mathbf{p}, z) \tag{169}$$

where we have used the fact that

$$\Sigma(\mathbf{0}, z) = 0. \tag{170}$$

On the same line we can give the imaginary part proportional to  $z^{(D-2)/2}$  at three loops because this contribution comes from the Dyson resummation of one- and two-loop self-energy:

$$\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \Sigma^{(3)}(\mathbf{p}, z + i\epsilon) = -\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \left[ \Sigma^{(1)} G_0 \Sigma^{(1)} G_0 \Sigma^{(1)} + 2\Sigma^{(1)} G_0 \Sigma^{(2)} \right] \tag{171}$$

$$\propto z^{(D-2)/2} \left( 3[\Sigma^{(1)}]^2 - 2V(\mathbf{p}, \mathbf{p}) \Sigma^{(2)} \right). \tag{172}$$

At this point we can give a general expression for the imaginary part proportional to  $z^{(D-2)/2}$  at any perturbative order:

$$\begin{aligned} \tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \Sigma^{(n)}(\mathbf{p}, z + i\epsilon) = & -\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \left[ \left( \sum_{k=2}^n \sum_{\substack{i_1, \dots, i_k \\ \sum i_\sigma = n; i_\sigma < n}} \prod_{\alpha=1}^k (\Sigma^{(i_\alpha)}(\mathbf{p}, z + i\epsilon) G_0(\mathbf{p}, z + i\epsilon)) \right) \right. \\ & \left. \times [G_0(\mathbf{p}, z + i\epsilon)]^{-1} \right]. \end{aligned} \quad (173)$$

From this expression we can prove by induction that the imaginary part of the self-energy proportional to  $z^{(D-2)/2}$  cannot appear at any order in perturbation theory. In fact, we have seen that it does not appear at one and two loops so we can prove that if it does not appear up to  $n$  loops it does not appear up to  $n + 1$  loops too. We can see that

$$\tilde{\text{Im}} \lim_{\epsilon \rightarrow 0} \Sigma^{(k)}(\mathbf{p}, z + i\epsilon) G_0(\mathbf{p}, z + i\epsilon) \propto z^{(D-2)/2} p^\gamma \quad (174)$$

where  $\gamma \geq 2$  and where we have showed only the term at lowest order in  $z$ . Moreover we have

$$\Sigma^{(k)}(\mathbf{p}, z) G_0(\mathbf{p}, z) \sim 1 + \mathcal{O}(p^2) \quad (175)$$

where we have neglected the higher order in  $z$ . It follows that the generic term in (173) is of order

$$z^{(D-2)/2} p^\beta \quad (176)$$

with  $\beta \geq 4$  because  $[G_0(\mathbf{p}, z)]^{-1} \sim p^2$ . This completes the proof.

## 6. Conclusions

In conclusion, we have given a detailed description of the perturbative high-density computation of the resolvent (and, in particular, the density of states) of Euclidean random matrices within two different formalisms. The combinatorial formalism of section 4 results in fewer diagrams and is probably more convenient when the goal is to obtain an expression of the self-energy at a given order. On the other hand, the field-theoretic formalism (section 5), though producing a higher number of diagrams, has allowed us to analyse the  $p \rightarrow 0$  behaviour at all orders in perturbation theory. This analysis shows that the imaginary part of the self-energy in the limit of small momenta (which controls the width of the Brillouin peak of the dynamic structure factor) has, in contrast to previous claims [15]–[18, 31], the structure

$$-\text{Im} \Sigma(\lambda, p) = \mathcal{B} \lambda^{(D-2)/2} p^4 + \mathcal{C} \lambda^{D/2} \frac{p^2}{c^2} + \dots, \quad (177)$$

where  $\mathcal{C}, \mathcal{B} > 0$  are amplitudes and  $c$  is the speed of sound. This implies in particular a



$p^4$  scaling for the Brillouin peak width, but it also shows that the structure of the theory is more complex than in the case of scattering from lattice models [30].

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### Appendix. Proof of equation (122)

The proof proceeds by induction. First note that the explicit computation in equations (117) and (119) already implies that equation (122) holds for  $k = 1$  and 2.

The cornerstone of the proof is a general result for the  $k$ -point correlation functions of  $\rho$  (rather than  $\delta\rho$ ). The sought correlation function, in the thermodynamic limit, is

$$\overline{\rho(\mathbf{y}_1)\rho(\mathbf{y}_2)\cdots\rho(\mathbf{y}_k)} = 1 + \sum_{\omega \in \mathcal{P}^{(k)}, \|\omega\| < k} \frac{1}{\rho^{k-\|\omega\|}} \left[ \prod_{l=1, \|\Omega_{l,\omega}\| > 1}^{\|\omega\|} \left( \prod_{r=1}^{\|\Omega_{l,\omega}\|-1} \delta(\mathbf{y}_{\alpha_r^{(l,\omega)}} - \mathbf{y}_{\alpha_{r+1}^{(l,\omega)}}) \right) \right]. \quad (\text{A.1})$$

Equation (A.1) looks very similar to equation (122), yet we note the following crucial differences.

- The partitions  $\omega$  belong to  $\mathcal{P}^{(k)}$  rather than to the restricted set  $\mathcal{H}^{(k)}$ . In particular, the term equal to 1 in equation (A.1) follows from the only partition  $\omega$  with  $\|\omega\| = k$ , namely  $\{\{1\}, \{3\}, \dots, \{k\}\}$ , which obviously does not belong to  $\mathcal{H}^{(k)}$ .
- In the innermost product in equation (A.1), a subset  $\Omega_{l,\omega}$  with just one element,  $\|\Omega_{l,\omega}\| = 1$ , merely contributes a factor of one. Hence, for all practical purposes, such a subset  $\Omega_{l,\omega}$  can be ignored.

To establish equation (A.1), first note that

$$\overline{\rho(\mathbf{y}_1)\rho(\mathbf{y}_2)\cdots\rho(\mathbf{y}_k)} = \frac{1}{\rho^k} \sum_{i_1, i_2, \dots, i_k=1}^N \overline{\delta(\mathbf{y}_1 - \mathbf{x}_{i_1})\delta(\mathbf{y}_2 - \mathbf{x}_{i_2})\cdots\delta(\mathbf{y}_k - \mathbf{x}_{i_k})}, \quad (\text{A.2})$$

where the average is taken with respect to the flat probability measure:

$$\frac{\prod_{i=1}^N d^D \mathbf{x}_i}{V^N}.$$

Now, for a given assignment of the  $k$  particle labels  $i_1, i_2, \dots, i_k$ , we declare that all terms with a coinciding particle label  $i_r$  form a subset  $\Omega_{l,\omega}$ . It is then obvious that every assignment of the  $k$  particle labels  $i_1, i_2, \dots, i_k$  defines a partition  $\omega$  in  $\mathcal{P}^{(k)}$ . Furthermore, a little reflection shows that all possible partitions in  $\mathcal{P}^{(k)}$  can be obtained in this way. Equation (A.1) follows from the following three facts about a given partition  $\omega$ :

- (1) There are  $N(N-1)\cdots(N-N_{\|\omega\|})$  possible assignments of the  $k$  particle labels  $i_1, i_2, \dots, i_k$  that yield the partition  $\omega$  (you are given  $N$  choices for the particle that appears in the subset  $\Omega_{1,\omega}$ ,  $N-1$  for that appearing in  $\Omega_{2,\omega}$ , and so forth).
- (2) A subset with a single element,  $\|\Omega_{l,\omega}\| = 1$ , contributes a factor  $1/V$ .

(3) A subset with more than one element,  $\|\Omega_{l,\omega}\| > 1$ , contributes a factor

$$\frac{1}{V} \prod_{r=1}^{\|\Omega_{l,\omega}\|-1} \delta(\mathbf{y}_{\alpha_r^{(l,\omega)}} - \mathbf{y}_{\alpha_{r+1}^{(l,\omega)}}).$$

Now consider a partition  $\omega$  that belongs to  $\mathcal{P}^{(k)}$  but does *not* belong to  $\mathcal{H}^{(k)}$ . Imagine that  $\omega$  contains  $k - s$  subsets  $\Omega_{l,\omega}$  with just one element. The values that  $s$  can take are  $s = 0, 2, 3, 4, \dots, k - 1$ . We are not interested in the trivial case  $s = 0$ , which corresponds to the partition  $\{\{1\}, \{3\}, \dots, \{k\}\}$ . Hence, for  $s > 0$ , we simply erase from  $\omega$  all the  $k - s$  subsets  $\Omega_{l,\omega}$  with  $\|\Omega_{l,s}\| = 1$ . The  $s$  integers

$$\Lambda = \{\beta_1^\Lambda, \beta_2^\Lambda, \dots, \beta_s^\Lambda\},$$

that belong to the remaining  $\Omega_{l,\omega}$ , form the *irreducible* set  $\Lambda$  associated with the partition  $\omega$ . The list of  $\Omega_{l,\omega}$  with  $\|\Omega_{l,\omega}\| > 1$  provides a partition  $\tilde{\omega}$  of  $\Lambda$ , which obviously belongs to  $\mathcal{H}^{(s,\Lambda)}$ . Furthermore,  $\|\omega\| = k - s + \|\tilde{\omega}\|$ , so we have for the prefactor in equation (A.1) that

$$\frac{1}{\rho^{k-\|\omega\|}} = \frac{1}{\rho^{s-\|\tilde{\omega}\|}}.$$

Hence, since  $s < k$ , the induction hypothesis implies that the added contribution in equation (A.1) of all the partitions sharing the same irreducible set,  $\Lambda$ , is

$$\overline{\delta\rho(\mathbf{y}_{\beta_1^\Lambda})\delta\rho(\mathbf{y}_{\beta_2^\Lambda})\cdots\delta\rho(\mathbf{y}_{\beta_s^\Lambda})}.$$

At this point, we may rewrite equation (A.1) as

$$\begin{aligned} \overline{\rho(\mathbf{y}_1)\rho(\mathbf{y}_2)\cdots\rho(\mathbf{y}_k)} &= 1 + \sum_{\omega \in \mathcal{H}^{(k)}} \frac{1}{\rho^{k-\|\omega\|}} \left[ \prod_{l=1}^{\|\omega\|} \left( \prod_{r=1}^{\|\Omega_{l,\omega}\|-1} \delta(\mathbf{y}_{\alpha_r^{(l,\omega)}} - \mathbf{y}_{\alpha_{r+1}^{(l,\omega)}}) \right) \right] \\ &+ \sum_{s=2}^{k-1} \sum_{\Lambda=\{\beta_1^\Lambda, \beta_2^\Lambda, \dots, \beta_s^\Lambda\}} \overline{\delta\rho(\mathbf{y}_{\beta_1^\Lambda})\delta\rho(\mathbf{y}_{\beta_2^\Lambda})\cdots\delta\rho(\mathbf{y}_{\beta_s^\Lambda})}. \end{aligned} \tag{A.3}$$

We finally note that, if one writes  $\rho(\mathbf{y}_r) = 1 + \delta\rho(\mathbf{y}_r)$ :

$$\begin{aligned} \overline{\rho(\mathbf{y}_1)\rho(\mathbf{y}_2)\cdots\rho(\mathbf{y}_k)} &= 1 + \overline{\delta\rho(\mathbf{y}_1)\delta\rho(\mathbf{y}_2)\cdots\delta\rho(\mathbf{y}_k)} \\ &+ \sum_{s=2}^{k-1} \sum_{\Lambda=\{\beta_1^\Lambda, \beta_2^\Lambda, \dots, \beta_s^\Lambda\}} \overline{\delta\rho(\mathbf{y}_{\beta_1^\Lambda})\delta\rho(\mathbf{y}_{\beta_2^\Lambda})\cdots\delta\rho(\mathbf{y}_{\beta_s^\Lambda})}. \end{aligned} \tag{A.4}$$

Comparison of equations (A.3) and (A.4) completes the proof.

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