FINITE-DIMENSIONAL POINTED HOPF ALGEBRAS OVER FINITE SIMPLE GROUPS OF LIE TYPE IV. UNIPOTENT CLASSES IN CHEVALLEY AND STEINBERG GROUPS

NICOLÁS ANDRUSKIEWITSCH, GIOVANNA CARNOVALE AND GASTÓN ANDRÉS GARCÍA

ABSTRACT. We show that all unipotent classes in finite simple Chevalley or Steinberg groups, different from $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$, collapse (i.e. are never the support of a finite-dimensional Nichols algebra), with a possible exception on one class of involutions in $\mathbf{PSU}_n(2^m)$.

CONTENTS

1. INTRODUCTION

1.1. **The main result and the context.** This is the fourth paper of our series on finite-dimensional complex pointed Hopf algebras whose group of group-likes is isomorphic to a finite simple group of Lie type $\boldsymbol{G}$. See Part I [1] for a comprehensive Introduction. As we explain in *loc. cit.*, the primary

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task is to study Nichols algebras over $\boldsymbol{G}$ with support in a conjugacy class $\mathcal{O}$ of $\boldsymbol{G}$. Actually there are group-theoretical criteria allowing to conclude that every Nichols algebra with support in a given conjugacy class $\mathcal{O}$ has infinite dimension. These criteria were developed in [4, 1, 3] and are recalled in §2.1. The verification of any of these criteria in any conjugacy class might be difficult. Let p be a prime number, $m \in \mathbb{N}$, $q = p^m$, $\mathbb{F}_q$ the field with q elements and $\Bbbk := \overline{\mathbb{F}_q}$. There are three families of finite simple groups of Lie type (according to the shape of the Steinberg endomorphism): Chevalley, Steinberg and Suzuki-Ree groups; see the list in [1, p. 38] and [16, 22.5] for details. Here are the contents of the previous papers:

- ⋄ In [1] we dealt with unipotent conjugacy classes in $\mathbf{PSL}_n(q)$, and as a consequence with the non-semisimple ones (since the centralizers of semisimple elements are products of groups with root system A_ℓ).
- ⋄ The paper [2] was devoted to unipotent conjugacy classes in $\mathbf{PSp}_{2n}(q)$.
- ⋄ The subject of [3] was the semisimple conjugacy classes in $\mathbf{PSL}_n(q)$. But we also introduced the criterium of type C, and applied it to some of the classes not reached with previous criteria in [1, 2].

In this paper we consider unipotent conjugacy classes in Chevalley and Steinberg groups, different from $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$. Concretely, these are the groups in Table 1. Notice that $\mathbf{PSU}_3(2)$ is not simple but needed for recursive arguments.

Table 1. Finite groups considered in this paper; q odd for $\mathbf{P\Omega}_{2n+1}(q)$; $q \geq 3$ for $G_2(q)$

Chevalley

$\boldsymbol{G}$	Root system
$\mathbf{P\Omega}_{2n+1}(q)$	B_n, $n \geq 3$
$\mathbf{P\Omega}^+_{2n}(q)$	D_n, $n \geq 4$
$G_2(q)$	G_2
$F_4(q)$	F_4
$E_j(q)$	E_6, E_7, E_8

Steinberg

$\boldsymbol{G}$	Root system
$\mathbf{PSU}_n(q)$	A_{n-1}, $n \geq 3$
$\mathbf{P\Omega}^-_{2n}(q)$	D_n, $n \geq 4$
${}^3D_4(q)$	D_4
${}^2E_6(q)$	E_6

As in [4, 2.2], we say that a conjugacy class $\mathcal{O}$ of a finite group G *collapses* if the Nichols algebra $\mathfrak{B}(\mathcal{O}, \mathbf{q})$ has infinite dimension for every finite faithful 2-cocycle $\mathbf{q}$. Our main result says:

Main Theorem. *Let $\boldsymbol{G}$ be as in Table 1. Let $\mathcal{O}$ be a non-trivial unipotent conjugacy class in $\boldsymbol{G}$. Then either $\mathcal{O}$ collapses, or else $\boldsymbol{G} = \mathbf{PSU}_n(q)$ with q even and $(2, 1, \dots, 1)$ the partition of $\mathcal{O}$.*

In the terminology of §2.1, the classes not collapsing in the Main Theorem are austere, see Lemma 5.16. This means that the group-theoretical criteria do not apply for it; however, we ignore whether these classes collapse by other reasons. The classes in $\mathbf{PSL}_n(q)$ or $\mathbf{PSp}_{2n}(q)$ not collapsing (by these methods) are listed in Table 3.

1.2. **The scheme of the proof and organization of the paper.** Let $\boldsymbol{G}$ be a finite simple group of Lie type. Then there is q as above, a simple simply connected algebraic group $\mathbb{G}_{\text{sc}}$ defined over $\mathbb{F}_q$ and a Steinberg endomorphism F of $\mathbb{G}_{\text{sc}}$ such that $\boldsymbol{G} = \mathbb{G}_{\text{sc}}^F/Z(\mathbb{G}_{\text{sc}}^F)$. We refer to [16, Chapter 21] for details. Conversely, $\boldsymbol{G} = \mathbb{G}_{\text{sc}}^F/Z(\mathbb{G}_{\text{sc}}^F)$ is a simple group, out of a short list of exceptions, see [16, Theorem 24.17]. For our inductive arguments, it is convenient to denote by $\boldsymbol{G}$ the quotient $\mathbb{G}_{\text{sc}}^F/Z(\mathbb{G}_{\text{sc}}^F)$ even when it is not simple. Often there is a simple algebraic group $\mathbb{G}$ with a projection $\boldsymbol{\pi} : \mathbb{G}_{\text{sc}} \to \mathbb{G}$ such that F descends to $\mathbb{G}$ and $[\mathbb{G}^F, \mathbb{G}^F]/\boldsymbol{\pi}(Z(\mathbb{G}_{\text{sc}}^F)) \simeq \boldsymbol{G}$.

The proof of the Main Theorem is by application of the criteria of type C, D or F (see §2.1), that hold by a recursive argument on the semisimple rank of $\mathbb{G}_{\text{sc}}$. The first step of the induction is given by the results on unipotent classes of $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$, while the recursive step is a reduction to Levi subgroups. Then we proceed group by group and class by class. The experience suggests that a general argument is not possible. There are some exceptions in low rank for which Levi subgroups are too small and we need the representatives of the classes to apply *ad-hoc* arguments.

Here is the organization of the paper: We recall some notations and facts in §2, where we also state the needed notation for groups of Lie type. In §3 we describe the reduction to Levi subgroups and collect the known results on unipotent classes of $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$.

Let $\mathcal{O}$ be a non-trivial unipotent class in a group $\boldsymbol{G}$ listed in Table 1. If $\mathcal{O}$ is not kthulhu then $\mathcal{O}$ collapses, cf. Theorem 2.4. The proof that $\mathcal{O}$ is not kthulhu is given in §4, respectively §5, when $\boldsymbol{G}$ is a Chevalley, respectively Steinberg, group.

Indeed, if $\boldsymbol{G} = \mathbf{P\Omega}_{2n+1}(q)$, $n \geq 3$, and q odd, the claim is Proposition 4.3. If $\boldsymbol{G} = \mathbf{P\Omega}^+_{2n}(q)$, $n \geq 4$, $E_6(q)$, $E_7(q)$, or $E_8(q)$, then the claim is Proposition 4.2. If $\boldsymbol{G} = F_4(q)$, the result follows from Lemmata 4.4 and 4.5; and if $\boldsymbol{G} = G_2(q)$, $q \geq 3$, the assertion follows from Lemmata 4.6 and 4.7.

In turn, $\mathbf{PSU}_n(q)$ is settled in Proposition 5.1; $\mathbf{P\Omega}^-_{2n}(q)$ in Proposition 5.17; ${}^2E_6(q)$ in Proposition 5.19; and ${}^3D_4(q)$ in Proposition 5.20.

In this way, the Theorem is proved.

1.3. **Applications and perspectives.** The results in this paper will be applied to settle the non-semisimple classes in Chevalley and Steinberg groups.

Next we will deal with unipotent and non-semisimple classes in Suzuki-Ree groups. These are too small to apply the recursive arguments introduced in this paper.

The semisimple conjugacy classes in $\boldsymbol{G}$ different from $\mathbf{PSL}_n(q)$ are more challenging. We expect that reducible classes would collapse while the irreducible ones would be kthulhu, as is the case for $\mathbf{PSL}_2(q)$ and $\mathbf{PSL}_3(q)$ (with some exceptions). Both cases require a deeper understanding of the classes, and in addition the irreducible case seems to need an inductive argument on the maximal subgroups.

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2. Preliminaries

If $a \leq b \in \mathbb{N}$, then $\mathbb{I}_{a,b}$ denotes $\{a, a+1, \dots, b\}$; for simplicity $\mathbb{I}_a = \mathbb{I}_{1,a}$.

2.1. **Glossary of racks.** See [3] for details and more information.

2.1.1. A *rack* is a set $X \neq \emptyset$ with a self-distributive operation $\triangleright : X \times X \to X$ such that $x \triangleright \underline{\quad}$ is bijective for every $x \in X$. The archetypical example is the conjugacy class $\mathcal{O}_z^G$ of an element z in a group G with the operation $x \triangleright y = xyx^{-1}$, $x, y \in \mathcal{O}_z^G$. A rack X is *abelian* if $x \triangleright y = y$, for all $x, y \in X$.

2.1.2. [4, Definition 3.5] A rack X is *of type D* if it has a decomposable subrack $Y = R \coprod S$ with elements $r \in R$, $s \in S$ such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$.

Lemma 2.1. [1, Lemma 2.10] *Let X and Y be racks, $y_1 \neq y_2 \in Y$, $x_1 \neq x_2 \in X$ such that $x_1 \triangleright (x_2 \triangleright (x_1 \triangleright x_2)) \neq x_2$, $y_1 \triangleright y_2 = y_2$. Then $X \times Y$ is of type D.* □

Remark 2.2. One of the hypothesis of Lemma 2.1 holds in the following setting. Let $\mathcal{O}$ be a real conjugacy class, i.e. $\mathcal{O} = \mathcal{O}^{-1}$, with no involutions. Then $y_1 \neq y_2 = y_1^{-1}$, that obviously commute.

2.1.3. [1, Definition 2.4] A rack X is *of type F* if it has a family of subracks $(R_a)_{a \in \mathbb{I}_4}$ and elements $r_a \in R_a$, $a \in \mathbb{I}_4$, such that $R_a \triangleright R_b = R_b$, for $a, b \in \mathbb{I}_4$, and $R_a \cap R_b = \emptyset$, $r_a \triangleright r_b \neq r_b$ for $a \neq b \in \mathbb{I}_4$.

2.1.4. [3, Definition 2.3] A rack X is *of type C* when there are a decomposable subrack $Y = R \coprod S$ and elements $r \in R$, $s \in S$ such that $r \triangleright s \neq s$,

$$R = \mathcal{O}_r^{\operatorname{Inn} Y}, \quad S = \mathcal{O}_s^{\operatorname{Inn} Y}, \quad \min\{|R|, |S|\} > 2 \text{ or } \max\{|R|, |S|\} > 4.$$

Here $\operatorname{Inn} Y$ is the subgroup of $\mathbb{S}_Y$ generated by $y \triangleright \underline{\quad}$, $y \in Y$.

2.1.5. Being of type C, D or F can be phrased in group terms, see [3]. Here is a new formulation suitable for later applications.

Lemma 2.3. *Let $\mathcal{O}$ be a conjugacy class in a group H. If there are $r, s \in \mathcal{O}$ such that $r^2 s \neq s r^2$, $s^2 r \neq r s^2$ and $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle}$, then $\mathcal{O}$ is of type C.*

Proof. We check that the conditions in [3, Lemma 2.8] hold with $H = \langle r, s \rangle = \langle \mathcal{O}_r^{\langle r,s \rangle}, \mathcal{O}_s^{\langle r,s \rangle} \rangle$. By hypothesis, $rs \neq sr$ and $\mathcal{O}_r^{\langle r,s \rangle} \neq \mathcal{O}_s^{\langle r,s \rangle}$. Now $r, s \triangleright r, s^2 \triangleright r$ are all distinct, so $|\mathcal{O}_r^{\langle r,s \rangle}| > 2$, and similarly for $\mathcal{O}_s^{\langle r,s \rangle}$. □

Theorem 2.4. [4, Theorem 3.6], [1, Theorem 2.8], [3, Theorem 2.9]. *A rack X of type D, F or C collapses.*

The proof rests on results from [7, 12, 13].

2.1.6. A rack is

- ◦ *kthulhu* if it is neither of type C, D nor F;
- ◦ *sober* if every subrack is either abelian or indecomposable;
- ◦ *austere* if every subrack generated by two elements is either abelian or indecomposable.

Clearly, sober implies austere and austere implies kthulhu.

The criteria of type C, D, F are very flexible:

Lemma 2.5. [4, 1, 3] *Let Y be either a subrack or a quotient rack of a rack X. If Y is not kthulhu, then X is not kthulhu.* □

2.2. **Conjugacy classes.**

2.2.1. Let $q = p^m$ be as above. We fix a simple algebraic group $\mathbb{G}$ defined over $\mathbb{F}_q$, a maximal torus $\mathbb{T}$, with root system denoted by Φ, and a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$. We denote by $\mathbb{U}$ the unipotent radical of $\mathbb{B}$ and by $\Delta \subset \Phi^+$ the corresponding sets of simple and positive roots. Also $\mathbb{U}^-$ is the unipotent radical of the opposite Borel subgroup $\mathbb{B}^-$ corresponding to Φ^-. We shall use the realisation of the associated root system and the numbering of simple roots in [5]. The coroot system of $\mathbb{G}$ is denoted by $\Phi^\vee = \{\beta^\vee \mid \beta \in \Phi\} \subset X_*(\mathbb{T})$, where $\langle \alpha, \beta^\vee\rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$, for all $\alpha \in \Phi$. Hence

$$\alpha(\beta^\vee(\zeta)) = \zeta^{\frac{2(\alpha,\beta)}{(\beta,\beta)}}, \qquad \alpha, \beta \in \Phi, \zeta \in \mathbb{F}_q^\times.$$

We denote by $\mathbb{G}_{\mathrm{sc}}$ the simply connected group covering $\mathbb{G}$.

For $\Pi \subset \Delta$, we denote by Φ_Π the root subsystem with base Π and $\Psi_\Pi := \Phi^+ - \Phi_\Pi$. For $\alpha \in \Phi$, we denote by $s_\alpha \in W = N_{\mathbb{G}}(\mathbb{T})/\mathbb{T}$ the reflection with respect to α. Also, $s_i = s_{\alpha_i}$, if α_i is a simple root with the alluded numeration. Also, there is a monomorphism of abelian groups $x_\alpha : \Bbbk \to \mathbb{U}$; the image $\mathbb{U}_\alpha$ of x_α is called a root subgroup. We adopt the normalization of x_α and the notation for the elements in $\mathbb{T}$ from [20, 8.1.4]. We recall the commutation rule: $t \triangleright x_\alpha(a) = t x_\alpha(a) t^{-1} = x_\alpha(\alpha(t)a)$, for $t \in \mathbb{T}$ and $\alpha \in \Phi$. In particular, if $t = \beta^\vee(\xi)$ for some $\xi \in \Bbbk^\times$, then $t \triangleright x_\alpha(a) = x_\alpha(\alpha(\beta^\vee(\xi))a) = x_\alpha(\xi^{\frac{2(\alpha,\beta)}{(\beta,\beta)}} a)$.

We denote by $\mathbb{P}$ a standard parabolic subgroup of $\mathbb{G}$, with standard Levi subgroup $\mathbb{L}$ and unipotent radical $\mathbb{V}$. Thus there exists $\Pi \subset \Delta$ such that $\mathbb{L} = \langle \mathbb{T}, \mathbb{U}_{\pm\gamma} \mid \gamma \in \Pi\rangle$.

If $u \in \mathbb{U}$ then for every ordering of Φ^+, there exist unique $c_\alpha \in \Bbbk$ such that $u = \prod_{\alpha\in\Phi^+} x_\alpha(c_\alpha)$. We define $\operatorname{supp} u = \{\alpha \in \Phi^+ \mid c_\alpha \neq 0\}$. In general the support depends on the chosen ordering of Φ^+. However, if $u \in \mathbb{V}$ as above, then $\operatorname{supp} u \subset \Psi_\Pi$ for every ordering of Φ^+.

2.2.2. In this paper we deal with Chevalley and Steinberg groups. Let F be a Steinberg endomorphism of $\mathbb{G}$; it is the composition of the split endomorphism Fr_q (the q-Frobenius map) with an automorphism induced by a Dynkin diagram automorphism ϑ. So, Chevalley groups correspond to $\vartheta = \mathrm{id}$. We assume that $\mathbb{T}$ and $\mathbb{B}$ are F-stable. Let $W^F = N_{\mathbb{G}^F}(\mathbb{T})/\mathbb{T}^F$. Thus $W^F \simeq W$ for Chevalley groups. For each $w \in W^F$, there is a representative $\dot{w}$ of w in $N_{\mathbb{G}^F}(\mathbb{T})$, cf. [16, Proposition 23.2]. Notice that $\dot{w} \triangleright (\mathbb{U}_\alpha) = \mathbb{U}_{w(\alpha)}$ for all $\alpha \in \Phi$. Hence, if $\alpha, \beta \in \Phi$ are ϑ-stable and have the same length, then $\mathbb{U}_\alpha^F$ and $\mathbb{U}_\beta^F$ are conjugated by an element in $N_{\mathbb{G}^F}(\mathbb{T})$ by [14, Lemma 10.4 C]. Let $\boldsymbol{G} = [\mathbb{G}^F, \mathbb{G}^F]/Z(\mathbb{G}^F)$.

2.2.3. We shall often use the Chevalley's commutator formula (2.1), see [22, pp. 22 and 24]. Let $\alpha, \beta \in \Phi$. If $\alpha + \beta$ is not a root, then $\mathbb{U}_\alpha$ and $\mathbb{U}_\beta$ commute. Assume that $\alpha + \beta \in \Phi$. Fix a total order in the set Γ of pairs $(i, j) \in \mathbb{N}^2$ such that $i\alpha + j\beta \in \Phi$. Then there exist $c_{ij}^{\alpha\beta} \in \mathbb{F}_q$ such that

$$x_\alpha(\xi)x_\beta(\eta)x_\alpha(\xi)^{-1}x_\beta(\eta)^{-1} = \prod_{(i,j)\in\Gamma} x_{i\alpha+j\beta}(c_{ij}^{\alpha\beta}\xi^i\eta^j), \qquad \forall \xi, \eta \in \Bbbk. \tag{2.1}$$

Definition 2.6. [2, Definition 3.3] Let $\alpha, \beta \in \Phi^+$ such that $\alpha+\beta \in \Phi$ but the pair α, β does not appear in Table 2. We fix an ordering of Φ^+. A unipotent conjugacy class $\mathcal{O}$ in $\boldsymbol{G}$ has the *$\alpha\beta$-property* if there exists $u \in \mathcal{O} \cap \mathbb{U}^F$ such that $\alpha, \beta \in \operatorname{supp} u$ and for any expression $\alpha + \beta = \sum_{1 \le i \le r} \gamma_i$, with $r > 1$ and $\gamma_i \in \operatorname{supp} u$, necessarily $r = 2$ and $\{\gamma_1, \gamma_2\} = \{\alpha, \beta\}$.

TABLE 2.

$p = 3$

Φ	α	β
G_2	α_1	$2\alpha_1 + \alpha_2$
	$2\alpha_1 + \alpha_2$	α_1
	$\alpha_1 + \alpha_2$	$2\alpha_1 + \alpha_2$
	$2\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$

$p = 2$

Φ	α	β
B_n, C_n, F_4	orthogonal	to each other
G_2	α_1	$\alpha_1 + \alpha_2$
	$\alpha_1 + \alpha_2$	α_1

Let $\alpha, \beta \in \Phi^+$. The scalar $c_{1,1}^{\alpha,\beta} \neq 0$ in (2.1) if $\alpha + \beta \in \Phi$ and the pair does not appear in Table 2.

Proposition 2.7. [2, Proposition 3.5] *Let $\boldsymbol{G}$ be a finite simple group of Lie type, with q odd. Assume $\mathcal{O}$ has the $\alpha\beta$-property, for some $\alpha, \beta \in \Phi^+$ such that $q > 3$ when $(\alpha, \beta) = 0$. Then $\mathcal{O}$ is of type D.* $\square$

Remark 2.8. Assume u satisfies the conditions in Definition 2.6. Then it is never an involution. Indeed if q is odd this is never the case. If q is even then the argument in the proof of [2, Proposition 3.5] shows that the coefficient of $x_{\alpha+\beta}$ in the expression of u^2 is nonzero.

2.2.4. Let us choose an ordering of the positive roots and let $w \in W$ and $u \in \mathbb{U}$ be such that $\Sigma := w(\operatorname{supp} u) \subset \Phi^+$. Then, $\dot{w} \triangleright u \in \mathbb{U}$ and there is an ordering of the positive roots for which Σ is the support of $\dot{w} \triangleright u$. If, in addition, $w\Sigma \not\subset \Psi_\Pi$ for some $\Pi \subset \Delta$, then by the discussion in 2.2.1, $\dot{w} \triangleright u \in \mathbb{U} - \mathbb{V}$.

2.2.5. We shall need a fact on root systems. Recall that there is a partial ordering $\preceq$ on the root lattice $\mathbb{Z}\Phi$ given by $\alpha \preceq \beta$ if $\beta - \alpha \in \mathbb{N}_0\Phi^+ = \mathbb{N}_0\Delta$.

Lemma 2.9. *Let $\gamma, \beta \in \Phi^+$ with $\beta \preceq \gamma$. Then there exists a sequence $\alpha_{i_1}, \dots, \alpha_{i_k} \in \Delta$ such that*

(1) $\forall j \in \mathbb{I}_k$ we have $\gamma_j := \beta + \alpha_{i_1} + \dots + \alpha_{i_j} \in \Phi^+$;

(2) $\gamma = \gamma_k$.

If, in addition, Φ is simply-laced, then $\gamma_j = s_{i_j} \cdots s_{i_1}\beta$ for every $j \in \mathbb{I}_k$.

Proof. (1) and (2) are consequences of [19, Lemma 3.2], with $\alpha_1 = \beta$, and the α_j being simple. Assume that Φ is simply-laced. Clearly, it is enough to prove it for a couple of roots. If $\alpha, \delta \in \Phi$ and $\alpha + \delta \in \Phi$, then $\Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\delta)$ is a root system of type A_2, so $s_\alpha(\delta) = \alpha + \delta$. The last claim follows. □

3. Unipotent classes in finite groups of Lie type

3.1. **Reduction to Levi subgroups.** We start by Lemma 3.2, that is behind the inductive step in most proofs below. We consider the following setting and notation, that we will use throughout the paper:

$\mathbb{P}_1, \dots, \mathbb{P}_k$ are standard F-stable parabolic subgroups of $\mathbb{G}$;
$\mathbb{P}_i = \mathbb{L}_i \ltimes \mathbb{V}_i$ are Levi decompositions, with $\mathbb{L}_i$ F-stable;
$U_i := (\mathbb{U} \cap \mathbb{L}_i)^F$, $U_i^- := (\mathbb{U}^- \cap \mathbb{L}_i)^F$, $P_i := \mathbb{P}_i^F$, $L_i := \mathbb{L}_i^F$, $V_i := \mathbb{V}_i^F$;
$\pi_i : P_i \to L_i$ is the natural projection;
$M_i = \langle U_i, U_i^- \rangle \leq L_i$; for $i \in \mathbb{I}_k$.

Remark 3.1. Assume that $\mathbb{G} = \mathbb{G}_{sc}$. If $\mathbb{L}_i$ is standard, then $M_i = [\mathbb{L}_i, \mathbb{L}_i]^F$.

Proof. Since $\mathbb{G}$ is simply-connected, so is $[\mathbb{L}_i, \mathbb{L}_i]$ (Borel-Tits, see [21, Corollary 5.4]). Then [16, Theorem 24.15] applies. □

Lemma 3.2. *Let $u \in \mathbb{U}^F$. Then $\pi_i(u) \in M_i$ for all $i \in \mathbb{I}_k$.*

(a) *If $\mathcal{O}^{M_i}_{\pi_i(u)}$ is not kthulhu for some $i \in \mathbb{I}_k$, then $\mathcal{O}^{L_i}_{\pi_i(u)}$, $\mathcal{O}^{P_i}_u$, and $\mathcal{O}^{\mathbb{G}^F}_u$ are not kthulhu either.*

(b) *Assume that*

(3.1) $$\text{No non-trivial unipotent class in } M_i \text{ is kthulhu}, \forall i \in \mathbb{I}_k.$$

If $u \notin \cap_{i\in\mathbb{I}_k} V_i$*, then* $\mathcal{O}_u^{\mathbb{G}^F}$ *is not kthulhu.*

(c) *Assume that* (3.1) *holds. Let* $\mathcal{O}$ *be a unipotent conjugacy class in* $\mathbb{G}^F$*. If* $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_{i\in\mathbb{I}_k} V_i$*, then* $\mathcal{O}$ *is not kthulhu, hence collapses.*

Proof. Since $\mathbb{U} \leq \mathbb{P}_i$, $u = u_1u_2$ with $u_1 \in \mathbb{L}_i$ and $u_2 \in \mathbb{V}_i \leq \mathbb{U}$. Hence $u_1 \in \mathbb{L}_i \cap \mathbb{U}$. Since clearly u_1 and u_2 are F-invariant, $u_1 = \pi_i(u) \in M_i$. Now (a) follows from Lemma 2.5 and implies (b), since $\pi_j(u) \neq 1$ for some $j \in \mathbb{I}_k$. (c) follows from (b) and Theorem 2.4 because $\mathcal{O} \cap \mathbb{U}^F \neq \emptyset$. □

3.2. **Unipotent classes in $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$.** We recall now results in the previous papers of the series that constitute the basis of the induction argument. We will also need some of the non-simple groups of Lie type of small rank and small characteristic listed in [2, 3.2.1].

The following Theorem collects information from [1, Table 2], [2, Lemma 3.12 & Tables 3, 4, 5] and [3, Tables 2 & 3].

Theorem 3.3. *Let* $\boldsymbol{G}$ *be either* $\mathbf{PSL}_n(q)$ *or* $\mathbf{PSp}_{2n}(q)$ *and let* $\mathcal{O} \neq \{e\}$ *be a unipotent conjugacy class in* $\boldsymbol{G}$*, not listed in Table 3. Then it is not kthulhu.*

TABLE 3. Kthulhu classes in $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$

$\boldsymbol{G}$	class	q
$\mathbf{PSL}_2(q)$	(2)	even, or 9, or odd not a square
$\mathbf{PSL}_3(2)$	(3)	2
$\mathbf{PSp}_{2n}(q)$, $n \geq 2$	$W(1)^{n-1} \oplus V(2)$	even
$\mathbf{PSp}_{2n}(q)$, $n \geq 2$	$(2, 1^{2n-2})$	9, or odd not a square
$\mathbf{PSp}_4(q)$	$W(2)$	even

We explain the notation of Table 3, see [1, 2] for further details:

(i) Unipotent classes in $\mathbf{PSL}_n(\Bbbk)$ are parametrized by partitions of n; i.e. $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$ and $\sum_j \lambda_j = n$. Thus, (n) is the regular unipotent class of $\mathbf{PSL}_n(\Bbbk)$. Unipotent classes in $\mathbf{PSL}_n(q)$ with the same partition are isomorphic as racks.

(ii) Unipotent classes in $\mathbf{PSp}_{2n}(\Bbbk)$, for q odd, are also parametrized by suitable partitions.

(iii) Unipotent classes in $\mathbf{PSp}_{2n}(\Bbbk)$, for q even, are parametrized by their *label*, which is the decomposition of the standard representation as a module for the action of an element in the conjugacy class:

(3.2) $$V = \bigoplus_{i=1}^{k} W(m_i)^{a_i} \oplus \bigoplus_{j=1}^{r} V(2k_j)^{b_j}, \qquad 0 < a_i,\ 0 < b_j \leq 2,$$

for $m_i, k_j \geq 1$. The block $W(m_i)$ corresponds to a unipotent class with partition (m_i, m_i), whereas the block $V(2k_j)$ corresponds to a unipotent class with partition $(2k_j)$.

(iv) The unipotent class in $\mathbf{PSp}_4(\Bbbk)$ with label $W(2)$, respectively, in $\mathbf{PSp}_{2n}(\Bbbk)$ with label $W(1)^{n-1} \oplus V(2)$ contain a unique unipotent class in $\mathbf{PSp}_4(q)$, respectively, $\mathbf{PSp}_{2n}(q)$.

Remark 3.4. Assume q is even. If $\mathcal{O}$ is a unipotent conjugacy class in $\mathbf{Sp}_{2n}(q)$ enjoying the $\alpha\beta$-property, for some α and β, then $\mathcal{O}$ is of type C, D, or F. Indeed, by Theorem 3.3 the kthulhu unipotent classes in $\mathbf{Sp}_{2n}(q)$ for q even consists of involutions. Remark 2.8 applies.

3.3. **Further remarks.** If a product $X = X_1 \times X_2$ of racks has a factor X_1 that is not kthulhu, then neither is X. Indeed, pick $x \in X_2$; then $X_1 \times \{x\}$ is a subrack of X and Lemma 2.5 applies (here as usual X_2 can be realized as a subrack of a group, so that $x \triangleright x = x$). The following results will be needed in order to deal with products of possibly kthulhu racks.

Lemma 3.5. *Let $\mathcal{O}$ be a unipotent conjugacy class in Table 3.*

(a) *There exist x_1, $x_2 \in \mathcal{O}$ such that $(x_1x_2)^2 \neq (x_2x_1)^2$.*

(b) *If $\boldsymbol{G} \neq \mathbf{PSL}_2(2), \mathbf{PSL}_2(3)$, then there exist $y_1, y_2 \in \mathcal{O}$ such that $y_1 \neq y_2$ and $y_1y_2 = y_2y_1$.*

Proof. By the isogeny argument [1, Lemma 1.2], we may reduce to classes in $\mathbf{SL}_n(q)$ or $\mathbf{Sp}_{2n}(q)$. Also, the classes in $\mathbf{PSp}_4(q)$ with label $W(2)$ and $W(1) \oplus V(2)$ are isomorphic as racks, [2, Lemma 4.26], so we need not to deal with the last row in Table 3.

If $\mathcal{O}$ is the class in $\mathbf{SL}_3(2)$, then $x_1 = \mathrm{id} + e_{1,2} + e_{2,3}$ and $x_2 = \sigma \triangleright x_1$, where $\sigma = e_{1,2} + e_{2,1} + e_{3,3}$, do the job for (a). For (b), take $y_1 = x_1$ and $y_2 = x_1^3 = x_1^{-1}$, that belongs to $\mathcal{O}$ by [1, Lemma 3.3].

If $\mathcal{O}$ is the class in $\mathbf{SL}_2(q)$, then $x_1 = \mathrm{id} + e_{1,2} \in \mathcal{O}$ and $x_2 = \sigma \triangleright x_1$, where $\sigma = e_{1,2} - e_{2,1}$ do the job for (a); while $y_1 = x_1$, and $y_2 = \mathrm{id} + a^2 e_{1,2}$, for $a \in \mathbb{F}_q$, $a^2 \neq 0, 1$, are as needed in (b) when $q > 3$.

Finally, let $\mathcal{O}$ be one of the classes in $\mathbf{Sp}_{2n}(q)$, cf. Table 3. Then $x_1 = \mathrm{id} + e_{1,2n} \in \mathcal{O}$ and $x_2 = \sigma \triangleright x_1$, where $\sigma = e_{1,2n} - e_{2n,1} + \sum_{j \neq 1,2n} e_{jj}$ do the job for (a). Let τ be the block-diagonal matrix $\tau = \mathrm{diag}(\mathtt{J}_2, \mathrm{id}_{2n-2}, \mathtt{J}_2)$, with $\mathtt{J}_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\tau \in \mathbf{Sp}_{2n}(q)$ and $y_1 := x_1$, $y_2 := \tau \triangleright y_1$ fulfil (b). $\square$

Here are results on regular unipotent classes needed later. Let $\mathbb{G}_{\mathrm{sc}}$ be a simply connected simple algebraic group and F a Steinberg endomorphism as before; let $\boldsymbol{G} = \mathbb{G}_{\mathrm{sc}}^F / Z(\mathbb{G}_{\mathrm{sc}}^F)$ but we do not assume that $\boldsymbol{G}$ is simple.

Proposition 3.6. [2, 3.7, 3.8, 3.11] *Let $\mathcal{O}$ be a regular unipotent class in $\boldsymbol{G}$. If any of the conditions below is satisfied, then $\mathcal{O}$ is of type D, or F.*

(1) $\boldsymbol{G} \neq \mathbf{PSL}_2(q)$ *is Chevalley and* $q \neq 2, 4$;
(2) $\boldsymbol{G} = \mathbf{PSU}_3(q)$, *with* $q \neq 2, 8$;
(3) $\boldsymbol{G} = \mathbf{PSU}_4(q)$, *with* $q \neq 2, 4$;
(4) $\boldsymbol{G} = \mathbf{PSU}_n(q)$, *with* $n \geq 5$ *and* $q \neq 2$;

In addition, every regular unipotent class in $\mathbf{GU}_n(q)$, *where* $1 < n$ *is odd and* $q = 2^{2h+1}$, $h \in \mathbb{N}_0$, *is of type D.* □

Finally, we quote [2, Lemma 4.8]:

Lemma 3.7. *Let $\mathcal{O}$ be a regular unipotent class in either $\mathbf{SL}_n(q)$, $\mathbf{SU}_n(q)$ or $\mathbf{Sp}_{2n}(q)$, q even. Then there are $x_1, x_2 \in \mathcal{O}$ such that $(x_1x_2)^2 \neq (x_2x_1)^2$.*

4. Unipotent classes in Chevalley groups

In this Section we deal with unipotent conjugacy classes in a finite simple Chevalley group $\boldsymbol{G} = \mathbb{G}_{\text{sc}}^F/Z(\mathbb{G}_{\text{sc}}^F)$, different to $\mathbf{PSL}_n(q)$ and $\mathbf{PSp}_{2n}(q)$, treated in [1, 2], see §3.2. For convenience, we shall work in $\mathbb{G}_{sc}^F$, cf. [1, Lemma 1.2]. Let

$$\Psi(\beta) = \{\gamma \in \Phi \mid \beta \preceq \gamma\}, \qquad \beta \in \Phi. \tag{4.1}$$

Let $u \in \mathbb{U}$ and $\beta \in \Phi^+$. Then the support $\operatorname{supp} u$ depends on a fixed ordering of Φ^+, but the assertion $\operatorname{supp} u \subset \Psi(\beta)$ does not. Indeed, passing from one order to another boils down to successive applications of the Chevalley formula (2.1), that do not affect the claim.

We denote by $\mathcal{O}$ a non-trivial unipotent conjugacy class in $\boldsymbol{G}$.

4.1. Unipotent classes in $\mathbf{P\Omega}_{2n}^+(q)$, $n \geq 4$; $E_6(q)$, $E_7(q)$ and $E_8(q)$.

We first deal with the case when Φ simply-laced, i.e. $\boldsymbol{G}$ is one of $\mathbf{P\Omega}_{2n}^+(q)$, $n \geq 4$; $E_6(q)$, $E_7(q)$ and $E_8(q)$.

Lemma 4.1. *Given $\beta \in \Phi^+ - \Delta$, there is $x \in \mathcal{O} \cap \mathbb{U}^F$ with $\operatorname{supp} x \not\subset \Psi(\beta)$.*

Proof. Let $u \in \mathcal{O} \cap \mathbb{U}^F$. If $\operatorname{supp} u \not\subset \Psi(\beta)$, then we are done. Assume that $\operatorname{supp} u \subset \Psi(\beta)$. We claim that there is $\tau \in N_{\mathbb{G}_{sc}^F}(\mathbb{T})$ such that

$$x := \tau \triangleright u \in \mathcal{O} \cap \mathbb{U}^F \qquad \text{and} \qquad \operatorname{supp} x \not\subset \Psi(\beta).$$

For every $\gamma \in \Psi(\beta)$ there is a unique k such that $\gamma = \beta + \alpha_{i_1} + \cdots + \alpha_{i_k}$ as in Lemma 2.9. Let m be the minimum k for $\gamma \in \operatorname{supp} u$. We call m the bound of u. We will prove the claim by induction on the bound m. If $m = 0$ then $\beta \in \operatorname{supp} u$ and since $\beta \notin \Delta$, there is a simple reflection s_i such that $s_i\beta = \beta - \alpha_i \in \Phi^+ - \Psi(\beta)$. Also, $s_i\gamma \in \Phi^+$ for every $\gamma \in \operatorname{supp} u$

because $s_i(\Phi^+ - \{\alpha_i\}) = \Phi^+ - \{\alpha_i\}$. In this case we take $\tau = \dot{s}_i$ to be any representative of s_i in $N_{\mathbb{G}_{sc}^F}(\mathbb{T})$.

Let now $m > 0$ and assume that the statement is proved for unipotent elements with bound $m-1$. Let $\gamma \in \operatorname{supp} u$ reach the minimum, i.e., be such that $\gamma = \beta + \alpha_{i_1} + \cdots + \alpha_{i_m}$ for some $\alpha_{i_j} \in \Delta$ chosen as in Lemma 2.9. Then $\gamma' = s_{i_m}\gamma = \beta + \alpha_{i_1} + \cdots + \alpha_{i_{m-1}} \in \Psi(\beta)$, and $s_{i_m}\alpha \in \Phi^+$ for every $\alpha \in \Psi(\beta)$ by construction. Let $\dot{s}_{i_m}$ be a representative of s_{i_m} in $N_{\mathbb{G}_{\mathrm{sc}}^F}(\mathbb{T})$. Then $u' = \dot{s}_{i_m} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and either $\operatorname{supp} u' \not\subset \Psi(\beta)$, or $\operatorname{supp} u' \subset \Psi(\beta)$, with bound at most $m-1$. In the first case, we conclude by setting $x = u'$. In the second case, we use the inductive hypothesis. □

Proposition 4.2. *$\mathcal{O}$ is not kthulhu.*

Proof. The basic idea of the proof is to apply Lemma 3.2 (c) to a series of standard F-stable parabolic subgroups $\mathbb{P}_i$ of $\mathbb{G}_{\mathrm{sc}}$ for which (3.1) holds. We show that for every $\mathcal{O}$ and for every $\boldsymbol{G}$, we have $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$. This follows from Lemma 4.1 by observing that in each case $\cap_i V_i$ is a product of root subgroups corresponding to roots in $\Psi(\beta)$ for some $\beta \in \Phi^+ - \Delta$. We analyze the different cases according to Φ.

D_n, $n \geq 4$. We consider the parabolic subgroups $\mathbb{P}_1$ and $\mathbb{P}_2$ such that $\mathbb{L}_1$ and $\mathbb{L}_2$ have root systems A_{n-1}, generated respectively by $\Delta - \alpha_{n-1}$ and $\Delta - \alpha_n$. Since $n \geq 4$, (3.1) holds by Theorem 3.3. Let $u \in V_1 \cap V_2$. Then $\alpha \in \operatorname{supp} u$ if and only if α contains α_{n-1} and α_n in its expression, i.e. $\alpha \in \Psi(\beta)$ for $\beta = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$.

E_6. We consider the parabolic subgroups $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ such that $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{L}_3$ have root systems D_5, D_5 and A_5, generated respectively by $\Delta - \alpha_1$, $\Delta - \alpha_6$ and $\Delta - \alpha_2$. By Theorem 3.3 and the result for D_n, (3.1) holds. Let $u \in V_1 \cap V_2 \cap V_3$. Then $\alpha \in \operatorname{supp} u$ if and only if $\alpha \in \Psi(\beta)$ for $\beta = \sum_{i=1}^6 \alpha_i$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$.

E_7. We consider the parabolic subgroups $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ such that $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{L}_3$ have root systems D_6, E_6 and A_6, generated respectively by $\Delta - \alpha_1$, $\Delta - \alpha_7$ and $\Delta - \alpha_2$. By Theorem 3.3 and the results for D_n and E_6, (3.1) holds. Let $u \in V_1 \cap V_2 \cap V_3$. Then $\alpha \in \operatorname{supp} u$ if and only if $\alpha \in \Psi(\beta)$ for $\beta = \sum_{i=1}^7 \alpha_i$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$.

E_8. We consider the parabolic subgroups $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_3$ such that $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{L}_3$ have root systems D_7, E_7 and A_7, generated respectively by $\Delta - \alpha_1$, $\Delta - \alpha_8$ and $\Delta - \alpha_2$. By Theorem 3.3 and the results for D_n and E_7, (3.1)

holds. Let $u \in V_1 \cap V_2 \cap V_3$. Then $\alpha \in \operatorname{supp} u$ if and only if $\alpha \in \Psi(\beta)$ for $\beta = \sum_{i=1}^{8} \alpha_i$. By Lemma 4.1, $\mathcal{O} \cap \mathbb{U}^F \not\subset \cap_i V_i$. $\square$

4.2. **Unipotent classes in $\mathbf{P\Omega}_{2n+1}(q)$.** Here we deal with $\mathbf{P\Omega}_{2n+1}(q)$, i.e. Φ is of type B_n, $n \geq 3$. In this case, q is always odd.

Proposition 4.3. *$\mathcal{O}$ is not kthulhu.*

Proof. We consider the standard F-stable parabolic subgroups $\mathbb{P}_1$ and $\mathbb{P}_2$ such that $\mathbb{L}_1$ and $\mathbb{L}_2$ have root systems A_{n-1} and C_2, generated respectively by $\Pi_1 := \Delta - \alpha_n$ and $\Pi_2 = \{\alpha_{n-1}, \alpha_n\}$. By Lemma 3.2 (a) and Theorem 3.3, if $\mathcal{O} \cap \mathbb{U}^F \not\subset V_1$ then $\mathcal{O}$ is not kthulhu. Let us thus consider $u \in \mathcal{O} \cap V_1$. Then $\operatorname{supp} u \subset \Psi_{\Pi_1} = \{\varepsilon_i, \varepsilon_j + \varepsilon_l \mid i, j, l \in \mathbb{I}_n, j < l\}$, since it must contain α_n. We will apply the argument in 2.2.4.

Assume first that $\operatorname{supp} u \subset \{\varepsilon_j + \varepsilon_l \mid j, l \in \mathbb{I}_n, j < l\}$. Let ℓ be the maximum l such that $\varepsilon_j + \varepsilon_l \in \operatorname{supp} u$ for some $j \in \mathbb{I}_{n-1}$. Then $s_{\varepsilon_\ell}(\operatorname{supp} u) \subset \Phi^+$. Let $\dot{s}_{\varepsilon_\ell}$ be a representative of s_{ε_ℓ} in $N_{\mathbb{G}^F_{\mathrm{sc}}}(\mathbb{T})$. Then $\dot{s}_{\varepsilon_\ell} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and $\varepsilon_j - \varepsilon_\ell \in \operatorname{supp}(\dot{s}_{\varepsilon_\ell} \triangleright u)$ for every j such that $\varepsilon_j + \varepsilon_\ell \in \operatorname{supp} u$. Hence $\dot{s}_{\varepsilon_\ell} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F - V_1$. By the previous argument, $\mathcal{O}$ is not kthulhu.

Assume next that there is some i such that $\varepsilon_i \in \operatorname{supp} u$. We can always assume $i = n$. Indeed, if $\varepsilon_n \notin \operatorname{supp} u$, we may replace u by $\dot{s}_{\varepsilon_i - \varepsilon_n} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$, where $\dot{s}_{\varepsilon_i - \varepsilon_n}$ is a representative of $s_{\varepsilon_i - \varepsilon_n}$ in $N_{\mathbb{G}^F_{\mathrm{sc}}}(\mathbb{T})$. Then $\pi_2(u) \in M_2$ lies in a non-trivial unipotent conjugacy class in a group isomorphic to $\mathbf{Sp}_4(q)$ and the short simple root lies in the support. A direct computation shows that a representative of this class in $\mathbf{Sp}_4(q)$ is as follows:

$$\begin{pmatrix} 1 & a & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a \neq 0.$$

Thus, its Jordan form has partition $(2, 2)$ and this class is not kthulhu by Theorem 3.3 (recall that q is odd). Then Lemma 3.2 applies. $\square$

4.3. **Unipotent classes in $F_4(q)$.** Here we deal with unipotent classes in $F_4(q)$. In this case the approach in Section 4.1 is not effective. Indeed, in characteristic 2, (3.1) does not hold for any of the standard parabolic subgroups. For this reason we shall use explicit representatives of unipotent classes and apply results from Theorem 3.3 and Proposition 4.3 for B_3, where q is assume to be odd.

We use the list of representatives of unipotent classes in $F_4(q)$ in [18, Tables 5,6] for q odd, see Table 4, respectively in [17, Theorem 2.1] for q even, see Table 5. We indicate the roots as in [17]: ε_i is indicated by i, $\varepsilon_i - \varepsilon_j$ is indicated by $i - j$, and $\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)$ is indicated by $1 \pm 2 \pm 3 \pm 4$. Thus the simple roots are $\alpha_1 = 2 - 3$, $\alpha_2 = 3 - 4$, $\alpha_3 = 4$, $\alpha_4 = 1 - 2 - 3 - 4$.

Table 4. Representatives of unipotent classes in $F_4(q)$ in odd characteristic; η, ξ and ζ are suitable elements in $\mathbb{F}_q^\times$

$x_1 = x_{1+2}(1)$
$x_2 = x_{1-2}(1)x_{1+2}(-1)$
$x_3 = x_{1-2}(1)x_{1+2}(-\eta)$
$x_4 = x_2(1)x_{3+4}(1)$
$x_5 = x_{2-3}(1)x_4(1)x_{2+3}(1)$
$x_6 = x_{2-3}(1)x_4(1)x_{2+3}(\eta)$
$x_7 = x_2(1)x_{1-2+3+4}(1)$
$x_8 = x_{2-3}(1)x_4(1)x_{1-2}(1)$
$x_9 = x_{2-3}(1)x_{3-4}(1)x_{3+4}(-1)$
$x_{10} = x_{2-3}(1)x_{3-4}(1)x_{3+4}(-\eta)$
$x_{11} = x_{2+3}(1)x_{1+2-3-4}(1)x_{1-2+3+4}(1)$
$x_{12} = x_{2-3}(1)x_4(1)x_{1-4}(1)$
$x_{13} = x_{2-3}(1)x_4(1)x_{1-4}(\eta)$
$x_{14} = x_{2-4}(1)x_{3+4}(1)x_{1-2}(-1)x_{1-3}(-1)$
$x_{15} = x_{2-4}(1)x_{3+4}(1)x_{1-2}(-\eta)x_{1-3}(-1)$
$x_{16} = x_{2-4}(1)x_{2+4}(-\eta)x_{1-2+3+4}(1)x_{1-3}(-1)$
$x_{17} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3+4}(1)x_{1-2}(-\eta)x_{1-3}(\xi)$
$x_{18} = x_2(1)x_{3+4}(1)x_{1-2+3-4}(1)x_{1-2}(-1)x_{1-3}(\zeta)$
$x_{19} = x_{2-3}(1)x_{3-4}(1)x_4(1)$
$x_{20} = x_2(1)x_{3+4}(1)x_{1-2-3-4}(1)$
$x_{21} = x_{2-4}(1)x_3(1)x_{2+4}(1)x_{1-2-3+4}(1)$
$x_{22} = x_{2-4}(1)x_3(1)x_{2+4}(\eta)x_{1-2-3+4}(1)$
$x_{23} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2}(1)$
$x_{24} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2}(\eta)$
$x_{25} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)$
$x_{26} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{1-2+3+4}(\zeta)$
$x_{27} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{1-2+3+4}(-\zeta)$

If q is odd, then the possible representatives are x_i, $i \in \mathbb{I}_{25}$, for $p \neq 3$, with two additional representatives x_{26}, x_{27} when $p = 3$.

Lemma 4.4. *If q is odd, then $\mathcal{O}$ is not kthulhu.*

Proof. A direct verification shows that all representatives for $i \geq 7$ enjoy the $\alpha\beta$-property with $(\alpha, \beta) \neq 0$; we list in Table 6 the roots α and β for each representative. By Proposition 2.7, $\mathcal{O}$ is of type D.

We next consider the representative x_1, that equals $x_\gamma(1)$ for a long root γ. By the discussion in §2.2.2, $\mathcal{O}_{x_1}$ contains an element in $\mathbb{U}^F_{\alpha_1}$, that lies in the subgroup of type A_2 generated by $\mathbb{U}_{\pm\alpha_1}, \mathbb{U}_{\pm\alpha_2}$. Theorem 3.3 applies.

Table 5. Representatives of unipotent classes in $F_4(q)$ in even characteristic; η and ζ are suitable elements in $\mathbb{F}_q^\times$

$x_1 = x_1(1)$
$x_2 = x_{1+2}(1)$
$x_3 = x_1(1)x_{1+2}(1)$
$x_4 = x_{2+3}(1)x_1(1)$
$x_5 = x_2(1)x_{2+3}(1)x_{1-3}(1)$
$x_6 = x_2(1)x_{2+3}(1)x_{1-3}(1)x_{1+3}(\eta)$
$x_7 = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)$
$x_8 = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1+4}(\eta)$
$x_9 = x_2(1)x_{1-2}(1)$
$x_{10} = x_2(1)x_{1-2}(1)x_{1+2}(\eta)$
$x_{11} = x_2(1)x_{3+4}(1)x_{1-4}(1)$
$x_{12} = x_2(1)x_{1-2+3+4}(1)x_{1-4}(1)$
$x_{13} = x_2(1)x_{2+3}(1)x_{1-2}(1)$
$x_{14} = x_2(1)x_{3+4}(1)x_{1-2}(1)$
$x_{15} = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1-3}(1)$
$x_{16} = x_2(1)x_{2+3}(1)x_{1-2+3+4}(1)x_{1-2}(1)$
$x_{17} = x_2(1)x_{2+3}(1)x_{1-2-3+4}(1)x_{1-2}(1)$
$x_{18} = x_2(1)x_{2+3}(1)x_{1-2-3+4}(1)x_{1-2}(1)x_{1-4}(\eta)$
$x_{19} = x_2(1)x_{3+4}(1)x_{1-2+3-4}(1)x_{1-2}(1)x_{1-3}(\zeta)$
$x_{20} = x_{1-2}(1)x_{2-3}(1)x_3(1)$
$x_{21} = x_{1-2}(1)x_{2-3}(1)x_3(1)x_{2+3}(\eta)$
$x_{22} = x_4(1)x_{2-4}(1)x_{1-2+3-4}(1)$
$x_{23} = x_4(1)x_{2-4}(1)x_{2+4}(\eta)x_{1-2+3-4}(1)$
$x_{24} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)$
$x_{25} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)x_{1-2}(\eta)$
$x_{26} = x_{2-4}(1)x_{3+4}(1)x_{1-2-3-4}(1)x_{1-2-3+4}(1)x_{1-2}(\eta)x_{1-3}(\eta)$
$x_{27} = x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-2-3+4}(1)$
$x_{28} = x_{2-4}(1)x_3(1)x_{3+4}(1)x_{2+4}(\eta)x_{1-2-3+4}(1)x_{1-2}(\eta)$
$x_{29} = x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)$
$x_{30} = x_{1-2}(1)x_{2-3}(1)x_{3-4}(1)x_4(1)x_{3+4}(\eta)$
$x_{31} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)$
$x_{32} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{3+4}(\eta)$
$x_{33} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{1-2}(\eta)$
$x_{34} = x_{2-3}(1)x_{3-4}(1)x_4(1)x_{1-2-3-4}(1)x_{3+4}(\eta)x_{1-2}(\eta)$

Finally, we deal with the x_i's, $i \in \mathbb{I}_{2,6}$. Let $\mathbb{L}_1$ be the standard Levi subgroup (of type B_3) generated by the root subgroups $\mathbb{U}_\gamma$, for $\gamma = \pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$. We claim that all x_i, $i \in \mathbb{I}_{2,6}$, are conjugated to elements in M_1; then the result follows by Proposition 4.3. Indeed, x_2, x_3 lie in $\mathbb{U}_{1-2}^F\mathbb{U}_{1+2}^F$;

TABLE 6. $\mathcal{O}_{x_i}$ with the $\alpha\beta$-property

i	α	β
7	2	1-2+3+4
8	1-2	2-3
9,10	2-3	3-4
11	1+2-3-4	1-2+3+4
12,13	4	1-4
14,15	2-4	1-2
16	2-4	1-2+3+4
17,21,22	2-4	1-2-3+4
18	2	1-2
19,23,24,25,(26,27)	2-3	3-4
20	2	1-2-3-4

thus conjugating by $s_{1-3}s_{2-4}$, we get a representative in $\mathbb{U}^F_{3-4}\mathbb{U}^F_{3+4}$. Also x_5, x_6 lie in $\mathbb{U}^F_{2-3}\mathbb{U}^F_{2+3}\mathbb{U}^F_4$, and $x_4 = x_2(1)x_{3+4}(1)$, so they all lie in M_1. $\square$

Lemma 4.5. *If q is even, then $\mathcal{O}$ is not kthulhu.*

Proof. The representative x_1, respectively x_2, is equal to $x_\gamma(1)$ for a short, respectively long, root γ. By the discussion in §2.2.2, $\mathcal{O}_{x_1}$ intersects $\mathbb{U}^F_{\alpha_3}$ and $\mathcal{O}_{x_2}$ intersects $\mathbb{U}^F_{\alpha_1}$. Let $M = \langle \mathbb{U}^F_{\pm\alpha_3}, \mathbb{U}^F_{\pm\alpha_4}\rangle$ and $M' = \langle \mathbb{U}^F_{\pm\alpha_1}, \mathbb{U}^F_{\pm\alpha_2}\rangle$, both of type A_2. Then $\mathcal{O}_{x_1} \cap M$, respectively $\mathcal{O}_{x_2} \cap M'$, is a unipotent class corresponding to the partition $(2, 1)$ in M, respectively M'. By Theorem 3.3, these classes are not kthulhu.

We consider now the classes labelled by $i \in \mathbb{I}_{20,34}$. Let $\mathbb{P}_1$ be the standard parabolic subgroup with standard Levi $\mathbb{L}_1$ as in the proof of Lemma 4.4. Set $y_i = \pi_1(x_i)$. Then the class $\mathcal{O}^{M_1}_{y_i}$ satisfies the $\alpha\beta$-property; we list in Table 7 the roots α and β for each representative. Since Φ_{Π_1} is of type B_3, the group $[\mathbb{L}_1, \mathbb{L}_1]$ is isogenous to $\mathbf{Sp}_6(\Bbbk)$. By Remark 3.4, $\mathcal{O}^{M_1}_{y_i}$ is not kthulhu, hence neither is $\mathcal{O}$.

We consider now the classes labelled by $i \in \mathbb{I}' = \{3, 4, 7, 8, 12\} \cup \mathbb{I}_{14,19}$. Let $\mathbb{P}_2$ be the standard parabolic subgroup with standard Levi $\mathbb{L}_2$ (of type C_3) associated with $\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4\}$; here $\Phi^+_{\Pi_2}$ consists of the roots $1 - 2$, 3, 4, 3 ± 4, $1 - 2 \pm 3 \pm 4$. Let $\beta_1 = \alpha_4$, $\beta_2 = \alpha_3$, $\beta_3 = \alpha_2$ be the simple roots of $\Phi^+_{\Pi_2}$. Set $z_i = \pi_2(x_i)$. Now $\mathcal{O}^{M_2}_{z_i}$ is a unipotent class in $\mathbf{Sp}_6(q)$. Let $\mathbb{I}'' = \mathbb{I}' - \{3, 4\}$. Table 8 lists the index $i \in \mathbb{I}''$, the support of z_i and the

TABLE 7. $\mathcal{O}_{y_i}^{M_1}$ with the $\alpha\beta$-property.

i	α	β
$i \in \mathbb{I}_{20,21}$	α_1	$\alpha_2 + \alpha_3$
$i \in \mathbb{I}_{22,23}$	α_3	$\alpha_1 + \alpha_2$
$i \in \mathbb{I}_{24,28}$	$\alpha_1 + \alpha_2$	$\alpha_2 + 2\alpha_3$
$i \in \mathbb{I}_{29,34}$	α_1	α_2

partition associated to $\mathcal{O}_{z_i}^{M_2}$, obtained from the Jordan form of z_i in $\mathbf{Sp}_6(\Bbbk)$. Since the partition is always different from $(2, 1^4)$, the label of the class in $\mathbf{Sp}_6(q)$ is never $W(1) \oplus V(2)$, whence $\mathcal{O}_{z_i}^{M_2}$ is not kthulhu by Theorem 3.3. The remaining classes in $\mathbb{I}'$ are represented by $x_3 = x_1(1)x_{1+2}(1)$ and $x_4 = x_{2+3}(1)x_1(1)$. Let $x = (\dot{s}_{1-3}\dot{s}_{2-4}) \triangleright x_3 \in \mathcal{O}_{x_3}^{\boldsymbol{G}}$ and $y = (\dot{s}_{2-3}\dot{s}_{1-2}\dot{s}_3) \triangleright x_4 \in \mathcal{O}_{x_4}^{\boldsymbol{G}}$. Then $x \in \mathbb{U}_3\mathbb{U}_{3+4}$, so $x \in \mathbb{U}_{\beta_2+\beta_3}^F \mathbb{U}_{2\beta_2+\beta_3}^F \subset M_2$, $y \in \mathbb{U}_{1-2}\mathbb{U}_3$, so $y \in \mathbb{U}_{2\beta_1+2\beta_2+\beta_3}^F \mathbb{U}_{\beta_2+\beta_3}^F \subset M_2$. The partition associated to x, respectively y, as unipotent element in $\mathbf{Sp}_6(q)$ is $(2, 2, 1, 1)$, respectively $(2, 2, 2)$. Hence, neither $\mathcal{O}_{x_3}^{\boldsymbol{G}}$ nor $\mathcal{O}_{x_4}^{\boldsymbol{G}}$ is kthulhu by Theorem 3.3.

TABLE 8. $\operatorname{supp} z_i$ and its partition

i	$\operatorname{supp} z_i$	partition
7,8,12,15	$\beta_1 + 2\beta_2 + \beta_3$	$(2, 2, 1, 1)$
14	$2\beta_1 + 2\beta_2 + \beta_3$, $2\beta_2 + \beta_3$,	$(2, 2, 1, 1)$
16	$2\beta_1 + 2\beta_2 + \beta_3$, $\beta_1 + 2\beta_2 + \beta_3$,	$(2, 2, 1, 1)$
17,18	$2\beta_1 + 2\beta_2 + \beta_3$, $\beta_1 + \beta_2$,	$(2, 2, 1, 1)$
19	$2\beta_1 + 2\beta_2 + \beta_3$, $2\beta_2 + \beta_3$, $\beta_1 + \beta_2 + \beta_3$	$(2, 2, 2)$

The x_i's for $i \in \mathbb{I}''' = \{5, 6, 9, 10, 11, 13\}$ lie in the subgroup $\mathbb{K}$ of type B_4 generated by the subgroups $\mathbb{U}_{\pm\alpha}$, $\alpha \in \{1-2, 2-3, 3-4, 4\}$. If $i \in \mathbb{I}'''$, $\mathcal{O}_{x_i}^{\mathbb{K}^F}$ has the $\alpha\beta$-property, see Table 9. Since $\mathbf{SO}_9(\Bbbk)$ is isogenous to $\mathbf{Sp}_8(\Bbbk)$, Remark 3.4 applies.

□

4.4. **Unipotent classes in $G_2(q)$.** Here we deal with unipotent classes in $G_2(q)$, $q > 2$. As for $F_4(q)$, we shall use explicit representatives of the classes, the parabolics being too small. The list of representatives can be found in [6] when $p > 3$ and in [9] otherwise; see (4.2), (4.3), (4.9).

Lemma 4.6. *If q is odd, then $\mathcal{O}$ is not kthulhu.*

TABLE 9. $\mathcal{O}_{x_i}^{\mathbb{K}^F}$ with the $\alpha\beta$-property, $i \in \mathbb{I}'' = \{5, 6, 9, 10, 11, 13\}$.

x_i	α	β
$5, 6$	2+3	1-3
$9, 10, 13$	2	1-2
11	1-4	3+4

Proof. Assume first $p > 3$. By [6, Theorems 3.1, 3.2, 3.9] every non-trivial class of p-elements in $\boldsymbol{G}$ is either regular or can be represented by an element of the following form, for suitable $a, b, c \in \mathbb{F}_q^\times$:

$$\begin{aligned} &x_{\alpha_2}(1), \quad x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b), \quad x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(-1)x_{3\alpha_1+\alpha_2}(c), \\ &x_{\alpha_1+\alpha_2}(1), \quad x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a). \end{aligned} \tag{4.2}$$

The regular classes are covered by Proposition 3.6 (1). The elements $x_{\alpha_2}(1)$ and $x_{\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b)$ lie in the subgroup of type A_2 generated by $\mathbb{U}_{\pm\alpha_2}^F$ and $\mathbb{U}_{\pm(3\alpha_2+\alpha_2)}^F$ and we apply Theorem 3.3. The classes represented by $x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(-1)x_{3\alpha_1+\alpha_2}(c)$ enjoy the $\alpha\beta$-property, so we invoke Proposition 2.7. We prove now that the class of $r = x_{\alpha_1+\alpha_2}(1)$ is of type D. We observe that there is an element $\sigma = \dot{s}_{\alpha_2} \in \boldsymbol{G} \cap N_{\mathbb{G}}(\mathbb{T})$ such that $s := \sigma \triangleright r = x_{\alpha_1}(\xi)$, $\xi \in \mathbb{F}_q^\times$. Then $sr \neq rs$ by the Chevalley commutator formula (2.1) and, as $rs, sr \in \mathbb{U}^F$ and p is odd, we have $(rs)^2 \neq (sr)^2$. In addition, $r, s \in \mathbb{P}_1^F$, for $\mathbb{P}_1$ the standard parabolic subgroup with Levi $\mathbb{L}_1$ associated with α_1. Since r lies in the unipotent radical $\mathbb{V}_1$ of $\mathbb{P}_1$ and s lies in $\mathbb{L}_1$, we have $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$.

Let $r = x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a)$; it lies in $\langle \mathbb{U}_{\pm\alpha_2}^F \rangle \times \langle \mathbb{U}_{\pm(2\alpha_1+\alpha_2)}^F \rangle$. We argue as in §3.3. As $q > 3$, Lemmata 3.5 and 2.1 apply whence $\mathcal{O}_r$ is of type D.

Assume now $p = 3$. By [9, 6.4] the non-trivial classes of p-elements in $\boldsymbol{G}$ are either regular or are represented by an element of the following form:

$$\begin{aligned} &x_{3\alpha_1+2\alpha_2}(1), && x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(a), \\ &x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1), && x_{2\alpha_1+\alpha_2}(1), \end{aligned} \tag{4.3}$$

for suitable $a \in \mathbb{F}_q^\times$. The regular classes are covered by Proposition 3.6 (1). The element $x_{3\alpha_1+2\alpha_2}(1)$ lies in the subgroup of type A_2 generated by $\mathbb{U}_{\pm\alpha_2}^F$ and $\mathbb{U}_{\pm(3\alpha_1+2\alpha_2)}^F$ and Theorem 3.3 applies.

We show that if $r = x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(a) \in \mathcal{O}$, then it is of type D. Indeed, let $s := \dot{s}_{\alpha_2} \triangleright r \in \mathbb{U}_{\alpha_1}^F \mathbb{U}_{3\alpha_1+2\alpha_2}^F$. Then $sr \neq rs$; since $sr, rs \in \mathbb{U}^F$, we have $(sr)^2 \neq (rs)^2$. Moreover, $r, s \in \mathbb{P}_1^F$ with $s \in \mathbb{L}_1$, $r \in \mathbb{V}_1$, with notation as for $p > 3$. Thus, $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$ and $\mathcal{O}$ is of type D.

Assume that $u = x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1) \in \mathcal{O}$. Conjugating by suitable elements in $N_{\boldsymbol{G}}(\mathbb{T})$ we find $r \in \mathcal{O} \cap \mathbb{U}_{\alpha_1}\mathbb{U}_{3\alpha_1+\alpha_2} \subset \mathbb{P}_1$, $r \notin \mathbb{V}_1$ and $s \in \mathcal{O} \cap \mathbb{U}_{\alpha_1+\alpha_2}\mathbb{U}_{\alpha_2} \subset \mathbb{V}_1$. By repeated use of (2.1), we see that the coefficient of $x_{\alpha_1+\alpha_2}$ in srs^{-1} is $\neq 0$, hence $rs \neq sr$, $(rs)^2 \neq (sr)^2$ and $\mathcal{O}$ is of type D.

Assume finally that $u = x_{2\alpha_1+\alpha_2}(1) \in \mathcal{O}$. Let $r = \dot{s}_{\alpha_1} \triangleright u \in \mathcal{O}_u^{\boldsymbol{G}} \cap \mathbb{U}_{\alpha_1+\alpha_2}$ and $s = \dot{s}_{\alpha_1+\alpha_2} \triangleright u \in \mathcal{O}_u^{\boldsymbol{G}} \cap \mathbb{U}_{\alpha_1}$. Then $rs, sr \in \mathbb{U}$, $(rs)^2 \neq (sr)^2$, and $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$, as $s \in \mathbb{L}_1$ and $r \in \mathbb{V}_1$, so $\mathcal{O}$ is of type D. □

In order to deal with some unipotent classes in $G_2(4)$ we will need a precise version of (2.1) for all pairs of positive roots. We shall use the relations from [9, II.2], that we write for convenience. They hold in general for q even, and we shall use them recalling that $a^3 = 1$ for every $a \in \mathbb{F}_4^\times$.

$$x_{\alpha_1}(a)x_{\alpha_2}(b) = x_{\alpha_2}(b)x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(ab)x_{2\alpha_1+\alpha_2}(a^2b)x_{3\alpha_1+\alpha_2}(a^3b) \tag{4.4}$$

$$x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(b) = x_{\alpha_1+\alpha_2}(b)x_{\alpha_1}(a)x_{3\alpha_1+\alpha_2}(a^2b)x_{3\alpha_1+2\alpha_2}(ab^2) \tag{4.5}$$

$$x_{\alpha_1}(a)x_{2\alpha_1+\alpha_2}(b) = x_{2\alpha_1+\alpha_2}(b)x_{\alpha_1}(a)x_{3\alpha_1+\alpha_2}(ab) \tag{4.6}$$

$$x_{\alpha_2}(a)x_{3\alpha_1+\alpha_2}(b) = x_{3\alpha_1+\alpha_2}(b)x_{\alpha_2}(a)x_{3\alpha_1+2\alpha_2}(ab) \tag{4.7}$$

$$x_{\alpha_1+\alpha_2}(a)x_{2\alpha_1+\alpha_2}(b) = x_{2\alpha_1+\alpha_2}(b)x_{\alpha_1+\alpha_2}(a)x_{3\alpha_1+2\alpha_2}(ab) \tag{4.8}$$

For all other pairs of positive roots the corresponding subgroups commute.

Lemma 4.7. *If $q > 2$ is even, then $\mathcal{O}$ is not kthulhu.*

Proof. By [9, 2.6] all non-trivial classes of 2-elements in $\boldsymbol{G}$ can be represented by an element of the following form, for suitable $a, b, c \in \mathbb{F}_q$:

$$\begin{aligned} &x_{2\alpha_1+\alpha_2}(1), \quad x_{3\alpha_1+2\alpha_2}(1), && x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a), \\ &x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b), && x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(c). \end{aligned} \tag{4.9}$$

Assume that $r = x_{2\alpha_1+\alpha_2}(1) \in \mathcal{O}$. It is enough to prove that $\mathcal{O}$ is of type C for $G_2(2)$, which is a non-simple subgroup of $G_2(q)$. We consider $\dot{s}_{\alpha_1+\alpha_2} \triangleright r = x_{\alpha_1}(1) \in \mathcal{O}$ and $s := x_{-\alpha_1}(1) \triangleright x_{\alpha_1}(1) = \dot{s}_{\alpha_1} \in \mathcal{O}_r^{G_2(2)}$. Let $H := \langle r, s, z = x_{\alpha_1+\alpha_2}(1)\rangle \leq \mathbb{P}_1$ (the parabolic subgroup associated with α_1), with $r \in \mathbb{V}_1$, $s \in \mathbb{L}_1$. Hence, $\mathcal{O}_r^H \neq \mathcal{O}_s^H$. By a direct computation,

$$\begin{aligned} s \triangleright r &= x_{\alpha_1+\alpha_2}(1) = z \neq r, & z \triangleright r &= r x_{3\alpha_1+2\alpha_2}(1), \\ r \triangleright s &= szr, & z \triangleright (szr) &= s x_{3\alpha_1+2\alpha_2}(1). \end{aligned}$$

So $H \leq \langle \mathcal{O}_r^H, \mathcal{O}_s^H\rangle \leq H$; $\{r, z, z \triangleright r\} \subset \mathcal{O}_r^H$ and $\{s, szr, sx_{3\alpha_1+2\alpha_2}(1)\} \subset \mathcal{O}_s^H$ hence $\mathcal{O}_r^{G_2(2)}$ is of type C by [3, Lemma 2.8].

Assume that $r = x_{3\alpha_1+2\alpha_2}(1) \in \mathcal{O}$. Now $r \in \mathbb{M} = \langle \mathbb{U}_{\pm\alpha_2}, \mathbb{U}_{\pm(3\alpha_1+\alpha_2)}\rangle$, of type A_2. Since $\mathcal{O}_r^{\mathbb{M}}$ has partition $(2,1)$, $\mathcal{O}$ is not kthulhu by Theorem 3.3 and [16, Theorem 24.15].

Assume $q > 4$. The classes represented by the $x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(a)$ for $a \in \mathbb{F}_q$ are regular, thus they are not kthulhu by Proposition 3.6. The classes of $x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b)$ and $x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(c)$ enjoy the $\alpha\beta$-property. By [2, Proposition 3.6], these classes are of type F.

Let now $q = 4$ and let ζ be a generator of $\mathbb{F}_4^{\times}$ so $\zeta^2+\zeta+1 = 0$ and $\zeta^3 = 1$. By [9] there are 2 regular unipotent classes, one represented by $x_{\alpha_1}(1)x_{\alpha_2}(1)$ and the other by $x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta)$. We shall apply Lemma 2.3 in order to show that these classes are of type C. For this, we need the following formula which can be retrieved applying (4.4) and (4.7).

$$\begin{aligned} x_{\alpha_1}(a)x_{\alpha_2}(b)x_{\alpha_1}(c)x_{\alpha_2}(d) &= x_{\alpha_1}(a+c)x_{\alpha_2}(b+d) \\ &\times x_{3\alpha_1+\alpha_2}(b)x_{3\alpha_1+2\alpha_2}(bd)x_{2\alpha_1+\alpha_2}(c^2b)x_{\alpha_1+\alpha_2}(bc), \end{aligned} \tag{4.10}$$

$a, b, c, d \in \mathbb{F}_q$. Let $r = x_{\alpha_1}(1)x_{\alpha_2}(1)$, $t := \alpha_1^{\vee}(\zeta)$, $s := t \triangleright r = x_{\alpha_1}(\zeta^2)x_{\alpha_2}(1) \in \mathcal{O}_r^{\boldsymbol{G}}$. By direct computation using (4.10) we see that

$$\begin{aligned} r^2 &= x_{3\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1) \\ s^2 &= x_{3\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta)x_{\alpha_1+\alpha_2}(\zeta^2). \end{aligned}$$

Using (4.8) and that $\xi^2 \neq \xi$, we see $r^2s^2 \neq s^2r^2$, hence $r^2s \neq sr^2$ and $s^2r \neq rs^2$. In addition, $\langle r, s\rangle \subseteq \mathbb{U}^F$ and $\mathbb{U}^F \triangleright r \subset r\langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \Delta\rangle$ and $\mathbb{U}^F \triangleright s \subset s\langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \Delta\rangle$, so $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$, whence $\mathcal{O}_r^{\boldsymbol{G}}$ is of type C.

Similarly, we consider now $r = x_{\alpha_1}(1)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta)$, $t := \alpha_1^{\vee}(\zeta)$ and $s := t \triangleright r = x_{\alpha_1}(\zeta^2)x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta^2) \in \mathcal{O}_r^{\boldsymbol{G}}$. In this case

$$\begin{aligned} r^2 &= x_{3\alpha_1+\alpha_2}(\zeta^2)x_{3\alpha_1+2\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1) \\ s^2 &= x_{3\alpha_1+\alpha_2}(\zeta^2)x_{3\alpha_1+2\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(\zeta)x_{\alpha_1+\alpha_2}(\zeta^2). \end{aligned}$$

As above we verify that $r^2s \neq s^2r$ and $s^2r \neq r^2s$ and that $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$ so $\mathcal{O}_r^{\boldsymbol{G}}$ is of type C.

We assume now that $x := x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(b) \in \mathcal{O}$, with $b \neq 0$. By [9, Proposition 2.6, page 499] if $q = 4$ we can take $b = \zeta$. We prove that this class is of type C. Set $r_\alpha := x_\alpha(1)x_{-\alpha}(1)x_\alpha(1) = \dot{s}_\alpha$, $\alpha \in \Phi^+$, see

[22, Lemma 19]. The elements

$$\begin{aligned} s = r_{\alpha_1} \triangleright x &= x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1)x_{\alpha_2}(\zeta) \\ &= x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_2}(\zeta)x_{3\alpha_1+2\alpha_2}(1), \\ r = r_{\alpha_2}r_{\alpha_1} \triangleright s &= x_{\alpha_1}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(\zeta) \end{aligned}$$

belong to $\mathcal{O}$. We claim that $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$. Indeed, r, $s \in \mathbb{P}_1$ with $r \notin \mathbb{V}_1$, $s \in \mathbb{V}_1$. A direct calculation shows that $r^2 = x_{3\alpha_1+\alpha_2}(1)$,

$$\begin{aligned} &r \triangleright s = x_{\alpha_1+\alpha_2}(1+\zeta)x_{2\alpha_1+\alpha_2}(1+\zeta)x_{\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta), \\ &r^2 \triangleright s = sx_{3\alpha_1+2\alpha_2}(\zeta), \\ &r^3 \triangleright s = r \triangleright (r^2 \triangleright s) = r \triangleright (sx_{3\alpha_1+2\alpha_2}(\zeta)) = (r \triangleright s)x_{3\alpha_1+2\alpha_2}(\zeta), \\ &s \triangleright (r \triangleright s) = x_{\alpha_1+\alpha_2}(1+\zeta)x_{2\alpha_1+\alpha_2}(1+\zeta)x_{\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+2\alpha_2}(1+\zeta). \end{aligned}$$

We see that all these are distinct, and different from s, by looking at the unique expression as a product of elements in root subgroups in the order:

$$\alpha_1 < \alpha_1+\alpha_2 < 2\alpha_1+\alpha_2 < \alpha_2 < 3\alpha_1+\alpha_2 < 3\alpha_1+2\alpha_2$$

Hence, $|\mathcal{O}_r^{\langle r,s\rangle}| \geq 5$ and $\mathcal{O}$ is of type C, by [3, Lemma 2.8], with $H = \langle r, s\rangle$.

Let now $t_2 := \alpha_1^\vee(\zeta)\alpha_2^\vee(\zeta)$, $t_3 := \alpha_2^\vee(\zeta)$, $t_4 := \alpha_1^\vee(\zeta)\alpha_2^\vee(\zeta^2)$ and set

$$\begin{aligned} x &= x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1), \\ x_1 &= r_{\alpha_2} \triangleright x = x_{\alpha_1}(1)x_{2\alpha_1+\alpha_2}(1) \in \mathcal{O}_x; \\ x_2 &= t_2 \triangleright x_1 = x_{\alpha_1}(\zeta)x_{2\alpha_1+\alpha_2}(\zeta), \\ x_3 &= t_3 \triangleright x_1 = x_{\alpha_1}(\zeta^2)x_{2\alpha_1+\alpha_2}(1), \\ x_4 &= t_4 \triangleright x_1 = x_{\alpha_1}(1)x_{2\alpha_1+\alpha_2}(\zeta). \end{aligned}$$

Let $Y_i = \mathbb{U}^F \triangleright x_i$, $i \in \mathbb{I}_4$. A direct computation shows that

$$\begin{aligned} Y_1 &= \bigcup_{f,\ell\in\mathbb{F}_4} x_{\alpha_1}(1)x_{\alpha_1+\alpha_2}(\ell)x_{2\alpha_1+\alpha_2}(\ell+1)x_{3\alpha_1+2\alpha_2}(f^2+f)\mathbb{U}^F_{3\alpha_1+\alpha_2}, \\ Y_2 &= \bigcup_{f,\ell\in\mathbb{F}_4} x_{\alpha_1}(\zeta)x_{\alpha_1+\alpha_2}(\ell\zeta)x_{2\alpha_1+\alpha_2}(\ell\zeta^2+\zeta)x_{3\alpha_1+2\alpha_2}(f^2\zeta+f\zeta)\mathbb{U}^F_{3\alpha_1+\alpha_2}, \\ Y_3 &= \bigcup_{f,\ell\in\mathbb{F}_4} x_{\alpha_1}(\zeta^2)x_{\alpha_1+\alpha_2}(\ell^2\zeta)x_{2\alpha_1+\alpha_2}(\ell\zeta+1)x_{3\alpha_1+2\alpha_2}(f^2\zeta^2+f)\mathbb{U}^F_{3\alpha_1+\alpha_2}, \\ Y_4 &= \bigcup_{f,\ell\in\mathbb{F}_4} x_{\alpha_1}(1)x_{\alpha_1+\alpha_2}(\ell)x_{2\alpha_1+\alpha_2}(\ell+\zeta)x_{3\alpha_1+2\alpha_2}(f^2+f\zeta)\mathbb{U}^F_{3\alpha_1+\alpha_2}. \end{aligned}$$

The union $Y = \bigcup_{i\in\mathbb{I}_4} Y_i$ is disjoint and a subrack of $\mathcal{O}_x$. We take

$$\begin{aligned}
&r_1 = x_1,\\
&r_2 = x_{\alpha_1}(\zeta)x_{\alpha_1+\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(1) \in \mathbb{U}^F \triangleright x_2, && (\ell = 1, f = 0),\\
&r_3 := x_{\alpha_1}(\zeta^2)x_{2\alpha_1+\alpha_2}(1) \in \mathbb{U}^F \triangleright x_3, && (\ell = f = 0),\\
&r_4 := x_{\alpha_1}(1)x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(\zeta^2) \in \mathbb{U}^F \triangleright x_4, && (\ell = 1, f = 0).
\end{aligned}$$

We claim that $x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(b)x_{2\alpha_1+\alpha_2}(c)$ and $x_{\alpha_1}(\widetilde{a})x_{\alpha_1+\alpha_2}(\widetilde{b})x_{2\alpha_1+\alpha_2}(\widetilde{c})$ do not commute, for $a, b, c, \widetilde{a}, \widetilde{b}, \widetilde{c} \in \mathbb{F}_q$ such that $c\widetilde{a}+\widetilde{a}^2 b \neq \widetilde{c}a+a^2\widetilde{b}$. This follows from the formula:

$$x_{\alpha_1}(a)x_{\alpha_1+\alpha_2}(b)x_{2\alpha_1+\alpha_2}(c)x_{\alpha_1}(\widetilde{a})x_{\alpha_1+\alpha_2}(\widetilde{b})x_{2\alpha_1+\alpha_2}(\widetilde{c}) =$$
$$x_{\alpha_1}(a+\widetilde{a})x_{\alpha_1+\alpha_2}(b+\widetilde{b})x_{2\alpha_1+\alpha_2}(c+\widetilde{c})x_{3\alpha_1+\alpha_2}(c\widetilde{a}+\widetilde{a}^2 b)x_{3\alpha_1+2\alpha_2}(b^2\widetilde{a}+c\widetilde{b}).$$

Hence, $r_ir_j \neq r_jr_i$ for $i \neq j$, $i, j \in \mathbb{I}_4$ and the class $\mathcal{O}_x$ is of type F.

By [9], the remaining class can be represented by any of

$$\begin{aligned}
&r_1 = x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(\zeta),\\
&r_2 = r_{\alpha_1} \triangleright r_1 = x_{\alpha_2}(\zeta)x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(1)x_{3\alpha_1+2\alpha_2}(\zeta).
\end{aligned}$$

Let $t := \alpha_1^\vee(\zeta)\alpha_2^\vee(\zeta^2)$ and

$$\begin{aligned}
&x := t \triangleright r_1 = x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta^2),\\
&y := t \triangleright r_2 = x_{\alpha_2}(\zeta^2)x_{\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+\alpha_2}(\zeta)x_{3\alpha_1+2\alpha_2}(1).
\end{aligned}$$

It is easier now to work with a different ordering of the positive roots:

$$\alpha_1 < \alpha_2 < 2\alpha_1+\alpha_2 < \alpha_1+\alpha_2 < 3\alpha_1+\alpha_2 < 3\alpha_1+2\alpha_2.$$

Let $Y_i = \mathbb{U}^F \triangleright r_i$, $i \in \mathbb{I}_2$, $Y_3 = \mathbb{U}^F \triangleright x$, $Y_4 = \mathbb{U}^F \triangleright y$. A direct computation shows that

$$\begin{aligned}
Y_1 &= \bigcup_{\ell\in\mathbb{F}_4} x_{\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1+\ell^2)x_{\alpha_1+\alpha_2}(\ell)x_{3\alpha_1+\alpha_2}(\zeta+\ell^3+\ell)\mathbb{U}^F_{3\alpha_1+2\alpha_2},\\
Y_2 &= \bigcup_{\ell\in\mathbb{F}_4} x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(\ell^2\zeta)x_{\alpha_1+\alpha_2}(1+\ell\zeta)x_{3\alpha_1+\alpha_2}(\ell^2+\ell^3\zeta+1)\mathbb{U}^F_{3\alpha_1+2\alpha_2},\\
Y_3 &= \bigcup_{\ell\in\mathbb{F}_4} x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(\zeta+\ell^2\zeta)x_{\alpha_1+\alpha_2}(\ell\zeta)x_{3\alpha_1+\alpha_2}(\ell^3\zeta+\ell\zeta+\zeta^2)\mathbb{U}^F_{3\alpha_1+2\alpha_2},\\
Y_4 &= \bigcup_{\ell\in\mathbb{F}_4} x_{\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(\ell^2\zeta^2)x_{\alpha_1+\alpha_2}(\ell\zeta^2+\zeta)x_{3\alpha_1+\alpha_2}(\ell^3\zeta^2+\ell^2\zeta+\zeta)\mathbb{U}^F_{3\alpha_1+2\alpha_2}.
\end{aligned}$$

The union $Y = \bigcup_{i\in\mathbb{I}_4} Y_i$ is disjoint and a subrack of $\mathcal{O}$. We take

$$\begin{aligned}
&r_3 := x_{\alpha_2}(\zeta)x_{2\alpha_1+\alpha_2}(1)x_{\alpha_1+\alpha_2}(1) \in Y_3, && (\ell = \zeta^2),\\
&r_4 := x_{\alpha_2}(\zeta^2)x_{2\alpha_1+\alpha_2}(\zeta^2)x_{\alpha_1+\alpha_2}(1)x_{3\alpha_1+\alpha_2}(\zeta^2) \in Y_4, && (\ell = 1).
\end{aligned}$$

By looking at the coefficient of $x_{3\alpha_1+2\alpha_2}$ in the expression of each product, we verify that $r_i \triangleright r_j \neq r_j \triangleright r_i$ if $i \neq j$, hence $\mathcal{O}$ is of type F. □

5. Unipotent classes in Steinberg groups

In this Section we deal with unipotent classes in Steinberg groups, i.e. $\mathbf{PSU}_n(q)$, $n \geq 3$; $\mathbf{P\Omega}^-_{2n}(q)$, $n \geq 4$; ${}^3D_4(q)$ and ${}^2E_6(q)$. In order to apply inductive arguments as in Section 4, we first need information about the unitary groups $\mathbf{PSU}_n(q)$, including the non-simple group $\mathbf{PSU}_3(2)$.

5.1. **Unipotent classes in unitary groups.** Here $\boldsymbol{G} = \mathbf{PSU}_n(q)$, $G = \mathbf{SU}_n(q)$, $n \geq 3$ and $\mathbb{G} = \mathbf{SL}_n(\Bbbk)$, for $n \geq 2$. For a clearer visibility of the behaviour of the conjugacy classes, we use the language of matrices and partitions. Here we choose $\mathbb{B}$, $\mathbb{U}$, as the subgroups of upper triangular, respectively unipotent upper triangular, matrices. We start by some notation and basic facts.

- ◇ $\mathsf{J}_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} = \mathsf{J}_n^{-1} \in \mathbf{GL}_n(\Bbbk)$.
- ◇ Fr_q is the Frobenius endomorphism of $\mathbf{GL}_n(\Bbbk)$ raising all entries of the matrix to the q-th power.
- ◇ $F : \mathbf{GL}_n(\Bbbk) \to \mathbf{GL}_n(\Bbbk)$, $F(X) = \mathsf{J}_n\, {}^t(\mathrm{Fr}_q(X))^{-1} \mathsf{J}_n$, $X \in \mathbf{GL}_n(\Bbbk)$.
- ◇ $\mathbf{GU}_n(q) = \mathbf{GL}_n(\Bbbk)^F$, $\mathbf{SU}_n(q) = \mathbf{SL}_n(\Bbbk)^F \leq \mathbf{SL}_n(q^2)$, [16, 21.14(2), 23.10(2)].
- ◇ To every unipotent class in $\mathbf{SU}_n(q)$ we assign the partition of n corresponding to the class in $\mathbf{GL}_n(q)$ it is embedded into.
- ◇ Every unipotent class in $\mathbf{GL}_n(\Bbbk)$ meets $\mathbf{GU}_n(q)$ in exactly one class, since $C_{\mathbf{GL}_n(\Bbbk)}(x)$ is connected for every x [15, 8.5], [21, I.3.5]. In other words, every partition comes from a class in $\mathbf{SU}_n(q)$.
- ◇ Since $\mathbf{SU}_n(q)$ is normal in $\mathbf{GU}_n(q)$, [1, Remark 2.1] says that all unipotent class in $\mathbf{SU}_n(q)$ with the same partition are isomorphic as racks.
- ◇ For $d \leq n$ with $d \equiv n \mod 2$ and $h = \frac{n-d}{2}$, we denote by $\mathbb{M}_d \leq \mathbb{G}$ the subgroup of matrices $\begin{pmatrix} \mathrm{id}_h & & \\ & A & \\ & & \mathrm{id}_h \end{pmatrix}$ with $A \in \mathbf{SL}_d(\Bbbk)$. So $\mathbb{M}_d^F \simeq \mathbf{SU}_d(q)$.
- ◇ For $c \leq \left[\frac{n}{2}\right]$ we denote by $\mathbb{H}_{2c} \leq \mathbb{G}$ the subgroup of matrices $\begin{pmatrix} A & & B \\ & \mathrm{id}_{n-2c} & \\ C & & D \end{pmatrix}$ for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbf{SL}_{2c}(\Bbbk)$. Then $\mathbb{H}_{2c}^F \simeq \mathbf{SU}_{2c}(q)$.
- ◇ If q is odd, then $G^{\mathrm{Fr}_q} = \mathbf{SO}_n(q)$. If q and n are even, then $G^{\mathrm{Fr}_q} = \mathbf{Sp}_n(q)$.

Here is the main result of this Subsection:

Proposition 5.1. *Let $\mathcal{O} \neq \{e\}$ be a unipotent class in $\boldsymbol{G} = \mathbf{PSU}_n(q)$ with partition λ, where λ is different from $(2, 1, \dots)$ if q is even. Then $\mathcal{O}$ is not kthulhu.*

Proof. First, we reduce our analysis to $G = \mathbf{SU}_n(q)$ by the isogeny argument [1, Lemma 1.2]. Thus, from now on $\mathcal{O}$ is a unipotent class in G. Second, we split the proof for q odd in §5.1.1 and for q even in §5.1.2. In each of these, we distinguish several cases according to the partition $\lambda = (\lambda_1, \dots, \lambda_n)$ associated to $\mathcal{O}$.

5.1.1. *Proof of Proposition 5.1 when q is odd.*

Lemma 5.2. *If $\lambda_1 \geq 3$, but $\lambda \neq (3,1)$ in $\mathbf{SU}_4(3)$, then $\mathcal{O}$ is not kthulhu.*

Proof. If λ_1 is even, we may find u in $\mathcal{O} \cap (\mathbb{H}^F_{\lambda_1} \times \mathbb{M}^F_{n-\lambda_1})$ such that its component in $\mathbb{H}^F_{\lambda_1}$ is regular. If λ_1 and n are odd, then we may find u in $\mathcal{O} \cap (\mathbb{H}^F_{n-\lambda_1} \times \mathbb{M}^F_{\lambda_1})$ whose component in $\mathbb{M}^F_{\lambda_1}$ is regular. In both cases Proposition 3.6 applies.

It remains the case when λ_1 is odd and n is even. Then there is $i > 1$ such that λ_i is odd, and $l = \lambda_1 + \lambda_i \geq 4$. We take i minimal with this property. Then, we may find u in $\mathcal{O} \cap (\mathbb{H}^F_l \times \mathbb{M}^F_{n-l})$, whose component in $\mathbb{H}^F_l$ has partition (λ_1, λ_i). We consider $H := \mathbb{H}^F_l \cap \mathbb{H}^{F Fr_q}_l \cong \mathbf{SO}_l(q)$. Since (λ_1, λ_i) is an orthogonal partition, we may assume $u \in H$. If $l \geq 6$ the class $\mathcal{O}^H_u$ is of type D by Theorem 3.3 and Proposition 4.3, since $\mathbf{SO}_6(\Bbbk)$ is $\mathbf{SL}_4(\Bbbk)$ up to isogeny.

Let $l = 4$ and $q > 3$. Now $\mathbf{SO}_l(\Bbbk)$ is $\mathbf{SL}_2(\Bbbk) \times \mathbf{SL}_2(\Bbbk)$ up to isogeny, the class $\mathcal{O}^H_u$ is isomorphic as a rack to the product $X \times X$ for X the non-trivial unipotent class in $\mathbf{SL}_2(q)$. By Lemma 2.1, $\mathcal{O}^H_u$ is of type D.

If $q = 3$ and $n > 4$, then the partition either contains the sub-partition $(3,3)$ or $(3,1,1,1)$. Reducing to the subgroup $\mathbb{M}_6$, we look at the classes $(3,3)$ and $(3,1,1,1)$ in $\mathbf{SU}_6(q)$. Since $\mathbf{SO}_6(q) < \mathbf{SU}_6(q)$ and $\mathbf{SO}_6(\Bbbk)$ is $\mathbf{SL}_4(\Bbbk)$ up to isogeny, these racks contain a subrack isomorphic to a non-trivial unipotent class in $\mathbf{SL}_4(q)$. Then Theorem 3.3 applies. □

Lemma 5.3. *If $\lambda = (3,1)$, $G = \mathbf{SU}_4(3)$, then $\mathcal{O}$ is of type D.*

Proof. Let ζ be a generator of $\mathbb{F}_9^\times$. We may assume that $r := \begin{pmatrix} 1 & \zeta & \zeta & 1 \\ 0 & 1 & 0 & -\zeta^3 \\ 0 & 0 & 1 & -\zeta^3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$. Let $t := \begin{pmatrix} 2 & \zeta^6 & \zeta^2 & \zeta \\ 2 & \zeta^5 & 0 & 0 \\ 0 & \zeta^2 & 2 & \zeta^5 \\ 0 & \zeta^6 & 1 & \zeta^7 \end{pmatrix} \in \mathbf{SU}_4(3)$ and let $s = t \triangleright r = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \in \mathcal{O}$. A direct computation shows that $(rs)^2 \neq (sr)^2$. The subgroup $H = \langle r, s \rangle \subset \{ \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \mid A, D \in \mathbf{SL}_2(9) \}$. If $s \in \mathcal{O}^H_r$, then $\left(\begin{smallmatrix} 0 & 1 \\ 2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & \zeta \\ 0 & 1 \end{smallmatrix}\right)$ would be conjugate in $\mathbf{SL}_2(9)$. But $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \triangleright \left(\begin{smallmatrix} 0 & 1 \\ 2 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ which is not conjugate to $\left(\begin{smallmatrix} 1 & \zeta \\ 0 & 1 \end{smallmatrix}\right)$ because ζ is not a square. Hence, $\mathcal{O}^H_r \neq \mathcal{O}^H_s$ and $\mathcal{O}$ is of type D. □

By Lemmata 5.2 and 5.3, there remain the partitions $(2^a, 1^b)$, $a > 0$.

Lemma 5.4. *If $q > 3$ and $\lambda_1 = 2$, then $\mathcal{O}$ is of type D.*

Proof. Assume that n is odd. Then the partition contains $(2,1)$ and we may find a representative whose component in $\mathbb{M}_3^F$ has partition $(2,1)$. It is therefore enough to prove the statement for $G = \mathbf{SU}_3(q)$ and $\lambda = (2,1)$. Let $r = \begin{pmatrix} 1 & 0 & a \\ & 1 & 0 \\ & & 1 \end{pmatrix} \in \mathcal{O}$ with $a \in \mathbb{F}_{q^2}^\times$, $a^q = -a$. As $\mathbb{F}_q^\times = \{\xi^{q+1} | \xi \in \mathbb{F}_{q^2}^\times\}$ and $q > 3$, we may pick $\xi \in \mathbb{F}_{q^2}^\times$ such that $-a^2\xi^{q+1} \in \mathbb{F}_q^\times - (\{2\} \cup (\mathbb{F}_q^\times)^2)$. Let $t \in G$ be the diagonal matrix $(\xi, \xi^{q-1}, \xi^{-q})$, $\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in G$ and

$$s := (\sigma t) \triangleright r = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ a\xi^{1+q} & 0 & 1 \end{pmatrix} \in \mathcal{O}.$$

Since $2 \neq -a^2\xi^{q+1}$, $(rs)^2 \neq (sr)^2$. Let $\eta \in \Bbbk$ be such that $\eta^2 = a^{-1}$. Conjugating by the diagonal matrix (η, η^{-1}) we have

$$H := \langle r, s \rangle \simeq \left\langle \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a\xi^{q+1} & 1 \end{pmatrix} \right\rangle \simeq \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ a^2\xi^{q+1} & 1 \end{pmatrix} \right\rangle.$$

By [23, Theorem 6.21, page 409], $H \simeq \mathbf{SL}_2(q)$. Since $-a^2\xi^{q+1}$ is not a square, $\mathcal{O}_r^H \neq \mathcal{O}_s^H$. Thus $\mathcal{O}$ is of type D.

Assume that n is even. Then the partition contains either $(2,2)$ or $(2,1,1)$ and we may use $\mathbb{M}_4$ to reduce to $\lambda = (2,2)$ or $(2,1,1)$ in $G = \mathbf{SU}_4(q)$. If $\lambda = (2,2)$, which is an orthogonal partition, then we may assume that the representative u lies in $\mathbf{SO}_4(q)$, and $\mathcal{O}_u^{\mathbf{SO}_4(q)} \cong X \times X$, where X is the non-trivial unipotent class in $\mathbf{SL}_2(q)$. Hence, it is of type D.

If $\lambda = (2,1,1)$, then we take $r = \begin{pmatrix} 1 & 0 & 0 & a \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \in \mathcal{O}$ for $a \in \mathbb{F}_{q^2}^\times$ satisfying $a^q = -a$. Let $t \in G$ be the diagonal matrix $(\xi, \xi^{-1}, \xi^q, \xi^{-q})$ for $\xi \in \mathbb{F}_{q^2}^\times$, such that $-a^2\xi^{q+1} \in \mathbb{F}_q^\times$ is not a square in $\mathbb{F}_q^\times$, and let $\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^{-q} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in G$ with $\zeta \in \mathbb{F}_{q^2}^\times$ such that $\zeta^q = -\zeta$. We consider

$$s := (\sigma t) \triangleright r = \begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ a\xi^{1+q} & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$$

and we proceed as for n odd. $\square$

Lemma 5.5. *If $q = 3$ and $\lambda_1 = \lambda_2 = 2$, then $\mathcal{O}$ is of type D.*

Proof. By reducing to the subgroup $\mathbb{H}_4$, it is enough to consider $\lambda = (2,2)$ in $G = \mathbf{SU}_4(3)$. Let ζ be a generator of $\mathbb{F}_9^\times$. Let

$$r = \begin{pmatrix} 1 & \zeta & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -\zeta^3 \\ & & & 1 \end{pmatrix} \in \mathcal{O}, \ \sigma = \begin{pmatrix} \zeta^2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \zeta^2 \end{pmatrix} \in G, \ s = \sigma \triangleright r = \begin{pmatrix} 1 & 0 & \zeta^3 & 0 \\ & 1 & 0 & -\zeta \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}.$$

It is easy to see that $(rs)^2 \neq (sr)^2$. In addition, $\langle r, s\rangle \subset \mathbb{U}^F$ and $\mathcal{O}_r^{\mathbb{U}^F} \neq \mathcal{O}_s^{\mathbb{U}^F}$, whence the statement. $\square$

Lemma 5.6. *If $\lambda = (2, 1, \dots)$ in $G = \mathbf{SU}_n(3)$, $n \geq 3$, then $\mathcal{O}$ is of type C.*

Proof. Let $\mathbb{F}_9^\times = \langle \zeta \rangle$. Without loss of generality we may assume that

$$r = \begin{pmatrix} 1 & & \zeta^2 \\ & \ddots & \\ & & 1 \end{pmatrix} = \mathrm{id}_n + \zeta^2 e_{1,n} \in \mathcal{O}.$$

We consider, for n odd, respectively even, the following element of $\mathbf{SU}_n(3)$:

$$\sigma := \begin{pmatrix} 0 & 0 & 0 & 0 & \zeta \\ 0 & \mathrm{id}_{\frac{n-3}{2}} & 0 & 0 & 0 \\ 0 & 0 & -\zeta^2 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{\frac{n-3}{2}} & 0 \\ \zeta^{-3} & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \tau := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \zeta \\ 0 & \mathrm{id}_{\frac{n-4}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta^{-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id}_{\frac{n-4}{2}} & 0 \\ \zeta^{-3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Accordingly, we set $s := \sigma \triangleright r$ or $s := \tau \triangleright r$. In both cases, $s = \begin{pmatrix} 1 & & \\ & \ddots & \\ \zeta^{-2} & & 1 \end{pmatrix} = \mathrm{id}_n + \zeta^{-2} e_{n,1}$. Then $rs \neq sr$. Let $H := \langle r, s\rangle$. We have

$$H \simeq \langle \begin{pmatrix} 1 & \zeta^2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \zeta^2 & 1 \end{pmatrix} \rangle \simeq \mathbf{SL}_2(3).$$

Conjugation by $\mathrm{diag}(\zeta^{-1}, \zeta)$ and [23, Theorem 6.21, page 409] give $H \simeq \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \rangle \simeq \mathbf{SL}_2(3)$, so $\mathcal{O}_r^H \neq \mathcal{O}_s^H$. We conclude by [3, Lemma 2.7]. $\square$

5.1.2. *Proof of Proposition 5.1 when q is even.*

Lemma 5.7. *If $\mathcal{O}$ is a regular unipotent class, then it is not kthulhu.*

Proof. By Proposition 3.6 it is enough to deal with regular unipotent classes in $\mathbf{SU}_3(q)$ for $q = 2, 8$, $\mathbf{SU}_4(q)$ for $q = 2, 4$ and $\mathbf{SU}_n(2)$ for $n \geq 5$.

(i) *Regular unipotent classes in $\mathbf{SU}_3(2^{2h+1})$, $h \in \mathbb{N}_0$, are of type D.*

It suffices to prove the claim for $G = \mathbf{SU}_3(2) \leq \mathbf{SU}_3(2^{2h+1})$. Let ζ be a generator of $\mathbb{F}_4^\times$. Consider the class $\mathcal{O}$ represented by $r = \left(\begin{smallmatrix} 1 & 1 & \zeta \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. Let $t = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & \zeta^2 \\ 1 & \zeta & \zeta \end{smallmatrix}\right) \in \mathbf{SU}_3(2)$ and $s := t \triangleright r = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \zeta^2 & 1 & 1 \end{smallmatrix}\right) \in \mathcal{O}$. By direct verification, $(rs)^2 \neq (rs)^2$. A computation with GAP shows that $\mathcal{O}_s^{\langle r,s\rangle} \neq \mathcal{O}_r^{\langle r,s\rangle}$.

(ii) *Regular unipotent classes in $\mathbf{SU}_n(q)$, $n \geq 4$ even, are not kthulhu.*

Indeed, by the Jordan form theory, $\mathcal{O}$ is represented by an element of a regular class in $\mathbf{Sp}_n(q) = \mathbf{SU}_n(q)^{\mathrm{Fr}_q}$. We conclude invoking Theorem 3.3.

(iii) *Regular unipotent classes in $\mathbf{SU}_n(2)$, $n \geq 5$ odd, are not kthulhu.*

By projecting a representative in $\mathbb{U}^F$ to $\mathbb{M}_5$, we obtain a regular unipotent class in the latter. Hence, it is enough to assume $n = 5$. Let $\mathbb{P}$ be the standard F-stable parabolic subgroup of $\mathbb{G}$ with Levi factor $\mathbb{L}$ corresponding to the simple roots α_2, α_3, and let $\pi\colon \mathbb{P}^F \to \mathbb{L}^F$ be the projection. Any $u \in \mathcal{O} \cap \mathbb{U}^F$ lies in $P := \mathbb{P}^F$ and $\pi(u)$ is regular in $\mathbb{L}^F$, which is the subgroup of matrices of the form $\begin{pmatrix} \det A & & \\ & A & \\ & & \det A \end{pmatrix}$, where $A \in \mathbf{GU}_3(2)$, so $\mathbb{L}^F \simeq \mathbf{GU}_3(2)$. By Proposition 3.6, $\mathcal{O}^{\mathbb{L}^F}_{\pi(u)}$ is not kthulhu. Then Lemma 2.5 applies. □

Now we argue inductively starting from Lemma 5.7.

Lemma 5.8. *If any of the following conditions holds, then $\mathcal{O}$ is not kthulhu.*

(a) *There is j such that $\lambda_j \geq 4$ is even.*
(b) *n is odd and there is j such that $\lambda_j \geq 3$ is odd.*
(c) *There is i such that $\lambda_i = \lambda_{i+1} > 2$.*
(d) *n is even and there are i, j such that $\lambda_j > \lambda_i \geq 3$ and are both odd.*

Proof. (a), (b): Apply Lemma 5.7 either to $\mathbb{H}_{\lambda_j}$ or to $\mathbb{M}_{\lambda_j}$.

(c): The class with partition (λ_i, λ_i) in $\mathbb{H}^F_{2\lambda_i}$ has a representative in $(\mathbb{H}^F_{2\lambda_i})^{\mathrm{Fr}_q} \simeq \mathbf{Sp}_{2\lambda_i}(q)$. Its class has label $W(\lambda_i)$ or $V(\lambda_i) \oplus V(\lambda_i)$ hence Theorem 3.3 applies.

(d): By considering the class with partition (λ_j, λ_i) in $\mathbb{M}_{\lambda_i+\lambda_j}$ we may assume $n = \lambda_i + \lambda_j$. Let $d = \lambda_j - \lambda_i$ and let $\mathbb{P}$ be the parabolic subgroup with standard Levi subgroup of type $A_{\lambda_i-1} \times A_{\lambda_i-1}$ associated with

$$\{\alpha_k \in \Delta \mid k \in \mathbb{I}_{\lambda_i-1} \cup \mathbb{I}_{\lambda_j+1,\lambda_j+\lambda_i-1}\}.$$

Then $M \simeq \mathbf{SL}_{\lambda_i}(q^2)$. By Lemma 3.2 and Theorem 3.3 it is enough to show that $\mathcal{O} \cap V = \emptyset$. Now, if $u \in V$ then it is of the form

$$\begin{pmatrix} \mathrm{id}_{\lambda_i} & A_1 & A_2 \\ 0 & B & A_3 \\ 0 & 0 & \mathrm{id}_{\lambda_i} \end{pmatrix},$$

for some upper-triangular $B \in \mathbf{SU}_d(q)$ and some matrices A_i, $i \in \mathbb{I}_3$. Hence,

$$\mathrm{rk}(u - \mathrm{id}) = \mathrm{rk}\begin{pmatrix} A_1 & A_2 \\ B - \mathrm{id}_d & A_3 \end{pmatrix} \leq d + \lambda_i. \tag{5.1}$$

On the other hand, if $u \in \mathcal{O}$ by Jordan form theory we have

$$\mathrm{rk}(u - \mathrm{id}) = \lambda_i + \lambda_j - 2 = 2\lambda_i + d - 2. \tag{5.2}$$

As $\lambda_i \geq 3$, conditions (5.1) and (5.2) are not compatible, so $\mathcal{O} \cap V = \emptyset$. □

Lemma 5.9. *If there is i such that $\lambda_i = \lambda_{i+1} = 2$, then $\mathcal{O}$ is not kthulhu.*

Proof. Looking at $\mathbb{H}^F_{2\lambda_i}$, we reduce to the partition $(2,2)$ in $\mathbf{SU}_4(q)$. Let u be a representative of a unipotent class with label $V(2)\oplus V(2)$ in $\mathbf{Sp}_4(q) = \mathbf{SU}_4(q)^{\mathrm{Fr}_q} \leq \mathbf{SU}_4(q)$. By Jordan form theory, we may assume that $u \in \mathcal{O}$. By Theorem 3.3, $\mathcal{O}_u^{\mathbf{Sp}_4(q)}$ is not kthulhu, whence the statement. □

By Lemmata 5.7, 5.8 and 5.9 there remain the partitions: $(2,1^a)$ for all $n \geq 3$, and $(\lambda_1, 2, 1^a)$, $(\lambda_1, 1^a)$ for $\lambda_1 > 1$ odd and n even.

Lemma 5.10. *If n is even and $\lambda = (\lambda_1, 2, 1^a)$, where $1 < \lambda_1$ is odd, then $\mathcal{O}$ is of type D.*

Proof. It is enough to consider $\lambda = (\lambda_1, 2, 1)$. We pick a representative of $\mathcal{O}$ lying in $\mathbb{H}^F_2 \times \mathbb{M}^F_{\lambda_1+1} \simeq \mathbf{SL}_2(q) \times \mathbf{SU}_{\lambda_1+1}(q)$. Then $\mathcal{O}$ contains a subrack isomorphic to $X \times Y$ where $X \neq \{e\}$ is a unipotent class in $\mathbf{SL}_2(q)$ and Y is a unipotent class with partition $(\lambda_1, 1)$ in $\mathbf{SU}_{\lambda_1+1}(q)$. The latter is not a class of involutions because $\lambda_1 > 2$. By [24, 1.4(ii)] and Remark 2.2 there are $y_1 \neq y_2 \in Y$ such that $y_1 y_2 = y_2 y_1$. Now Lemmata 3.7 and 2.1 apply. □

Lemma 5.11. *If n is even and $\lambda = (\lambda_1, 1, \dots)$ for some $3 < \lambda_1$ odd, then $\mathcal{O}$ is not kthulhu.*

Proof. It is enough to deal with the partition $(\lambda_1, 1)$. Set $d := (\lambda_1+1)/2 > 2$. Let $\mathbb{P}$ be the parabolic subgroup with standard Levi subgroup associated with $\Delta - \alpha_d$. Then $M \simeq \mathbf{SL}_d(q^2)$. We claim that $\mathcal{O} \cap V = \emptyset$. Indeed, if $u \in V$ then it is of the form

$$\begin{pmatrix} \mathrm{id}_d & A \\ 0 & \mathrm{id}_d \end{pmatrix},$$

for some A, so $\mathrm{rk}(u-1) \leq d$. If, in addition, $u \in \mathcal{O}$, then $\mathrm{rk}(u - \mathrm{id}) = \lambda_1 - 1$. This is impossible because $\lambda_1 > 3$. Lemma 3.2 and Theorem 3.3 apply. □

Lemma 5.12. *If $n \geq 6$ is even and $\lambda = (3, 1, 1, 1, \dots)$, then $\mathcal{O}$ is of type D.*

Proof. It is enough to deal with the partition $(3,1,1,1)$ in $\mathbf{SU}_6(q)$. Let $x \in \mathbb{F}^\times_{q^2} - \mathbb{F}^\times_q$ and let $r = \begin{pmatrix} 1 & x & 0 & 0 & 1 & x \\ & 1 & 0 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & x^q \\ & & & & & 1 \end{pmatrix} \in \mathcal{O}$,

$$\sigma = \begin{pmatrix} 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \end{pmatrix} = s_1 s_5 s_2 s_4 \in \mathbf{SU}_6(q), \quad s = \sigma \triangleright r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & x & 1 & x & 0 \\ & & 1 & 0 & 1 & 0 \\ & & & 1 & x^q & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \in \mathcal{O}.$$

Since $r, s \in \mathbb{U}$, the discussion in [1, 3.1] shows that $\mathcal{O}_r^{\langle r,s\rangle} \neq \mathcal{O}_s^{\langle r,s\rangle}$. By looking at the $(1,5)$-entry, we see that $(rs)^2 \neq (sr)^2$ and $\mathcal{O}$ is of type D. □

Lemma 5.13. *If $q > 4$ and $\lambda = (3, 1)$ in $\mathbf{SU}_4(q)$, then $\mathcal{O}$ is of type F.*

Proof. Let $x, y \in \mathbb{F}_{q^2}$ such that $x^q y + y^q x \neq 0$ and $\zeta_i \in \mathbb{F}_q^{\times}$, for $i \in \mathbb{I}_4$, satisfying $\zeta_i \neq \zeta_j$ if $i \neq j$. Let

$$r = \begin{pmatrix} 1 & x & y & xy^q \\ & 1 & 0 & y^q \\ & & 1 & x^q \\ & & & 1 \end{pmatrix}, \quad t_i = \begin{pmatrix} 1 & & & \\ & \zeta_i & & \\ & & \zeta_i^{-1} & \\ & & & 1 \end{pmatrix}, \quad r_i := t_i \triangleright r = \begin{pmatrix} 1 & x\zeta_i^{-1} & y\zeta_i & xy^q \\ & 1 & 0 & y^q\zeta_i \\ & & 1 & x^q\zeta_i^{-1} \\ & & & 1 \end{pmatrix}.$$

Then $r, r_i \in \mathcal{O}$, since $t_i \in \mathbf{SU}_4(q)$. Let $H := \langle r_1, r_2, r_3, r_4 \rangle \subset \mathbb{U}^F$. By the discussion in [1, 3.1] that $\mathcal{O}_{r_i}^H \neq \mathcal{O}_{r_j}^H$ if $i \neq j$. Then $r_i r_j = r_j r_i$ if and only if $(x^q y + xy^q)(\zeta_i \zeta_j^{-1} + \zeta_i^{-1}\zeta_j) = 0$, if and only if $\zeta_i = \zeta_j$. □

Lemma 5.14. *The unipotent classes of type $(3, 1)$ in $\mathbf{SU}_4(2)$ are of type D.*

Proof. Let ζ be a generator of $\mathbb{F}_4^{\times}$. We may assume that $r = \begin{pmatrix} 1 & 1 & \zeta & \zeta \\ 0 & 1 & 0 & \zeta^2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$. Let $t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \in \mathbf{SU}_4(2)$ and $s := t \triangleright r = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \zeta^2 & \zeta & 1 & 1 \\ 0 & \zeta & 0 & 1 \end{pmatrix} \in \mathcal{O}$. Then $(rs)^2 \neq (rs)^2$. By GAP we see that $\mathcal{O}_s^{\langle r,s \rangle} \neq \mathcal{O}_r^{\langle r,s \rangle}$ and the claim follows. □

Lemma 5.15. *The unipotent classes of type $(3, 1)$ in $\mathbf{SU}_4(4)$ are of type D.*

Proof. Let ζ be a generator of $\mathbb{F}_{16}^{\times}$. We may assume that $r = \begin{pmatrix} 1 & 1 & \zeta & \zeta \\ 0 & 1 & 0 & \zeta^4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{O}$. Let $t = \begin{pmatrix} \zeta^{11} & \zeta^2 & \zeta^5 & \zeta^{14} \\ \zeta^{11} & \zeta^2 & \zeta^8 & \zeta^2 \\ 0 & \zeta^{14} & \zeta^9 & \zeta^{11} \\ 0 & \zeta^{14} & \zeta^9 & \zeta^6 \end{pmatrix} \in \mathbf{SU}_4(4)$ and $s := t \triangleright r = \begin{pmatrix} 0 & 1 & 0 & \zeta^{12} \\ 1 & 0 & \zeta^3 & \zeta^{10} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{O}$. We check at once that $(rs)^2 \neq (rs)^2$, and with GAP that $\mathcal{O}_s^{\langle r,s \rangle} \neq \mathcal{O}_r^{\langle r,s \rangle}$. □

The Proposition is now proved. □

The remaining classes could not be reached with our methods.

Lemma 5.16. *If $\lambda = (2, 1^a)$ in $\mathbf{SU}_n(q)$, then $\mathcal{O}$ is austere, hence kthulhu.*

Proof. We show that any subrack generated by two elements is either abelian or indecomposable. Let $r, s \in \mathcal{O}$, $rs \overset{\star}{\neq} sr$. We may assume $r = \mathrm{id}_n + ae_{1,n} = x_\beta(a)$ where β is the highest positive root in Φ and $a \in \mathbb{F}_q^{\times}$. Let $g \in G$ be such that $s = grg^{-1}$. By [16, 24.1] there are $u, v \in \mathbb{U}^F$, and $\sigma \in G \cap N(\mathbb{T})$ such that $g = u\sigma v$. As $F(\sigma) = \sigma$, the coset $\overline{\sigma} = \sigma\mathbb{T} \in W$ lies in $W^F \simeq \mathbb{S}_n^F$ which is the centralizer of the permutation

$$(1, n)(2, n-1) \cdots ([\frac{n}{2}], n + 1 - [\frac{n}{2}]);$$

hence, either $\overline{\sigma}(\{1,n\}) = \{1,n\}$ or $\overline{\sigma}(\{1,n\}) \cap \{1,n\} = \emptyset$. Since r is central in $\mathbb{U}^F$, $s = u\sigma r\sigma^{-1}u^{-1} = ux_{\overline{\sigma}(\beta)}(a')u^{-1}$ for some $a' \in \mathbb{F}_q$. Since $ru = ur$ and $rv = vr$, $\star$ holds if and only if $r \neq \sigma r\sigma^{-1}$. Thus, $\overline{\sigma}(1) = n$ and $\overline{\sigma}(n) = 1$, so σ is of the form

$$\sigma = \begin{pmatrix} 0 & 0 & \xi \\ 0 & A & 0 \\ \xi^{-q} & 0 & 0 \end{pmatrix}, \qquad \text{where } A \in \mathbf{GU}_{n-2}(q), \qquad \xi \in \mathbb{F}_q^{\times},\ \xi^{q-1} = \det A.$$

Then $\sigma r\sigma^{-1} = \mathrm{id}_n + a\xi^{-1-q}e_{n,1}$, so

$$H := \langle r, s\rangle \simeq \langle \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right), \begin{pmatrix} 1 & 0 \\ \xi^{-q-1}a & 1 \end{pmatrix}\rangle \subset \mathbf{SL}_2(q).$$

Since the non-trivial unipotent class in $\mathbf{SL}_2(q)$ is sober [1, 3.5], $\mathcal{O}_r^H = \mathcal{O}_s^H$. □

5.2. **Unipotent classes in $\mathbf{P\Omega}_{2n}^{-}(q)$, $n \geq 4$.** In this subsection $\boldsymbol{G} = \mathbf{P\Omega}_{2n}^{-}(q)$, $n \geq 4$. We shall use the knowledge of unipotent conjugacy classes in $\mathbf{PSL}_n(q)$ and $\mathbf{PSU}_n(q)$ and apply inductive arguments.

Here $\mathbb{G}$ is assumed simply-connected. The root system of $\mathbb{G}$ is of type D_n, and the Dynkin diagram automorphism ϑ interchanges α_{n-1} and α_n; it fixes the basis vectors ε_j for $j \in \mathbb{I}_{n-1}$, and maps ε_n to $-\varepsilon_n$. Here is the main result of this Subsection:

Proposition 5.17. *Let $\mathcal{O}$ be a non-trivial unipotent class in $\mathbf{P\Omega}_{2n}^{-}(q)$ with $n \geq 4$. Then $\mathcal{O}$ is not kthulhu.*

We split the proof for q odd in §5.2.1 and for q even in §5.2.2.

5.2.1. *Proof of Proposition 5.17 when q is odd.*

Proof. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be the standard F-stable parabolic subgroups with F-stable Levi factors $\mathbb{L}_1$ and $\mathbb{L}_2$ associated respectively with $\Pi_1 := \Delta - \{\alpha_{n-1}, \alpha_n\}$ (of type A_{n-2}), and $\Pi_2 := \{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$ (of type A_3). Then

$$\Phi_{\Pi_1}^+ = \{\varepsilon_i - \varepsilon_j \mid i < j \in \mathbb{I}_{n-1}\},\ \Phi_{\Pi_2}^+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j \in \mathbb{I}_{n-2,n}\},$$
$$\Psi_{\Pi_1} = \Phi^+ \smallsetminus \Phi_{\Pi_1}^+ = \{\varepsilon_i + \varepsilon_j, \varepsilon_k - \varepsilon_n \mid i < j \in \mathbb{I}_n,\ k \in \mathbb{I}_{n-1}\},$$
$$\Psi_{\Pi_2} = \{\varepsilon_i \pm \varepsilon_j \mid i < j,\ i \in \mathbb{I}_{n-3}, j \in \mathbb{I}_n\}.$$

By Lemma 3.2, Theorem 3.3 and Proposition 5.1, it is enough to show that $\mathcal{O} \cap \mathbb{U}^F \not\subset V_1 \cap V_2$. Assume that there is $u \in \mathcal{O} \cap V_1 \cap V_2$; then

$$\operatorname{supp} u \subset \Psi_{\Pi_1} \cap \Psi_{\Pi_2} = \{\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_n \mid i < j,\ i \in \mathbb{I}_{n-3}, j \in \mathbb{I}_n\}.$$

We consider various different cases.

(i) $\varepsilon_i - \varepsilon_n \in \operatorname{supp} u$ for some $i \in \mathbb{I}_{n-3}$.

Then $s_{\varepsilon_i-\varepsilon_{n-2}}(\operatorname{supp} u) \subseteq \Phi^+$. Since $s_{\varepsilon_i-\varepsilon_{n-2}} \in W^F$, it has a representative $\dot{s}_{\varepsilon_i-\varepsilon_{n-2}} \in N_{\mathbb{G}^F}(\mathbb{T})$; hence $\dot{s}_{\varepsilon_i-\varepsilon_{n-2}} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and $\varepsilon_{n-2} - \varepsilon_n \in \operatorname{supp}(\dot{s}_{\varepsilon_i-\varepsilon_{n-2}} \triangleright u)$. Thus $\dot{s}_{\varepsilon_i-\varepsilon_{n-2}} \triangleright u \in \mathbb{U}^{\mathbb{F}} \cap \mathcal{O} - V_2$.

(ii) $\varepsilon_i - \varepsilon_n \notin \operatorname{supp} u$ for all $i \in \mathbb{I}_{n-3}$.

Then there exist $k \in \mathbb{I}_{n-3}$ and j such that $\varepsilon_k + \varepsilon_j \in \operatorname{supp} u$. Let

$$\ell = \max\{j \mid \varepsilon_k + \varepsilon_j \in \operatorname{supp} u \text{ for some } k\}.$$

If $\ell = n$, then pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T}) \cap \mathbb{L}_2$ of $s_{\varepsilon_{n-1}-\varepsilon_n} s_{\varepsilon_{n-1}+\varepsilon_n} \in W^F$. Thus $\sigma \triangleright u \in \mathcal{O} \cap V_2$ and $\varepsilon_k - \varepsilon_n \in \operatorname{supp}(\sigma \triangleright u)$ for all k such that $\varepsilon_k + \varepsilon_n \in \operatorname{supp} u$. Therefore, either $\operatorname{supp}(\sigma \triangleright u) \not\subset V_1$, and we are done, or $\operatorname{supp}(\sigma \triangleright u) \subset V_1$ and $\varepsilon_k - \varepsilon_n \in \operatorname{supp}(\sigma \triangleright u)$, and we fall in (i).

If $\ell = n-1$, then pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T}) \cap \mathbb{L}_2$ of $s_{\varepsilon_{n-2}+\varepsilon_{n-1}} \in W^F$. As above, $\sigma \triangleright u \in \mathcal{O} \cap V_2$, and $\varepsilon_i - \varepsilon_{n-2} \in \operatorname{supp}(\sigma \triangleright u) \cap \Phi_{\Pi_1}$ for some $i < n-2$. That is, $\operatorname{supp}(\sigma \triangleright u) \not\subset V_1$.

Finally, if $\ell < n-1$, then we pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T})$ of $s_{\varepsilon_\ell - \varepsilon_{n-1}}$. Then $\operatorname{supp} \sigma \triangleright u \subset V_1 \cap V_2$, and we fall in the case $\ell = n-1$. □

5.2.2. *Proof of Proposition 5.17, q even.* Here Lemma 3.2 does not apply in its full strength because of the existence of kthulhu classes in $\mathbf{PSU}_4(q)$, q even, and in $\mathbf{PSL}_3(2)$. We proceed by induction on n. The case $n = 4$, Lemma 5.18 below, requires a special treatment.

Lemma 5.18. *If $\boldsymbol{G} = \mathbf{P\Omega}_8^-(q)$ with q even, then $\mathcal{O}$ is not kthulhu.*

Proof. Let us consider the F-stable standard parabolic subgroups $\mathbb{P}_1$, $\mathbb{P}_2$ with standard Levi subgroups $\mathbb{L}_1$ and $\mathbb{L}_2$ associated with the sets $\Pi_1 = \{\alpha_1, \alpha_2\}$ and $\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4\}$, respectively. Let $u \in \mathcal{O} \cap \mathbb{U}^F$. We analyse different situations, according to $\Delta \cap \operatorname{supp} u$. Recall that, u being F-invariant, the simple root $\alpha_3 \in \operatorname{supp} u$ if and only if $\alpha_4 \in \operatorname{supp} u$.

(i) $\alpha_2, \alpha_3, \alpha_4 \in \operatorname{supp} u$.

The projection $\pi_2(u) \in L_2$ is regular, thus $\mathcal{O}^{M_2}_{\pi_2(u)}$ is isomorphic as a rack to a unipotent class in $\mathbf{SU}_4(q)$ of partition (4) and Proposition 5.1 applies.

(ii) $\Delta \cap \operatorname{supp} u = \{\alpha_1, \alpha_3, \alpha_4\}$ or $\Delta \cap \operatorname{supp} u = \{\alpha_3, \alpha_4\}$.

Then $\mathcal{O}^{M_2}_{\pi_2(u)}$ has partition $(2,2)$ or $(3,1)$ and Proposition 5.1 applies.

(iii) $\Delta \cap \operatorname{supp} u = \{\alpha_1\}$ or $\Delta \cap \operatorname{supp} u = \{\alpha_2\}$.

Here $\pi_1(u) \in L_1$ is not regular, hence $\mathcal{O}^{M_1}_{\pi_1(u)}$ is isomorphic as a rack to a unipotent class in $\mathbf{SL}_3(q)$ with partition $\neq (3)$; Theorem 3.3 applies.

(iv) $\Delta \cap \operatorname{supp} u = \{\alpha_1, \alpha_2\}$: either $\alpha_2 + \alpha_3 + \alpha_4 \in \operatorname{supp} u$ or not.

We may assume that $\alpha_2 + \alpha_3 \notin \operatorname{supp} u$, by conjugating with a suitable element in $(\mathbb{U}_{\alpha_3}\mathbb{U}_{\alpha_4})^F$ and using (2.1). If $\alpha_2 + \alpha_3 + \alpha_4 \in \operatorname{supp} u$, then $\mathcal{O}^{M_2}_{\pi_2(u)} \simeq \mathcal{O}^{\mathbf{SU}_4(q)}_v$, where $\operatorname{rk}(v - \mathrm{id}) = 2$ and $(v - \mathrm{id})^2 = 0$, which is not kthulhu since its partition is $(2, 2)$. If $\alpha_2 + \alpha_3 + \alpha_4 \notin \operatorname{supp} u$, then pick a representative $\sigma \in N_{\mathbb{G}^F}(\mathbb{T})$ of $s_3 s_4 \in W$. Then $\sigma \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and $\Delta \cap \operatorname{supp}(\sigma \triangleright u) = \{\alpha_1\}$ so we reduce to (iii).

(v) $\Delta \cap \operatorname{supp} u = \emptyset$ and $\alpha_1 + \alpha_2 \in \operatorname{supp} u$ or $\alpha_2 + \alpha_3 \in \operatorname{supp} u$.

In the first case, $\mathcal{O}^{M_1}_{\pi_1(u)}$ has type $(2, 1)$, and Theorem 3.3 applies. In the second, also $\alpha_2 + \alpha_4 \in \operatorname{supp} u$ and $\mathcal{O}^{M_2}_{\pi_2(u)}$ has type $(2, 2)$. Indeed, $\mathcal{O}^{M_2}_{\pi_2(u)} \simeq \mathcal{O}^{\mathbf{SU}_4(q)}_v$, where $\operatorname{rk}(v - \mathrm{id}) = 2$ and $(v - \mathrm{id})^2 = 0$. We invoke Proposition 5.1.

(vi) $(\Delta \cup \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4\}) \cap \operatorname{supp} u = \emptyset$.

Let $\dot{s}_i \in N_{\mathbb{G}^F}(\mathbb{T})$ be a representative of s_i, $i \in \mathbb{I}_2$. If $\alpha_1 + \alpha_2 + \alpha_3 \in \operatorname{supp} u$, then also $\alpha_1 + \alpha_2 + \alpha_4 \in \operatorname{supp} u$. Now $\dot{s}_1 \triangleright u \in \mathbb{U}^F \cap \mathcal{O}$, $\Delta \cap \operatorname{supp}(\dot{s}_1 \triangleright u) = \emptyset$ and $\alpha_2 + \alpha_3 \in \operatorname{supp}(\dot{s}_1 \triangleright u)$, so we fall in (v). Let σ be as in (iv). If $\alpha_2 + \alpha_3 + \alpha_4 \in \operatorname{supp} u$, then $\sigma \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and $\alpha_2 \in \operatorname{supp}(\sigma \triangleright u)$ and we are in case (iii).

(vii) $\operatorname{supp} u \subset \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$.

If $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \operatorname{supp} u$, then $\dot{s}_1 \triangleright u$ is as in case (vi); while if $\operatorname{supp} u = \{\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\}$, then $\operatorname{supp}(\dot{s}_2 \triangleright u) = \{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4\}$. □

We now proceed with the recursive step and assume that all non-trivial unipotent classes in a twisted group with root system D_{n-1} are not kthulhu.

Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be the standard parabolic subgroups with F-stable standard Levi subgroups $\mathbb{L}_1$ and $\mathbb{L}_2$ associated with the sets $\Pi_1 = \{\alpha_i \mid i \in \mathbb{I}_{n-2}\}$ and $\Pi_2 = \{\alpha_i \mid i \in \mathbb{I}_{2,n}\}$, of type A_{n-2} and D_{n-1} respectively. By Lemma 3.2 in order to prove the inductive step, it is enough to show that no non-trivial unipotent class $\mathcal{O}$ in $\mathbb{G}^F$ satisfies $\mathcal{O} \cap \mathbb{U}^F \subset V_1 \cap V_2$. As usual let

$$\Phi_{\Pi_1} = \{\varepsilon_i - \varepsilon_j \mid i < j \in \mathbb{I}_{n-1}\}, \qquad \Phi_{\Pi_2} = \{\varepsilon_i \pm \varepsilon_j \mid i < j \in \mathbb{I}_{2,n}\},$$
$$\Psi_{\Pi_1} = \{\varepsilon_i - \varepsilon_n, \varepsilon_j + \varepsilon_k \mid i \in \mathbb{I}_{n-1}, j < k \in \mathbb{I}_n\}, \ \Psi_{\Pi_2} = \{\varepsilon_1 \pm \varepsilon_j \mid j \in \mathbb{I}_{2,n}\}.$$

Let $u \in \mathcal{O} \cap V_1 \cap V_2$. Then $\operatorname{supp} u \subset \Psi_{\Pi_1} \cap \Psi_{\Pi_2} = \{\varepsilon_1 - \varepsilon_n, \varepsilon_1 + \varepsilon_j \mid j \in \mathbb{I}_{2,n}\}$. Let $\dot{s}_i \in N_{\mathbb{G}^F}(\mathbb{T})$ be a representative of $s_i \in W^F$, $i \in \mathbb{I}_2$. If $\operatorname{supp} u \neq \{\varepsilon_1 + \varepsilon_2\}$ then $\dot{s}_1 \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$, but $\dot{s}_1 \triangleright u \notin V_2$ (look at its support). If, instead, $\operatorname{supp} u = \{\varepsilon_1 + \varepsilon_2\}$ then $\dot{s}_1 \dot{s}_2 \triangleright u \in \mathcal{O} \cap \mathbb{U}^F \cap \mathbb{U}_{\varepsilon_2 + \varepsilon_3}$, so $\dot{s}_1 \dot{s}_2 \triangleright u \notin V_1$.

This finishes the proof for q even and Proposition 5.17 is now proved. □

5.3. **Unipotent classes in $^2E_6(q)$.** We deal now with the group $^2E_6(q)$. Here the Dynkin diagram automorphism ϑ interchanges α_1 with α_6 and α_3 with α_5. Here is the main result of this Subsection:

Proposition 5.19. *Let $\mathcal{O} \neq \{e\}$ be a unipotent class in ${}^2E_6(q)$. Then $\mathcal{O}$ is not kthulhu.*

We give the proof for q odd in §5.3.1 and for q even in §5.3.2. Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be the F-stable standard parabolic subgroups with standard Levi subgroups $\mathbb{L}_1$ and $\mathbb{L}_2$ associated with $\Pi_1 = \Delta - \{\alpha_2\}$ (of type A_5) and $\Pi_2 = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ (of type D_4). Then Ψ_{Π_1}, respectively Ψ_{Π_2}, consists of all positive roots containing α_2, respectively at least one of α_1 and α_6.

5.3.1. *Proof of Proposition 5.19, q odd.* Here Lemma 3.2 (c) applies softly to the F-stable parabolic subgroups.

Proof. By Lemma 3.2, Propositions 5.1 and 5.17, it is enough to show that $\mathcal{O} \cap \mathbb{U}^F \not\subset V_1 \cap V_2$. Let $\beta = \sum_{i=1}^4 \alpha_i$, $\gamma = \sum_{i=1}^6 \alpha_i$; thus $\vartheta(\beta) = \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6$. Let $u \in \mathcal{O} \cap \mathbb{U}^F$ lying in $V_1 \cap V_2$. Then

$$\operatorname{supp} u \subset \Psi_{\Pi_1} \cap \Psi_{\Pi_2} = \Psi(\beta) \cup \Psi(\vartheta(\beta)) = \Sigma \cup \vartheta(\Sigma) \cup \Psi(\gamma);$$

here $\Sigma = \{\beta_j \mid j \in \mathbb{I}_{0,3}\}$ and $\Psi(\gamma) = \{\gamma_j \mid j \in \mathbb{I}_{0,6}\}$, where

$$\begin{array}{llll} \beta_0 = \beta, & \beta_1 = s_5\beta_0; & \beta_2 = s_4\beta_1; & \beta_3 = s_3\beta_2; \\ \gamma_0 = \gamma, & \gamma_1 = s_4\gamma_0; & \gamma_2 = s_3\gamma_1; & \gamma_3 = s_5\gamma_1; \\ \gamma_4 = s_5\gamma_2 = s_3\gamma_3; & \gamma_5 = s_4\gamma_4; & \gamma_6 = s_2\gamma_5. & \end{array}$$

Let $\dot{w} \in N_{\mathbb{G}^F}(\mathbb{T})$ be a representative of $w \in W^F$. If either β_j or $\vartheta(\beta_j) \in \operatorname{supp} u$ for $j \in \mathbb{I}_{0,3}$, then $\dot{w}_j \triangleright u \in \mathcal{O} \cap \mathbb{U}^F - V_1$, where $w_0 = w_1 = s_2$, $w_2 = s_2s_4$, $w_3 = s_2s_4s_5s_3$. Thus we may assume that $\operatorname{supp} u \subset \Psi(\gamma)$.

If $\gamma_0 \in \operatorname{supp} u$, then $\dot{s}_2 \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$, $\gamma - \alpha_2 \in \operatorname{supp}(\dot{s}_2 \triangleright u) - \Psi_{\Pi_1}$. Now we argue inductively. Suppose that $\gamma_i \in \operatorname{supp} u$ for some $i \in \mathbb{I}_{0,j-1}$ implies that $\mathcal{O}$ is not kthulhu. Assume that $\gamma_i \notin \operatorname{supp} u$ for $i \in \mathbb{I}_{0,j-1}$ and $\gamma_j \in \operatorname{supp} u$. We claim that there is $\omega_j \in W^F$ with $\dot{\omega}_j \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ and either $\operatorname{supp}(\dot{\omega}_j \triangleright u) \not\subset \Psi(\gamma)$ (a case settled above), or $\gamma_l \in \operatorname{supp}(\dot{\omega}_j \triangleright u)$ for some $l \in \mathbb{I}_{0,j-1}$, where the recursive hypothesis applies. The claim holds, taking $\omega_1 = \omega_5 = s_4$, $\omega_2 = \omega_3 = s_1s_6$, $\omega_4 = s_3s_5$, $w_6 = s_2$. □

5.3.2. *Proof of Proposition 5.19, q even.* Here, the use of Lemma 3.2 is hampered by the presence of kthulhu classes in $\mathbf{PSU}_6(q)$.

Proof. As we have shown in the odd case, §5.3.1, there is $u \in \mathcal{O} \cap \mathbb{U}^F$ such that $u \notin V_1 \cap V_2$. If $u \notin V_2$, then the result follows from Proposition 5.17. Let us assume that $u \in V_2 - V_1$. In particular, α_3, α_4, α_5, $\alpha_3 + \alpha_4$, $\alpha_4 + \alpha_5$, $\alpha_3 + \alpha_4 + \alpha_5 \notin \operatorname{supp} u$. Then $\mathcal{O}_{\pi_1(u)}^{M_1}$ is non-trivial.

If $\operatorname{supp} u \cap \Phi_{\Pi_1} \neq \{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6\}$, then $\mathcal{O}^{M_1}_{\pi_1(u)} \simeq \mathcal{O}^{\mathbf{SU}_6(q)}_v$ where $\operatorname{rk}(v-\mathrm{id}) = 2$, hence its associated partition is not $(2,1,1,1,1)$. By Lemma 5.1, $\mathcal{O}^{M_1}_v$ and $\mathcal{O}$ are not kthulhu.

If $\operatorname{supp} u \cap \Phi_{\Pi_1} = \{\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6\}$, then $\dot{w} \triangleright u \in \mathcal{O} \cap \mathbb{U}^F$ where $w = s_1 s_6 \in W^F$. But $\alpha_3+\alpha_4+\alpha_5 \in \operatorname{supp}(\dot{w} \triangleright u)$, hence $\dot{w} \triangleright u \notin V_2$ and we are done. □

5.4. **Unipotent classes in $^3D_4(q)$.** We deal now with triality; F arises from the graph automorphism ϑ of order 3 determined by $\vartheta(\alpha_1) = \alpha_3$. We assume that $\mathbb{G} = \mathbb{G}_{\mathrm{sc}}$. We fix and ordering of the ϑ-orbits in Φ^+. Let

$$y_\alpha(\xi) := x_\alpha(\xi) x_{\vartheta\alpha}(\xi^q) x_{\vartheta^2\alpha}(\xi^{q^2}), \qquad \alpha \in \Phi, \vartheta(\alpha) \neq \alpha, \qquad \xi \in \mathbb{F}_{q^3}.$$

Every element in $\mathbb{U}^F$ can be uniquely written as a product of elements $y_\alpha(\xi)$, $\vartheta\alpha \neq \alpha$, $\xi \in \mathbb{F}_{q^3}$, and $x_\beta(\zeta)$, $\vartheta\beta = \beta$, $\zeta \in \mathbb{F}_q$. Let

$$\Upsilon = \langle x_{\pm\gamma}(\xi), y_{\pm\delta}(\xi) \mid \vartheta(\gamma) = \gamma, \vartheta(\delta) \neq \delta, \xi \in \mathbb{F}_q^\times \rangle \leq \mathbb{G}^F. \tag{5.3}$$

The generators in (5.3) are the non-trivial elements in the root subgroups with respect to $\mathbb{T}^F \cap \mathbb{T}^{\mathrm{Fr}_q}$. It is known that $\Upsilon \simeq G_2(q) \simeq G^{\mathrm{Fr}_q}$.

Proposition 5.20. *Every unipotent class $\mathcal{O} \neq \{e\}$ in $^3D_4(q)$ is not kthulhu.*

Proof. By the isogeny argument we work in $G = \mathbb{G}^F_{sc}$ [1, Lemma 1.2]. We analyse different cases separately, according to q being odd, even and > 2, or 2.

(i) *q is odd.*

The list of representatives of the unipotent classes in $^3D_4(q)$ appears in [10, Table 3.1]; they all have one of the following forms:

$$\begin{aligned} &x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1), && x_{\alpha_2}(1) y_{\alpha_1+\alpha_2+\alpha_3}(-1), && u = x_{\alpha_2}(1) y_{\alpha_1+\alpha_2+\alpha_3}(\zeta), \\ &y_{\alpha_1+\alpha_2+\alpha_3}(1), && y_{\alpha_1}(1) x_{\alpha_2}(1), && r = y_{\alpha_1}(1) y_{\alpha_1+\alpha_2}(a), \end{aligned}$$

where $\zeta \in \mathbb{F}_{q^3}$ is not a square and $a \in \mathbb{F}_{q^3} - \mathbb{F}_q$. So all classes but those of u and r have a representative in $\Upsilon \simeq G_2(q)$, hence they are not kthulhu by Lemma 4.6. Now $u \in H = \langle \mathbb{U}^F_{\pm\alpha_2}, y_{\pm(\alpha_1+\alpha_2+\alpha_3)}(b) \mid b \in \mathbb{F}_{q^3}^\times \rangle$, which is isogeneous to $\mathbf{SL}_2(q) \times \mathbf{SL}_2(q^3)$. Since $\mathcal{O}^H_u$ is the product of two non-trivial racks and $q^3 > 3$, $\mathcal{O}^H_u$ is of type D by Lemmata 2.1 and 3.5.

Assume that $r \in \mathcal{O}$. Let ξ be a generator of $\mathbb{F}_{q^3}^\times$,

$$\eta = \xi^{q-1}, \quad t = \alpha_1^\vee(\eta)\alpha_3^\vee(\eta^q)\alpha_4^\vee(\eta^{q^2}), \quad s = t \triangleright r = y_{\alpha_1}(\eta^2) y_{\alpha_1+\alpha_2}(a\eta^2).$$

By [10, Table 3.2], for every $b, c \in \mathbb{F}_{q^3}^{\times}$ we have

$$\begin{aligned} y_{\alpha_1+\alpha_2}(b)y_{\alpha_1}(c) = y_{\alpha_1}(c)&y_{\alpha_1+\alpha_2}(b)y_{\alpha_1+\alpha_2+\alpha_3}(bc^q+cb^q)\\ &\times x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(-(bc^{q^2+q}+b^qc^{q^2+1}+b^{q^2}c^{q+1}))\\ &\times x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(-(cb^{q^2+q}+c^qb^{q^2+1}+c^{q^2}b^{q+1})). \end{aligned} \tag{5.4}$$

Using (5.4) we verify that the coefficient of $y_{\alpha_1+\alpha_2+\alpha_3}$ in the expression of rs, respectively sr, equals $a\eta^{2q}+a^q\eta^2$, respectively $a^q\eta^{2q}+a\eta^2$. These coefficients are equal if and only if $(a^q-a)(\eta^{2q}-\eta^2)=0$. As $\eta^{2(q-1)}\neq 1$ and $a^q\neq a$, we have $rs\neq sr$, with $rs, sr\in\mathbb{U}^F$. Thus, $(sr)^2\neq(rs)^2$, as q is odd. Comparing the coefficients of x_{α_1} in the expressions of r and s as products of elements in root subgroups, we see that

$$\mathbb{U}^F \triangleright r \subset x_{\alpha_1}(1)\langle \mathbb{U}_\beta | \beta\in\Phi^+-\{\alpha_1\}\rangle,\ \mathbb{U}^F \triangleright s \subset x_{\alpha_1}(\eta^2)\langle \mathbb{U}_\beta | \beta\in\Phi^+-\{\alpha_1\}\rangle.$$

So $\mathcal{O}_r^{\langle r,s\rangle}\neq\mathcal{O}_s^{\langle r,s\rangle}$, whence $\mathcal{O}_r$ is of type D.

(ii) *$q>2$ is even.*

The list of representatives of the unipotent classes in G appears in [8], see [11, Table A2]. For suitable $\zeta,\zeta'\in\mathbb{F}_q$, the representatives are of the form

$$\begin{aligned} &u_1 = x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1), && u_2 = x_{\alpha_2}(1)x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(1),\\ &u_3 = y_{\alpha_1+\alpha_2+\alpha_3}(1), && u_4 = y_{\alpha_1+\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(1)x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(\zeta),\\ &u_5 = y_{\alpha_1}(1)x_{\alpha_2}(1), && u_6 = y_{\alpha_1}(1)x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(\zeta'),\\ &u_7 = y_{\alpha_1}(1)y_{\alpha_1+\alpha_2}(a), && a\in\mathbb{F}_{q^3}-\mathbb{F}_q. \end{aligned}$$

All classes except those like $\mathcal{O}_{u_7}$ are represented by $v\in\Upsilon\simeq G_2(q)$; thus, these are not kthulhu by Lemma 4.7. We deal with $\mathcal{O}_{u_7}$. Let $\gamma_j=\sum_{i=1}^j\alpha_i$ for shortness. We use (5.4) and the following relations from [10], cf. [11]:

$$\begin{aligned} y_{\alpha_1}(b)y_{\gamma_3}(c) &= y_{\gamma_3}(c)y_{\alpha_1}(b)x_{\gamma_4}(c^qb+c^{q^2}b^q+cb^{q^2})\\ y_{\gamma_2}(b)y_{\gamma_3}(c) &= y_{\gamma_3}(c)y_{\gamma_2}(b)x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(c^qb+c^{q^2}b^q+cb^{q^2}),\\ x_{\alpha_2}(d)x_{\gamma_4}(e) &= x_{\gamma_4}(e)x_{\alpha_2}(d)x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(de),\\ y_{\alpha_1}(b)x_{\alpha_2}(d) &= x_{\alpha_2}(d)y_{\alpha_1}(b)y_{\gamma_2}(bd)y_{\gamma_3}(db^{q+1})x_{\gamma_4}(db^{q^2+q+1}); \end{aligned}$$

here $b,c\in\mathbb{F}_{q^3}^{\times}$ and $d,e\in\mathbb{F}_q^{\times}$. Let $\mathbf{C}\leq\mathbb{F}_{q^3}^{\times}$ be the cyclic subgroup of order q^2+q+1 and $\mathbf{D}:=\mathbf{C}\cap\mathbb{F}_q^{\times}$, a cyclic group of order $(q-1,3)$. Thus $|\mathbf{C}/\mathbf{D}|\geq 4$. Let ξ_i, $i\in\mathbb{I}_4$, be representatives of 4 distinct cosets in $\mathbf{C}/\mathbf{D}$ and let

$$t_i := \alpha_1(\xi_i)\alpha_3(\xi_i^q)\alpha_4(\xi_i^{q^2}),\quad r_i := t_i\triangleright u_7 = y_{\alpha_1}(\xi_i^2)y_{\alpha_1+\alpha_2}(a\xi_i^2)\in\mathcal{O}_r\cap\mathbb{U}^F.$$

Since $\mathbb{U}^F \triangleright r_i \subset y_{\alpha_1}(\xi_i^2)\langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \Delta\rangle$, we have $\mathcal{O}_{r_i}^{\langle r_1,r_2,r_3,r_4\rangle} \neq \mathcal{O}_{r_j}^{\langle r_1,r_2,r_3,r_4\rangle}$ for $i \neq j$. In addition by (5.4) we see that

$$r_i r_j \in y_{\alpha_1}(\xi_i^2+\xi_j^2) y_{\gamma_2}(a(\xi_i^2+\xi_j^2)) y_{\gamma_3}(a\xi_i^2\xi_j^{2q} + a^q\xi_i^{2q}\xi_j^2)\, \mathbb{U}_{\gamma_4}^F \mathbb{U}_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}^F$$

The coefficients of y_{γ_3} in the expressions of $r_i r_j$ and $r_j r_i$ are equal iff $(a+a^q)(\xi_i^2\xi_j^{2q} + \xi_j\xi_i^{2q}) = 0$, iff $(\xi_i\xi_j^{-1})^{2(q-1)} = 1$ (since $a \notin \mathbb{F}_q$), iff $i = j$ by our choice of the ξ_i's. Hence, $r_i \triangleright r_j \neq r_j$ for $i \neq j$ and $\mathcal{O}_{u_7}$ is of type F.

(iii) $q = 2$.

The description of the representatives is the same as in (ii) with $\zeta = 0$ and $\zeta' = 1$, see [11, §3], so that

$$u_4 = y_{\alpha_1+\alpha_2}(1) y_{\alpha_1+\alpha_2+\alpha_3}(1), \qquad u_6 = y_{\alpha_1}(1) x_{\alpha_2}(1) y_{\alpha_1+\alpha_2+\alpha_3}(1).$$

We do not have information on the unipotent classes of $G_2(2)$ yet, so we have to argue differently. However, the argument for u_7 is exactly as for $q > 2$. Now $u_1 \in \langle \mathbb{U}_{\pm\alpha_2}^F, \mathbb{U}_{\pm(\alpha_1+2\alpha_2+\alpha_3+\alpha_4)}^F \rangle$, a subgroup of type A_2, but it is not regular there. Hence $\mathcal{O}_{u_1}$ is of not kthulhu by Theorem 3.3 and [16, Theorem 24.15].

By [11, Tables A.8], we have $r := y_{\alpha_1}(1) y_{\gamma_3}(1) x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1) \in \mathcal{O}_{u_2}$. Let $\xi \in \mathbb{F}_8^\times$ such that $\xi^3 = \xi + 1$. Then the roots in $\mathbb{F}_8^\times$ of the polynomial $X^4 + X^2 + X$ are ξ, ξ^2 and ξ^4. Their inverses, together with 1, are the roots of the polynomial $X^4 + X^2 + X + 1$. Let $\mathbb{P}_1$ be the parabolic subgroup with standard Levi subgroup associated with $\{\alpha_1, \alpha_3, \alpha_4\}$, and, for $i \in \mathbb{I}_4$, let

$$\begin{aligned} t_i &:= \alpha_1^\vee(\xi^i)\alpha_3^\vee(\xi^{2i})\alpha_4^\vee(\xi^{4i}), \\ r_i &:= t_i \triangleright r = y_{\alpha_1}(\xi^{2i}) y_{\gamma_3}(\xi^{6i}) x_{\alpha_1+2\alpha_2+\alpha_3+\alpha_4}(1) \in \mathcal{O}_{u_2}, \text{ so} \\ \mathbb{U}^F &\triangleright r_a \subset y_{\alpha_1}(\xi^{2i}) V_1. \end{aligned}$$

Then, $\mathcal{O}_{r_i}^{\langle r_1,r_2,r_3,r_4\rangle} \neq \mathcal{O}_{r_j}^{\langle r_1,r_2,r_3,r_4\rangle}$ for $i \neq j$. In addition,

$$r_i r_j = y_{\alpha_1}(\xi^{2i} + \xi^{2j}) y_{\gamma_3}(\xi^{6i} + \xi^{6j}) x_{\gamma_4}(\xi^{4(j-i)} + \xi^{2(j-i)} + \xi^{j-i}).$$

Let $i \neq j$. The coefficient of x_{γ_4} in the expression of $r_i r_j$ is 0 if and only if $\xi^{j-i} \in \{\xi, \xi^2, \xi^4\}$ if and only if the coefficient of x_{γ_4} in the expression of $r_j r_i$ is 1. Thus, $\mathcal{O}_{u_2}$ is of type F.

Let now $r_1 = u_3$. Let σ and τ in Υ be representatives of $s_1s_3s_4, s_2 \in W^F$, respectively. Let $\mathbb{P}_2$ be the F-stable parabolic subgroup with standard Levi subgroup associated with α_2. We consider the following elements in $\mathcal{O} \cap V_2$:

$$\begin{aligned} r_2 &= \sigma \triangleright r_1 = y_{\alpha_1+\alpha_2}(1), \qquad r_3 = \tau \triangleright r_2 = y_{\alpha_1}(1) \\ r_4 &= x_{\alpha_2}(1) \triangleright r_3 = y_{\alpha_1}(1) y_{\alpha_1+\alpha_2}(1) y_{\alpha_1+\alpha_2+\alpha_3}(1) x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}(1). \end{aligned}$$

Let $Z = \langle \mathbb{U}_\gamma \mid \gamma \in \Phi^+ - \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}\rangle$. Then

$$V_2 \triangleright r_1 \subset y_{\alpha_1+\alpha_2+\alpha_3}(1)Z, \qquad V_2 \triangleright r_2 \subset y_{\alpha_1+\alpha_2}(1)Z,$$
$$V_2 \triangleright r_3 \subset y_{\alpha_1}(1)Z, \qquad V_2 \triangleright r_4 \subset y_{\alpha_1}(1)y_{\alpha_1+\alpha_2}(1)Z.$$

Hence, the classes $\mathcal{O}_{r_i}^{\langle r_1,r_2,r_3,r_4\rangle}$ for $i \in \mathbb{I}_4$ are disjoint. A direct computation shows that $r_ir_j \neq r_jr_i$ for $i \neq j$, so $\mathcal{O}_{u_3}$ is of type F.

We deal now with u_4. Let ξ, $\mathbb{P}_1$ and $\mathbb{P}_2$ be as above and let

$$t_1 := \alpha_1^\vee(\xi^3)\alpha_3^\vee(\xi^6)\alpha_4^\vee(\xi^5), \qquad t_2 := \alpha_1^\vee(\xi)\alpha_3^\vee(\xi^2)\alpha_4^\vee(\xi^4),$$
$$r_1 := t_1 \triangleright u_4 = y_{\gamma_2}(\xi^6)y_{\gamma_3}(\xi^4), \qquad r_2 := x_{\alpha_2}(1)y_{\gamma_3}(1)y_{\gamma_4}(1),$$
$$r_3 := y_{\alpha_1}(1)y_{\gamma_3}(1), \qquad r_4 := t_2 \triangleright r_3 = y_{\alpha_1}(\xi^2)y_{\gamma_3}(\xi^{-1}).$$

Then, $r_i \in \mathcal{O}_{u_4} \cap \mathbb{U}^F$, [11, Tables A.2, A.4, A.8, A.12]. In addition,

$$\mathbb{U}^F \triangleright r_1 \subset V_1 \cap V_2, \qquad \mathbb{U}^F \triangleright r_2 \subset x_{\alpha_2}(1)V_1 \cap V_2,$$
$$\mathbb{U}^F \triangleright r_3 \subset y_{\alpha_1}(1)V_1 \cap V_2, \qquad \mathbb{U}^F \triangleright r_4 \subset y_{\alpha_1}(\xi^2)V_1 \cap V_2.$$

Hence, for $H = \langle r_i \mid i \in \mathbb{I}_4\rangle$ we have $\mathcal{O}_{r_i}^H \neq \mathcal{O}_{r_j}^H$ for $i, j \in \mathbb{I}_4$, with $i \neq j$. A direct computation shows that $r_ir_j \neq r_jr_i$, for $i \neq j$, so $\mathcal{O}_{u_4}$ is of type F.

Finally, we treat simultaneously the classes of u_5 and u_6, that are of the form $x = y_{\alpha_1}(1)x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(\epsilon)$ with $\epsilon \in \{0, 1\}$ respectively. Let $\mathbf{C}$ be as in the odd case and let $(\xi_i)_{i\in\mathbb{I}_4}$ be a family of distinct elements in $\mathbf{C}$. Set

$$t_i := \alpha_1^\vee(\xi_i)\alpha_3^\vee(\xi_i^q)\alpha_4^\vee(\xi_i^{q^2}),$$
$$r_i := t_i \triangleright x = y_{\alpha_1}(\xi_i^2)x_{\alpha_2}(1)y_{\alpha_1+\alpha_2+\alpha_3}(\epsilon\xi_i^{1+q-q^2}) \in \mathcal{O}_x \cap \mathbb{U}^F.$$

Let $Q = \langle r_1, r_2, r_3, r_4\rangle$. Since $\mathbb{U}^F \triangleright r_i \subset y_{\alpha_1}(\xi_i^2)x_{\alpha_2}(1)\langle \mathbb{U}_\gamma |\ \gamma \in \Phi^+ - \Delta\rangle$, we have $\mathcal{O}_{r_i}^Q \neq \mathcal{O}_{r_j}^Q$ for $i \neq j$. The coefficient of $y_{\alpha_1+\alpha_2}$ in the expression of r_ir_j equals ξ_i^2, hence $r_ir_j \neq r_jr_i$ for $i \neq j$. Hence $\mathcal{O}_{u_5}$ and $\mathcal{O}_{u_6}$ are of type F. □

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N. A. : FaMAF, Universidad Nacional de Córdoba. CIEM – CONICET. Medina Allende s/n (5000) Ciudad Universitaria, Córdoba, Argentina

E-mail address: `andrus@famaf.unc.edu.ar`

G. C.: Dipartimento di Matematica, Università degli Studi di Padova, via Trieste 63, 35121 Padova, Italia

E-mail address: carnoval@math.unipd.it

G. A. G.: Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata. CONICET. C. C. 172, (1900) La Plata, Argentina.

E-mail address: ggarcia@mate.unlp.edu.ar