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Weighted Multifractal Spectrum of V-Statistics

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Abstract
We analyze and describe the weighted multifractal spectrum of V-statistics. The description will be possible when the condition of "weighted saturation" is fulfilled. This means that the weighted topological entropy of the set of generic points of measure μ equals the measure-theoretic entropy of μ. Zhao et al. (J Dyn Differ Equ 30:937–955, 2018) proved that for any ergodic measure weighted saturation is verified, generalizing a result of Bowen. Here we prove that under a property of "weighted specification" the saturation holds for any measure. From this we obtain the description of the spectrum of V-statistics. This generalizes the variational result that Fan, Schmeling and Wu obtained for the non-weighted case (arXiv:1206.3214v1, 2012).

Keywords V-statistics · Weighted multifractal spectrum · Weighted saturation · Weighted specification

Mathematics Subject Classification 37B40, 37C45

1 Introduction

The multiple ergodic averages can be seen as a dynamical version of the Szemeredi theorem in combinatorial number theory. This kind of interplay was studied by Furstenberg [9]. He analyzed ergodic averages in a measure-preserving probability space $(X, \mathcal{B}, \mu, f)$ of the form

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$$\frac{1}{N-M}\sum_{n=M}^{N-1}\mu\left(A\cap f^{n}A\cap\cdots\cap f^{kn}A\right), \tag{1.1}$$

where $A\in\mathcal{B}$ and $j\in\mathbf{N}$. Furstenberg proved that if $\mu(A)>0$ then

$$\liminf_{N\to\infty}\frac{1}{N-M}\sum_{n=M}^{N-1}\mu\left(A\cap f^{n}A\cap\cdots\cap f^{jn}A\right)>0.$$

From this can be proved, by arguments of Ergodic Theory, the Szemeredi theorem which in short says that if S is a set of integers with positive upper density then S contains arithmetic progressions of arbitrary length.

The V-statistics, thus called after the article by Fan et al. [5], are multiergodic averages of the following form: let (X, f) be a topological dynamical system with X a compact metric space and f a continuous map, let $X^{r}=X\times\cdots\times X$ be the product of r-copies of X with $r\geq 1$. If $\Phi: X^{r}\to\mathbf{R}$ is a continuous map, then we can define

$$V_{\Phi}(n,x)=\frac{1}{n^{r}}\sum_{1\leq i_{1},\ldots,i_{r}\leq n}\Phi\left(f^{i_{1}}(x),\ldots,f^{i_{r}}(x)\right). \tag{1.2}$$

These averages are called the *V-statistics of order r with kernel Φ*.

Ergodic limits of the form

$$\lim_{n\to\infty}\frac{1}{n^{r}}\sum_{1\leq i_{1},\ldots,i_{r}\leq n}\Phi\left(f^{i_{1}}(x),\ldots,f^{i_{r}}(x)\right),$$

were studied among others by Furstenberg [9], Bergelson [2] and Bourgain [3].

The multifractal decomposition for the spectra of V-statistics is

$$E_{\Phi}(\alpha)=\left\{x:\lim_{n\to\infty}V_{\Phi}(n,x)=\alpha\right\}.$$

Hereafter (X_{i}, d_{i}, f_{i}), $i=1,2,\ldots,k$, with $k\geq 2$, will denote a finite family of dynamical systems with each (X_{i}, d_{i}) a compact metric space and $f_{i}: X_{i}\to X_{i}$ a continuous map. The family of dynamical systems are considered such that each (X_{i+1}, f_{i+1}) is a factor of (X_{i}, f_{i}). The factor map is defined $\pi_{i}: X_{i}\to X_{i+1}$ so $f_{i+1}\circ\pi_{i}=\pi_{i}\circ f_{i}$, $i=1,2,\ldots,k$ and allows to define composition maps $\tau_{i}: X_{1}\to X_{i+1}$, by $\tau_{i}=\pi_{i}\circ\cdots\circ\pi_{1}$.

Let $\mathbf{a}=(a_{1},\ldots,a_{k})\in\mathbf{R}^{k}$ and let $\Phi_{1},\Phi_{2}\ldots,\Phi_{k}\in C\left(X_{1}^{r}\right)$, The **a**-*weighted V-statistics of order r with kernel* $\Phi_{1},\Phi_{2}\ldots,\Phi_{k}$ are defined as

$$V^{\mathbf{a}}_{\{\Phi_{1},\Phi_{2}\ldots,\Phi_{k}\}}(n,x)=\sum_{j=1}^{k}\frac{1}{\left(s_{j}(n)\right)^{r}}V_{\Phi_{j}}\left(s_{j}(n),x\right). \tag{1.3}$$

with $s_{j}(n)=\left\lfloor\left(a_{1}+\cdots+a_{j}\right)n\right\rfloor$ where $\lfloor z\rfloor$ denotes the largest integer $\leq z$ (floor function).

The **a**-*weighted multifractal decomposition* can be defined as

$$K^{\mathbf{a}}_{\{\Phi_{1},\Phi_{2}\ldots,\Phi_{k}\},\alpha}=\left\{x\in X_{1}:\lim_{n\to\infty}V^{\mathbf{a}}_{\{\Phi_{k}\}}(n,x)=\alpha\right\}. \tag{1.4}$$

Now we recall the definition of **a**-weighted measure-theoretic entropy and **a**-weighted topological entropy. Let (X_{i}, d_{i}, f_{i}) be a finite family of dynamical systems like above. If $\mu\in\mathcal{M}(X_{1}, f_{1})$ (where $\mathcal{M}(X_{1}, f_{1})$ is the set of all f_{1}-invariant measures) then let $(\tau_{i-1})_{*}(\mu)$ be the push-forward of the measure μ, i.e. $(\tau_{i-1})_{*}(\mu)(E)=\mu\left(\tau_{i-1}^{-1}(E)\right)$ for any $E\subset X_{i}$.

Definition 1 The **a**-*weighted measure-theoretic entropy* of μ with respect to (X_1, f_1) is

$$h_{\mu}^{\mathbf{a}}(f_1) = \sum_{i=1}^{k} a_i h_{(\tau_{i-1})_*(\mu)}(f_i), \tag{1.5}$$

where $h_{(\tau_{i-1})_*(\mu)}(f_i)$ is the usual measure-theoretic entropy of $(\tau_{i-1})_*(\mu)$ with respect to (X_i, f_i).

In X_1 we consider, for $\varepsilon > 0$, $n \in \mathbf{N}$, the following **a**-metric:

$$d_n^{\mathbf{a}}(x, y) = \max_{i=1,2,..,k} \left\{ d_{i,t_i(n)}(\tau_{i-1}(x), \tau_{i-1}(y)) \right\},$$

where $d_{i,t_i(n)}$ is the metric in X_i given by

$$d_{i,t_i(n)}(\tau_{i-1}(x), \tau_{i-1}(y)) = \max_{j=0,1,,...,t_i(n)-1} \left\{ d_i\left(f_i^j(\tau_{i-1}(x)),\ f_i^j(\tau_{i-1}(y)) \right) \right\}.$$

with $t_j(n) = \lceil (a_1 + \cdots + a_j) n \rceil$; here $\lceil z \rceil$ denotes the smallest integer $\geq z$ (ceiling function).

The ball $B_{n,\varepsilon}^{\mathbf{a}}(x)$, with centre x and radius ε in the $d_n^{\mathbf{a}}$-metric is called the **a**-*weighted Bowen ball.*

Definition 2 For $\varepsilon > 0$ and $n_j \in \mathbf{N}$ let

$$T_{n_j,\varepsilon}^{\mathbf{a}} = \left\{ A_j \subset X_1 : A_j \subset B_{n_j,\varepsilon}^{\mathbf{a}}(x),\ \text{for some } x \in X_1 \right\}$$

and define

$$\Lambda^{\mathbf{a}}(Z, \varepsilon, s, N) = \inf \left\{ \sum_j \exp(-s n_j) \right\}$$

where $Z \subset X_1$, $N \in \mathbf{N}$, $s \geq 0$ and the infimum is taken over the whole collection of sets

$$\left\{ (n_j, A_j) : n_j \geq N,\ A_j \in T_{n_j,\varepsilon}^{\mathbf{a}} \right\}$$

for which $\bigcup_j A_j \supset Z$.

The limit

$$\Lambda^{\mathbf{a}}(Z, s, \varepsilon) = \lim_{N\to\infty} \Lambda^{\mathbf{a}}(Z, s, N, \varepsilon),$$

does exist since $\Lambda^{\mathbf{a}}(Z, s, N, \varepsilon)$ is not increasing with respect to N.

There is a number $\overline{s}$ such that $\Lambda^{\mathbf{a}}(Z, s, \varepsilon)$ jumps from $+\infty$ to 0. Define

$$h^{\mathbf{a}}(Z, \varepsilon) = \overline{s} = \sup \left\{ s :\ \Lambda^{\mathbf{a}}(Z, s, \varepsilon) = +\infty \right\} = \inf \left\{ s : \Lambda^{\mathbf{a}}(Z, s, \varepsilon) = 0 \right\}.$$

The value

$$h^{\mathbf{a}}(Z) = \lim_{\varepsilon\to 0} h^{\mathbf{a}}(Z, \varepsilon),$$

which exists since $h^{\mathbf{a}}(Z, \varepsilon)$ is not decreasing with respect to ε, is the **a**-*Bowen weighted topological entropy of* Z.

Definition 3 Let (X_i, d_i, f_i), $i = 1, 2, \ldots, k$, be dynamical systems. By $\mathcal{E}_n(x)$, $x \in X_1$we denote the sequence of measures

$$\mathcal{E}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f_1^i(x)} \in \mathcal{M}(X_1, f_1),$$

where δ is the point mass measure. If $V(x)$ denotes the set of weak limits measures of the sequence $\{\mathcal{E}_n(x)\}$ then the set of generic points of a measure $\mu \in \mathcal{M}(X_1, f_1)$ is the set

$$G(\mu) = \{x \in X_1 : V(x) = \{\mu\}\}$$

Since X_1 is compact then $V(x) \neq \varnothing$ and if μ is ergodic then $\mu(G(\mu)) = 1$.

Definition 4 **A** finite family of dynamical systems (X_i, d_i, f_i) is **a**-*saturated* if $h_\mu^{\mathbf{a}}(f) = h^{\mathbf{a}}(G(\mu))$ for any $\mu \in \mathcal{M}(X_1, f_1)$.

In [17] Zhao, Chen, Zhou and Yin proved that if (X_i, d_i, f_i) is a finite family of dynamical system, then $h_\mu^{\mathbf{a}}(f_1) = h^{\mathbf{a}}(G(\mu))$ for any ergodic measure $\mu \in \mathcal{M}(X_1, f_1)$. This generalizes a Bowen theorem in [4] for the non-weighted case.

The main result to be proved is

Theorem 1.1 *Let (X_i, d_i, f_i), $i = 1, 2, \ldots, k$, with $k \geq 2$, be a finite family of dynamical systems like above, let $\Phi_1, \Phi_2 \ldots, \Phi_k \in C(X_1^r)$, $r \geq 1$. If the* **a**-*saturation property is verified then*

$$h^{\mathbf{a}}(K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha}) = \sup\left\{h_\mu^{\mathbf{a}}(f_1) : \mu \in \mathcal{M}(X_1, f_1) \text{ and } \sum_{j=1}^{k}\int_{X_1^r}\Phi_j d\mu^{\otimes r} = \alpha\right\},$$

where $\mu^{\otimes r}$ means $\mu \times \ldots \times \mu$, r-times.

Fan et al. [5] have obtained this variational principle for saturated dynamical systems in the non-weighted case i.e. $\mathbf{a} = (1, 0, \ldots, 0)$. This generalizes in turn the variational principle established by Takens and Verbitski for $r = 1$ [14]. Fan et al. [6] proved that saturatedness is verified for dynamical systems with the specification property. Thus, to have a condition for fulfilling the hypothesis of the theorem 1.1, we consider a notion of weighted specification. The definition of weighted specification will be given in the next section. Finally we point out that a weighed variational principle for $r = 1$, was presented in [1], the description is for the dimension spectrum and for shift spaces with specification, the saturatedness is not used in that article, in which besides is developed a weighted thermodynamic formalism. In [8] is established a variational principle for $\mathbf{a} = (a_1, a_2) \in \mathbf{R}^2$.

Theorem 1.2 *Let (X_i, d_i, f_i), $i = 1, 2, \ldots, k$, with $k \geq 2$, be a finite family of dynamical systems satisfying* **a**-*specification then $h_\mu^{\mathbf{a}}(f_1) = h^{\mathbf{a}}(G(\mu))$.*

In fact in [17] was proved that $h_\mu^{\mathbf{a}}(f_1) \geq h^{\mathbf{a}}(G(\mu))$ for any invariant measure, and that the reverse is valid for any ergodic measure μ. Therefore we must prove that $h_\mu^{\mathbf{a}}(f_1) \leq h^{\mathbf{a}}(G(\mu))$ for any $\mu \in \mathcal{M}(X_1, f_1)$.

For non-weighted V-statistics we studied [12] the *irregular part* of the spectrum, or *historic set*, say the set of points x for which $\lim_{n\to\infty} V_\Phi(n, x)$ does not exist. We also have analyzed the saturatedness, and consequently the validity of the variational principle, under a weak form of the specification property, known as *non-uniform specification* condition. This concept was introduced by Varandas [15] and is satisfied, for instance, by non-uniformly quadratic maps and for the so called Viana maps, which are a robust class of multidimensional non-uniformly hyperbolic functions [15]. So we think that the condition of weighted specification may be awakened to obtain the weighted versions of saturatedness and of the variational principle.

2 Proof of the Theorem 1.2

To prove theorem 1.2 we follow a similar scheme that [6], we begin by extending a result of Katok [10] which gives a formula for the entropy of ergodic measures by mean of a counting of dynamical balls needed to covering the space. Next we use an argument of box-counting for the set of generic point like in [6] which is based on ideas of [14].

We have a weighted version of the Shannon-Mcmillan theorem [8]. Before stating it recall some notation. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ be a measurable partition of a measure space X, by $\mathcal{A}^n = \mathcal{A}^n(X, f)$ is denoted the partition by "names" of length n, the name of a point x is the string $(\ell_0, \ldots, \ell_{n-1})$ such that $x \in A_{\ell_0}, f(x) \in A_{\ell_1,\ldots,} f^{n-1}(x) \in A_{\ell_{n-1}}$. The members of the partition $\mathcal{A}^n$ is formed are the sets with the same name. By $\mathcal{A}^n(x)$ is denoted the member of $\mathcal{A}^n$ containing x. The quantity of information of the partition $\mathcal{A}$ with respect to the measure μ is $H_\mu(\mathcal{A}) = -\sum_{j=1}^{m} \mu(A_j) \log \mu(A_j)$. Finally if $\mathcal{A}, \mathcal{B}$ are elements in a σ-algebra of X then $\mathcal{A} \bigvee \mathcal{B} = \{A \cap B : A \in \mathcal{A},\ B \in \mathcal{B}\}$.

Theorem 2.1 (Weighed Shannon-Mcmillan theorem) [8] *Let (X, f) be a dynamical system, and μ an ergodic element of $\mathcal{M}(X, f)$. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be measurable partition of X such that $H_\mu(\mathcal{A}_i) < \infty$ is finite for each $i = 1, 2, \ldots, k$. If $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbf{R}^k$ then*

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu\left(\bigvee_{i=1}^{k} \mathcal{A}_i^{\lceil (a_1,\ldots,a_i)n\rceil - 1}(x)\right) = \sum_{i=1}^{k} a_i h_\mu\left(f, \bigvee_{j=i}^{k} \mathcal{A}_j\right). \tag{2.1}$$

Proposition 2.2 *Let (X_i, d_i, f_i), $i = 1, 2, \ldots, k$, with $k \geq 2$, be a finite family of dynamical systems, let μ be a probability ergodic f_1-invariant measure on X_1. For $\varepsilon, \delta > 0$, let $r_n^{\mathbf{a}}(\mu, \varepsilon, \delta)$ be the minimal number of balls $B_{n,\varepsilon}^{\mathbf{a}}$ whose union has μ-measure $> 1 - \delta$. Then, for each $\delta > 0$, is valid*

$$h_\mu^{\mathbf{a}}(f_1) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log r_n^{\mathbf{a}}(\mu, \varepsilon, \delta). \tag{2.2}$$

The case $\mathbf{a} = (1, 0, \ldots, 0)$ is a result due to Katok [10].

Proof Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be measurable partitions of $X_1, X_2, \ldots, X_k$ respectively, with $H_{(\tau_{i-1})_*(\mu)}(\mathcal{A}_i) < \infty, i = 1, 2, \ldots, k$. For $\varepsilon > 0$, let us choose partition with $diam \mathcal{A}_i < \varepsilon/2$ and such that any $\bigvee_{i=1}^{k} \mathcal{A}_i^{\lceil (a_1,\ldots,a_i)n\rceil - 1}$ be contained in a ball in the metric $d_{i,\lceil (a_1,\ldots,a_i)n\rceil}$. For $\varepsilon, \delta > 0$ let us consider the set

$$\begin{aligned} C_{n,\varepsilon,\delta}^{\mathbf{a}} = \Bigg\{ x : \mu\left(\bigvee_{i=1}^{k} \tau_{i-1}^{-1}\left(\mathcal{A}_i^{\lceil (a_1,\ldots,a_i)n\rceil - 1}\right)(x)\right) \\ \geq \exp\left[-n\left(\sum_{i=1}^{k} a_i h_\mu\left(f_1, \bigvee_{j=i}^{k} \tau_{j-1}^{-1}(\mathcal{A}_j)\right)\right) + \delta\right]\Bigg\}. \end{aligned} \tag{2.3}$$

By the weighted Shannon-Mcmillan theorem (recall that μ is ergodic) holds $\mu\left(C^{\mathbf{a}}_{n,\varepsilon,\delta}\right) \to 1$, as $n \to \infty$ and for any $\delta > 0$. So that for enough large n we have $\mu\left(C^{\mathbf{a}}_{n,\varepsilon,\delta}\right) > 1-\delta$. By the election of the partitions, the set $C^{\mathbf{a}}_{n,\varepsilon,\delta}$ contains at most $\exp\left[-n\left(\sum_{i=1}^{k} a_i h_\mu\left(f_1, \bigvee_{j=i}^{k} \tau_{j-1}^{-1}\left(\mathcal{A}_j\right)\right)+\delta\right)\right]$ elements of the partition $\bigvee_{i=1}^{k} \tau_{j-1}^{-1}\left(\mathcal{A}_i^{\lceil (a_1,\ldots,a_i)n\rceil -1}\right)$ and can be covered by this number of balls in the metric $d_{i,\lceil (a_1,\ldots,a_i)n\rceil}$. Therefore

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\log r^{\mathbf{a}}_n(\mu,\varepsilon,\delta) \le \sum_{i=1}^{k} a_i h_\mu\left(f_1, \bigvee_{j=i}^{k}\tau_{j-1}^{-1}\left(\mathcal{A}_j\right)\right)+\delta$$

$$\le \sum_{i=1}^{k} h_{(\tau_{i-1})_*(\mu)}\left(f_i,\mathcal{A}_i\right)+\delta \le \sum_{i=1}^{k} h_{(\tau_{i-1})_*(\mu)}\left(f_i\right)+\delta = h^{\mathbf{a}}_\mu\left(f_1\right)+\delta.$$

Since δ is arbitrary small we have

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\log r^{\mathbf{a}}_n(\mu,\varepsilon,\delta) \le h^{\mathbf{a}}_\mu\left(f_1\right).$$

To prove the opposite inequality we begin considering the symbolic spaces

$$\Sigma_N = \left\{x = (x_i)_{i\in\mathbf{N}},\ x_i \in \{1,\ldots,N\}\right\}$$

and

$$\Sigma_{n,N} = \left\{x = (x_i)_{i\in\{1,\ldots,n\}},\ x_i \in \{1,\ldots,N\}\right\}.$$

Recall the definition of the Hamming metric in $\Sigma_{n,N}$,

$$\rho^H_{n,N}(x,\overline{x}) = \frac{1}{n}\sum_{i=0}^{n-1}\left(1-\delta_{x_{ii},\overline{x_i}}\right). \tag{2.4}$$

For $x \in \Sigma_{n,N}$ denote by $B^H_r(x)$ the ball of radius r centered in x in the Hamming metric. Let $B(r,N,n) = card\ B^H_r(x)$, this value depends only on r, n and N, and holds [10]

$$B(r,N,n) = \sum_{m=0}^{[nr]}(N-1)^m\binom{m}{n},$$

so by the Stirling formula

$$\lim_{n\to\infty}\frac{1}{n}\log B(r,N,n) = r\log(N-1) - r\log r - (1-r)\log(1-r). \tag{2.5}$$

Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k$ be finite partitions of $X_1, X_2, \ldots, X_k$ respectively, with the notation $\mathcal{A}_i = \left\{A^i_1, A^i_2, \ldots, A^i_N\right\}$, with $\mu\left(\tau_{i-1}^{-1}\left(\partial\mathcal{A}_i\right)\right) = 0$, $i = 1,2,\ldots,k$. Let $x \in X_1$, so $\tau_{i-1}(x) \in X_i$, the name of $\tau_{i-1}(x)$ with respect to the partition $\mathcal{A}_i$ and the map f_i of length $t_i(n) := \lceil (a_1,\ldots,a_i)\,n\rceil$ will be the string $L_{\mathbf{a},i}\left(\tau_{i-1}(x)\right) = \left(\ell_0,\ldots,\ell_{t_i(n)-1}\right)$ such that $f_i^j\left(\tau_{i-1}(x)\right) \in A^i_{\ell_j}$, $j = 0,1,\ldots,t_i(n)-1$. Thus we can define an application $x \mapsto L_{\mathbf{a},i}\left(\tau_{i-1}(x)\right)$ and consider the semi-metric in each X_i given by

$$D^{\mathbf{a}}_{n,N,i}\left(\tau_{i-1}(x), \tau_{i-1}(y)\right) = \rho^H_{n,N} L_{\mathbf{a},i}\left(\tau_{i-1}(x)\right), L_{\mathbf{a},i}\left(\tau_{i-1}(y)\right)), \tag{2.6}$$

and for $x, y \in X_1$ set

$$D^{\mathbf{a}}_{n,N} = D^{\mathbf{a}}_{n,N,\{\mathcal{A}_i\}}(x, y) = \max_{i=1,2,\ldots,k} \left\{ D^{\mathbf{a}}_{n,N,i}\left(\tau_{i-1}(x), \tau_{i-1}(y)\right) \right\}. \tag{2.7}$$

For any $\mu \in \mathcal{M}(X_1, f_1)$, it may be assumed that $(\tau_{i-1})_*(\mu)$ is such that $(\tau_{i-1})_*(\mu)(E) = \mu\left(\tau_{i-1}^{-1}(E)\right) > 0$ for any non-empty $E \subset X_i$. For each partition $\mathcal{A}_i$ its boundary is defined as $\partial\mathcal{A}_i = \bigcup_j \partial A^i_j$. Let $\gamma > 0$ and let, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, N$, be

$$U_{\gamma,i}\left(A^i_j\right) = \left\{ x \in \tau_{i-1}^{-1}\left(A^i_j\right) : \text{ there is a } y \in X_1 - \tau_{i-1}^{-1}\left(A^i_j\right) : d_i\left(\tau_{i-1}(x), \tau_{i-1}(y)\right) < \gamma \right\}$$

and

$$U_{\gamma,i}(\mathcal{A}_i) = \bigcup_{j=1}^{N} U_{\gamma,i}\left(A^i_j\right). \tag{2.8}$$

It holds

$$\bigcap_{\gamma>0} U_{\gamma,i}(\mathcal{A}_i) = \partial\mathcal{A}_i$$

and

$$\lim_{\gamma\to 0} \mu\left(\left(U_{\gamma,i}\right)\right) = \mu\left(\partial\mathcal{A}_i\right).$$

Let $\varepsilon > 0$, there is a $\gamma \in (0, \varepsilon)$ such that $\mu\left(U_{\gamma.i}((\mathcal{A}_i)\right) < \varepsilon^2/4$. Define

$$V^{\mathbf{a}}_{n,,\varepsilon} = \left\{ x \in X_1 : \frac{1}{s_i(n)} \sum_{j=0}^{t_i(n)-1} I_{U_{\gamma,i}(\mathcal{A}_i)}\left(f_i^j\left(\tau_{i-1}(x)\right)\right) < \varepsilon/2,\ i = 1, 2, \ldots, k. \right\},$$

with I_E the characteristic function of the set E.

We have $\mu\left(X_1 - V^{\mathbf{a}}_{n,,\varepsilon}\right) < \varepsilon/2$. If $x, y \in X_1$ with $d_{i,t_i(n)}\left(\tau_{i-1}(x), \tau_{i-1}(y)\right) < \gamma$, $i = 1, 2, \ldots, k$ then for any $j = 0, 1, .., t_i(n) - 1$ the points $f_i^j\left(\tau_{i-1}(x)\right)$ and $f_i^j\left(\tau_{i-1}(y)\right)$ belong to the same member of $\mathcal{A}_i$ or are in $U_{\gamma.i}((\mathcal{A}_i)$. If $x \in V^{\mathbf{a}}_{n,,\varepsilon}$ and y is such that $d_{i,t_i(n)}\left(\tau_{i-1}(x), \tau_{i-1}(y)\right) < \gamma$, $i = 1, 2, \ldots, k$ then $D^{\mathbf{a}}_{n,N,i}\left(\tau_{i-1}(x), \tau_{i-1}(y)\right) < \varepsilon/2, i = 1, 2, \ldots, k$. So that if $B^{\mathbf{a}}_{n,,\varepsilon}$ is a ball of radius γ in the metric $d^{\mathbf{a}}_{n,}$ then $B^{\mathbf{a}}_{n,,\varepsilon}$ $\cap V^{\mathbf{a}}_{n,,\varepsilon}$ is contained in some ball $\widehat{B^{\mathbf{a}}_{n,,\varepsilon/2}}$ of radius $\varepsilon/2$ in the metric $D^{\mathbf{a}}_{n}$,

Let E_n be a subset of X_1 such that it is covered by a system $\mathcal{B}$ of balls of radius γ in the metric $d^{\mathbf{a}}_{n,\varepsilon}$ and with $\mu(E_n) > 1 - \delta$ so $\mu\left(E_n \cap B^{\mathbf{a}}_{n,,\varepsilon}\right) > 1 - \varepsilon/2 - \delta$. Let us consider a system $\mathcal{B}$ containing a number of $r^{\mathbf{a}}_n(\mu, \gamma, \delta)$ balls, If we consider partitions $\mathcal{A}_i$ with $diam < \varepsilon/2$ then each element of $\bigvee_{i=1}^k \tau_{j-1}^{-1}\left(\mathcal{A}_i^{\lceil(a_1,\ldots,a_i)n\rceil-1}\right)$ is contained is some ball $B^{\mathbf{a}}_{n,,\varepsilon}$. Thus since $B^{\mathbf{a}}_{n,,\varepsilon} \cap V^{\mathbf{a}}_{n,,\varepsilon} \subset \widehat{B^{\mathbf{a}}_{n,,\varepsilon}}$ for some balls $B^{\mathbf{a}}_{n,,\varepsilon}$, $\widehat{B^{\mathbf{a}}_{n,,\varepsilon}}$ we can consider a set $F_n \subset E_n \cap B^{\mathbf{a}}_{n,,\varepsilon}$with $\mu(F_n) > \dfrac{1-\delta}{4}$, and for n enough a big a part of F_n can be covered by elements $\mathcal{U} \in \bigvee_{i=1}^k \tau_{j-1}^{-1}\left(\mathcal{A}_i^{\lceil(a_1,\ldots,a_i)n\rceil-1}\right)$. Therefore by the Shannon-Mcmillan theorem (weighted version) we have

$$\mu(\mathcal{U}) < \exp\left[-n\left(\sum_{i=1}^k a_i h_\mu\left(f_1, \bigvee_{j=i}^k \tau_{j-1}^{-1}(\mathcal{A}_j)\right) - \varepsilon\right)\right].$$

Besides the number of such an elements is equal or greater than

$$\left(\frac{1-\delta}{4}\right)\exp\left[n\left(\sum_{i=1}^{k}a_i h_\mu\left(f_1,\bigvee_{j=i}^{k}\tau_{j-1}^{-1}\left(\mathcal{A}_j\right)\right)-\varepsilon\right)\right],$$

so

$$r_n^{\mathbf{a}}\left(\mu,\gamma,\delta\right)>\frac{\left(\frac{1-\delta}{4}\right)\exp\left[n\left(\sum_{i=1}^{k}a_i h_\mu\left(f_1,\bigvee_{j=i}^{k}\tau_{j-1}^{-1}\left(\mathcal{A}_j\right)\right)-\varepsilon\right)\right]}{\max_{i=1,2,\ldots,k}B(\varepsilon/2,N,s_i(n))}.$$

We also know that by the Stirling formula

$$B(\varepsilon/2,N,t_i(n))=\sum_{m=0}^{[(\varepsilon/2)t_i(n)]}(N-1)^m\binom{t_i(n)}{m}$$

then,

$$\lim_{n\to\infty}\frac{1}{t_i(n)}\log B(\varepsilon/2,N,t_i(n))=\varepsilon/2\log(N-1)-\varepsilon/2\log\varepsilon/2-(1-\varepsilon/2)\log(1-\varepsilon/2).$$

Recall that $\gamma\in(0,\varepsilon)$, hence

$$\lim_{n\to\infty}\frac{1}{n}\log r_n^{\mathbf{a}}\left(\mu,\varepsilon,\delta\right)\geq\sum_{i=1}^{k}a_i h_\mu\left(f_1,\bigvee_{j=i}^{k}\tau_{j-1}^{-1}\left(\mathcal{A}_j\right)\right)=\sum_{i=1}^{k}h_{(\tau_{i-1})_*(\mu)}\left(f_i,\mathcal{A}_i\right).$$

We are considering partitions with the property $\mu\left(\tau_{i-1}^{-1}\left(\partial\mathcal{A}_i\right)\right)=0$, and enough small diameter, therefore the entropies $h_{(\tau_{i-1})_*(\mu)}\left(f_i,\mathcal{A}_i\right)$ and $h_{(\tau_{i-1})_*(\mu)}\left(f_i\right)$ are arbitrary closed for any $i=1,2,\ldots,k$ and so we have

$$h_\mu^{\mathbf{a}}\left(f_1\right)\leq\lim_{\varepsilon\to0}\lim_{n\to\infty}\frac{1}{n}\log r_n^{\mathbf{a}}\left(\mu,\varepsilon,\delta\right).$$

□

According to [7] an alternative definition of the set of generic points it can be presented: let $\left\{p_j\right\}$ be a sequence of numbers with $\sum_{i=1}^{\infty}p_j=1$ and let $\{r_i\}$ be a sequence in ℓ^∞. The sequence $\left\{r_j=r_{n,j}\right\}_i$ converges to $\boldsymbol{\alpha}=(\alpha_j)\in\ell^\infty$ in the weak $*$- topology if and only if $\lim_{n\to\infty}\left|r_{n,j}-\alpha_j\right|=0$. Let $\{\Phi_1,\Phi_2,\ldots\}$ be a fixed dense subset in unit ball of $C(X)$ and $\Psi:X_1\to\ell^\infty$, with $\Psi=\{\Phi_1,\Phi_2,\ldots\}$. For a fixed $\mu\in\mathcal{M}(X,f)$, let $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots)$, with $\alpha_i=\int\Phi_i d\mu$. Thus

$$G\left(\mu\right)=\left\{x\in X_1:\lim_{n\to\infty}\sum_{j=1}^{\infty}p_{j,i}\left|\frac{S_n\left(\Phi_j\left(x\right)\right)}{n}-\alpha_i\right|=0\right\}=_{not}X_\Psi(\boldsymbol{\alpha}),\tag{2.9}$$

with $S_n\left(\Phi_i\left(x\right)\right)=\sum_{j=0}^{n-1}\left(\Phi_i\left(f_1^j\left(x\right)\right)\right)$.

The following metric in $\mathcal{M}(X_1,f_1)$ is compatible with the star weak topology in this space:

$$D\left(\mu,\nu\right)=\sum_{j=1}^{\infty}p_j\left|\int\Phi_j d\mu-\int\Phi_j d\nu\right|.$$

By a theorem of Young [16], we have the following approximation property, for any $\mu \in \mathcal{M}(X_1, f_1)$, $0 < \delta < 1$, $0 < \gamma < 1$, there is a measure ν such that $\nu = \sum_{j=1}^{t} \lambda_i \nu_i$, where each ν_j is ergodic and $\sum_{j=1}^{t} \lambda_j = 1$, and such that $\sum_{j=1}^{\infty} p_i \left| \int \varphi_i d\mu - \int \varphi_i d\nu \right| < \delta$.

Definition 5 A sequence of systems $(X_1, d_1, f_1), \ldots, (X_k, d_k, f_k)$ satisfy **a**-*specification* or *weighted specification* for $\mathbf{a} = (a_1, \ldots, a_k)$ if for any $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that, for any sequence of integer intervals $I_1 = [a_1, b_1], \ldots, I_s = [a_s, b_s]$ with $dist\left(I_i, I_j\right) > m(\varepsilon)$ $(i \neq j)$ and any points sequence $x_1, x_2, \ldots, x_k \in X_1$, there is a point $z \in X_1$ for which

$$\max_{i=1,,\ldots,k} \left\{ d_i \left(f_i^{a_\ell + j} (\tau_i(z), f_i^j (\tau_i(x_r)) \right) \right\} < \varepsilon$$

for any $\ell = 1, \ldots, s$; $r = 1, \ldots, t$ and $j = 0, 1, \ldots, \left\lceil \left(a_1 + \cdots + a_j\right) |I_\ell| \right\rceil$.

Examples of systems with **a**-specification are the full shift systems. More general shifts satisfy weighted specification if a condition on the dynamics is imposed. In some cases it is implied by the topological mixing condition. Other examples are Manneville-Pomeau maps systems [13] and families of logistic maps with an adequate choice of the parameters. The β-shift maps are also examples. We discuss with more detail these examples later on.

Let $\delta > 0$, $\alpha_j = \int \Phi_j d\mu$ and set

$$X_\Psi(\alpha, \delta, n) := \left\{ x \in X_1 : \sum_{j=1}^{\infty} p_j \left| \frac{S_n\left(\Phi_j(x)\right)}{n} - \alpha_j \right| < \delta \right\},$$

let $N_n^{\mathbf{a}}(\boldsymbol{\alpha}, \varepsilon, \delta)$ be the, minimal, number of balls $B_{n,,\varepsilon}^{\mathbf{a}}$ needed to cover $X_\Psi(\boldsymbol{\alpha}, \delta, n)$, then define

$$\Lambda_\Psi^{\mathbf{a}}(\boldsymbol{\alpha}) := \limsup_{n\to\infty} \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \frac{1}{n} \log N_n^{\mathbf{a}}(\alpha, \varepsilon, \delta) \tag{2.10}$$

Proposition 2.3 (Weighted entropy distribution principle) *Let* (X_i, d_i, f_i), $i = 1, 2, \ldots, k$, *be a finite sequence of dynamical systems, let* $\mu \in \mathcal{M}(X_1, f_1)$ *and* $Z \subset X_1$, *with* $\mu(Z) > 0$. *If for any* $\varepsilon > 0$. *for any ball* $B_{n,\varepsilon}^{\mathbf{a}}(x)$ *with* $B_{n,\varepsilon}^{\mathbf{a}}(x) \cap Z \neq \varnothing$ *and for a constant* s *holds* $\mu\left(B_{n,\varepsilon}^{\mathbf{a}}(x)\right) \leq C(\varepsilon)\exp(-ns)$, *for some constant* $C(\varepsilon) > 0$, *then* $h^{\mathbf{a}}(Z) \geq s$.

Proof Let $T_{n,\varepsilon}^{\mathbf{a}} = \left\{ A \subset X_1 : A \subset B_{n,\varepsilon}^{\mathbf{a}}(x),\ \text{for some } x \in X_1 \right\}$ and

$$\Gamma = \left\{ \left(n_j, A_j\right) : A_j \in T_{n_j,\varepsilon}^{\mathbf{a}},\ Z \subset \bigcup_{(n_j, A_j)\in\Gamma} A_j \right\}.$$

We may assume that the balls of the covering satisfy $B_{n,\varepsilon}^{\mathbf{a}}(x) \cap Z \neq \varnothing$. If $\left(n_j, A_j\right) \in \Gamma$ then

$$\sum_j \exp(-ns) \geq \frac{1}{C\varepsilon)} \sum_j \mu\left(B_{n_j,\varepsilon}^{\mathbf{a}}(x)\right) \geq \frac{1}{C\varepsilon)} \mu\left(\cup B_{n_j,\varepsilon}^{\mathbf{a}}(x)\right) \geq \frac{1}{C\varepsilon)} \mu(Z) > 0.$$

Hence for an integer N and $n_j \geq N$ we have $\Lambda^{\mathbf{a}}(Z, s, N, \varepsilon)$ and so $h^{\mathbf{a}}(Z) \geq s$. ◻

Proposition 2.4 $\Lambda_\Psi^{\mathbf{a}}() \leq h^{\mathbf{a}}(G(\mu))$.

Proof As we mentioned earlier we use the constructions of [6] based on techniques from [14]. Let $\{W_\ell\}_{\ell\geq 1}$ be a sequence of finite sets contains in X_1, let us consider sequence of integers $\{n_\ell\}$ such that for a fixed $\varepsilon > 0$ holds

$$d_{i,t_i(n_\ell)}\left(\tau_{i-1}(x), \tau_{i-1}(y)\right) > 5\varepsilon,\ i = 1, 2, \ldots, k \text{ and for any } x, y \in W_\ell,\ x \neq y.$$

For $\varepsilon > 0$ sufficiently small can be found a sequence $\{\delta_\ell\}$, with $\delta_\ell \searrow 0$ such that $W_\ell \subset X_\Psi(\alpha, \delta_\ell, n_\ell)$. besides, by the definition of $\Lambda_\Psi^{\mathbf{a}}(\boldsymbol{\alpha})$, we can choose the sets W_ℓ such that $M_\ell = card W_\ell \geq \exp\left[n_\ell\ \left(\Lambda_\Psi^{\mathbf{a}}(\boldsymbol{\alpha}) - \gamma\right)\right]$, for any $\gamma > 0$.

Let us consider a sequence of integers $\{N_\ell\}$, with $N_1 = 1$. Then, for fixed ℓ, select N_ℓ points $x_1, x_2, \ldots, x_{N_\ell} \in W_\ell$. so by the weighted specification property we can choose a point $y = y(x_1, x_2, \ldots, x_{N_\ell})$ such that

$$d_{i,t_i(n_\ell)}\left(\tau_{i-1}\left(f_1^{a_s}(y)\right), \tau_{i-1}(x_s)\right) < \varepsilon/2^\ell.\ s = 1, 2, \ldots, N_\ell,\ i = 1, 2, \ldots, k,$$

and where $a_s = (s-1)(n_\ell + m_\ell)$, with $m_\ell = m_\ell\left(\varepsilon/2^\ell\right)$ given by the definition of **a**-specification. The if $\left(x_1, x_2, \ldots, x_{N_\ell}\right) \in W_\ell^{N_\ell}$, $\left(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{N_\ell}}\right) \in W_\ell^{N_\ell}$ with $\left(x_1, x_2, \ldots, x_{N_\ell}\right) \neq \left(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{N_\ell}}\right)$ then

$$d_{i,t_i(t_\ell)}\left(\tau_{i-1}\left(y(x_1, x_2, \ldots, x_{N_\ell})\right), \tau_{i-1}\left(\overline{y}\left(\left(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{N_\ell}}\right)\right)\right)\right) > 4\varepsilon,$$

with $b_\ell = a_{N_\ell} + n_\ell = N_\ell n_\ell + (N_\ell - 1)\, m_\ell$. This is seen in the following way: take $x_s \neq \overline{x_s}$, for some s, we have

$$\begin{aligned}5\varepsilon &\leq d_{i,t_i(n_\ell)}\left(\tau_{i-1}(x_s), \tau_{i-1}(\overline{x_s})\right) \leq d_{i,t_i(n_\ell)}\left(\tau_{i-1}(x_s),\ \tau_{i-1}\left(f_1^{a_s}(y)\right)\right) \\ &\quad + d_{i,t_i(n_\ell)}\left(\tau_{i-1}\left(f_1^{a_s}(y)\right), \tau_{i-1}\left(f_1^{a_s}(\overline{y})\right)\right) + \ d_{i,t_i(n_\ell)}\left(, \tau_{i-1}\left(f_1^{a_s}(\overline{y})\right), \tau_{i-1}(\overline{x_s})\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + d_{i,t_i(b_\ell)}(\tau_{i-1}(y), \tau_{i-1}(\overline{y})).\end{aligned}$$

Then are defined the sets $D_1 = W_\ell$,

$$D_\ell = \left\{y(x_1, x_2, \ldots, x_{N_\ell}) : \left(x_1, x_2, \ldots, x_{N_\ell}\right) \in W_\ell^{N_\ell},\ i = 1, 2, \ldots, k\right\}.$$

Let $H_1 = D_1$, $h_1 = n_1$, and recursively define sets $H_{\ell+1}$ and numbers $h_{\ell+1}$. $\ell \geq 2$, as follows:

For each $x \in H_\ell$, $y \in D_{\ell+1}$ can be choose, by the weighted specification property a point $z = z(x, y) \in X_1$, such that

$$d_{i,t_i(h_1)}(\tau_{i-1}(z), \tau_{i-1}(x) < \varepsilon/2^{\ell+1},$$

for any $i = 1, 2, \ldots, k$ and

$$d_{i,t_i(b_{\ell+1})}(\tau_{i-1}\left(f_1^{h_\ell + m_{\ell+1}}(z)\right), \tau_{i-1}(y)) < \varepsilon/2^{\ell+1}, i = 1, 2, \ldots, k.$$

Then set

$$H_{\ell+1} = \{z(x, y) : x \in H_\ell,\ y \in D_{\ell+1}\}, \tag{2.11}$$

and

$$h_{\ell+1} = h_\ell + m_{\ell+1} + b_{\ell+1}. \tag{2.12}$$

Thus if $y, \overline{y} \in D_{\ell+1}$ with $y \neq \overline{y}$ then

$$d_{i,t_i(h_\ell)}(\tau_{i-1}(z(x, y)), \tau_{i-1}(z(\overline{x, y}))) > 3\varepsilon, \ell \geq 1.$$

Besides

$$d_{i,t_i(b_\ell)}(\tau_{i-1}(z(x,y)),\tau_{i-1}(z(\overline{x,y}))) < \varepsilon/2^\ell . d_{i,t_i(b_\ell)}(\tau_{i-1}(z(x,y)),\tau_{i-1}(z(\overline{x,y}))) < \varepsilon/2^\ell .$$

Now define

$$F_\ell = \bigcup_{x\in H_\ell} \overline{\left\{y : d_{i,t_i(h_\ell)}(\tau_{i-1}(x),\tau_{i-1}(y)) < \varepsilon/2^{\ell+1},\ i=1,2,\ldots,k\right\}}. \tag{2.13}$$

and

$$F = \bigcap_{\ell\geq 1} F_\ell . \tag{2.14}$$

There are two facts about F :

(i) It can be constructed a measure m concentrated on F, i.e. $m(F)=1$.
(ii) $h^{\mathbf{a}}(F) \geq \Lambda_\Psi^{\mathbf{a}}(\boldsymbol{\alpha})$

The proof of he fact $i)$ his is done following [14]. Let

$$m_\ell = \frac{1}{card H_\ell}\sum_{x\in H_\ell}\delta_x .$$

The sequence $\{m_\ell\}$ weakly converges to a limit m, concentrated on F, i.e. $m(F)=1$. To prove this we must see that for any $\gamma>0$, there is a $L(\gamma)$ such that for any $\ell_1,\ell_2>L$

$$\left|\int \varphi dm_{\ell_1} - \int \varphi dm_{\ell_1}\right| < \varepsilon \quad \text{for any } \varphi\in C(X_1).$$

We may assume that $\ell_1>\ell_2$, so we have

$$\begin{aligned}\left|\int \varphi dm_{\ell_1} - \int \varphi dm_{\ell_2}\right| &\leq \left|\frac{1}{card H_{\ell_1}}\sum_{x\in H_{\ell_1}}\varphi(x) - \frac{1}{card H_{\ell_2}}\sum_{z\in H_{\ell_2}}\varphi(z)\right|\\ &\leq \frac{1}{card H_{\ell_1}}\sum_{x\in H_{\ell_1}}|\varphi(x)-\varphi(z)|,\end{aligned}$$

with $z=z(x)\in H_{\ell_2}$, chosen like in the construction of such a space, i.e. $d_{i,s_i(h_{\ell_1})}(\tau_{i-1}(z),\tau_{i-1}(x) < \varepsilon/2^{\ell_1+1}$, for any $i=1,2,\ldots,k$. Thus by choosing a L and $\ell_1,\ell_2>L$, we get

$$\left|\int \varphi dm_{\ell_1} - \int \varphi dm_{\ell_2}\right| < \sup\left\{|\varphi(x)-\varphi(z)| : d_{i,t_i(h_{\ell_1})}(\tau_{i-1}(z),\tau_{i-1}(x) < \varepsilon/2^{\ell_1+1}\right\},$$

therefore for a given $\gamma>0$, $\ell_1,\ell_2>L$ can be made $\left|\int\varphi dm_{\ell_1} - \int\varphi dm_{\ell_2}\right| < \gamma$.

The uniqueness of the measure m is given by the Riesz theorem, in fact if we consider the positive functional $I(\varphi)=\lim_{n\to\infty}\int\varphi dm_\ell$, by the mentioned theorem there exist an

unique measure m such that $I(\varphi)=\int\varphi dm$.

By construction of the fractal set F has $m_{\ell+p}(F_{\ell+p})=1$, for any $p\geq 0$. The F_ℓ are closed, so by the property of the weak convergence we have

$m(F_\ell)\geq \limsup_{p\to\infty} m_{\ell+p}(F_{\ell+p})=1$ and therefore $m(F_\ell)=1$. Since $F=\bigcap_{\ell\geq 1}F_\ell$ we get $m(F)=1$.

For proving the fact $ii)$ is used the weighted entropy distribution principle to obtain a bound for $h^{\mathbf{a}}(F)$. To do this it may be estimated the m-measure of any ball $B^{\mathbf{a}}_{n_j,\varepsilon}$ such that $B^{\mathbf{a}}_{n_j,\varepsilon} \cap F \neq \varnothing$.

Let n be enough large and $x \in X_1$ with $B^{\mathbf{a}}_{n_j,\varepsilon}(x) \cap F \neq \varnothing$. By the definition of the sequence of measures $\{m_\ell\}$ with weak limit m, we have

$$m\left(B^{\mathbf{a}}_{n_j,\varepsilon}(x)\right) \leq \liminf_{\ell\to\infty} m_\ell\left(B^{\mathbf{a}}_{n_j,\varepsilon}(x)\right) = \liminf_{\ell\to\infty} \frac{1}{card H_\ell} \sum_{z\in H_{\ell B^{\mathbf{a}}_{n_j,\varepsilon}(x)}} \delta_x$$

$$= \liminf_{\ell\to\infty} \frac{1}{card H_\ell} card\left\{z \in H_\ell \cap B^{\mathbf{a}}_{n_j,\varepsilon}(x)\right\}.$$

Once constructed the sets H_ℓ and the measure m, like in [14], can be proved that

$$card\left(H_\ell \cap B^{\mathbf{a}}_{n_j,\varepsilon}(x)\right) \leq 1,$$

and so $m_\ell\left(B^{\mathbf{a}}_{n_j,\varepsilon}(x)\right) \leq \dfrac{1}{card H_\ell}$.

Let $\ell = \ell(n)$ and $0 \leq p = p(n) \leq N_{\ell+1}$ such that

$$h_\ell + p\left(m_{\ell+1} + n_{\ell+1}\right) < n \leq h_\ell + (p+1)\left(m_{\ell+1} + n_{\ell+1}\right),$$

if $z_1, z_2 \in H_{\ell+1} \cap B^{\mathbf{a}}_{n_j,\varepsilon}(x)$ then

$$z_1 = z\left(x, y(x_1, x_2, \ldots, x_{N_{\ell+1}})\right), z_1 = z\left(\overline{x}, \overline{y}\left(\left(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{N_{\ell+1}}}\right)\right)\right),$$

with $\left(x_1, x_2, \ldots, x_{N_\ell}\right), \left(\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{N_{\ell+1}}}\right) \in W^{N_{\ell+1}}_{\ell+1}$. Like in [14], can be proved that $x_1 = x_2$ and $x_i = \overline{x_i}$, $i = 1, 2, \ldots, p$. Thus for all the points in $H_{\ell+1} \cap B^{\mathbf{a}}_{n_j,\varepsilon}(x)$ the x and the $\left(x, y(x_1, x_2, \ldots, x_p)\right)$ are the same, and hence there are at most $M^{N_{\ell+1}-p}_{\ell+1}$ of these points. Therefore

$$m_{\ell+1}\left(B^{\mathbf{a}}_{n_j,\varepsilon}(x)\right) \leq \frac{1}{card H_\ell} \frac{M^{N_{\ell+1}-p}_{\ell+1}}{M^{N_{\ell+1}}_{\ell+1}} = \frac{1}{(card H_\ell)\, M^p_{\ell+1}}. \tag{2.15}$$

Thus, for $p \geq 1$

$$m_{\ell+p}\left(B^{\mathbf{a}}_{n_j,\varepsilon/2}(x)\right) \leq \frac{1}{(card H_\ell)\, M^p_{\ell+1}}. \tag{2.16}$$

Recall that we chosen the sets W_ℓ, such that $M_\ell = card W_\ell \geq \exp\left[n_\ell\ \left(\Lambda^{\mathbf{a}}_\Psi(\boldsymbol{\alpha}) - \gamma\right)\right]$, for any $\gamma > 0$ and for the sequence of numbers $\{n_\ell\}$ given earlier. Let $s = \Lambda^{\mathbf{a}}_\Psi(\boldsymbol{\alpha}) - \gamma$, so

$$\begin{aligned}(card H_\ell)\, M^p_{\ell+1} &= M_1^{N_1} M_2^{N_{21}} \ldots M_\ell^{N_\ell} M^p_{\ell+1} \geq \exp\left[\sum_{i=1}^{\ell} N_i n_i p + p n_{\ell+1}\right]\\ &\geq exp\left[(s - \gamma/2)\left(N_1 n_1 + \cdots + N_\ell\left(n_\ell + m_\ell\right) + p\left(n_{\ell+1} + m_{\ell+1}\right)\right)\right]\\ &\geq exp\left[(s-\gamma)\, n\right].\end{aligned}$$

Thus, for n large enough, $\ell \to \infty$ get

$$m\left(B^{\mathbf{a}}_{n_j,\varepsilon/2}(x)\right) \leq exp\left[(s-\gamma)\, n\right],$$

therefore, because the estimation of the ball intersecting F, with $m(F) = 1$, and since γ is arbitrary small, by the weighted entropy distribution principle we obtain

$$h^{\mathbf{a}}(F) \geq s = \Lambda^{\mathbf{a}}_\Psi(\boldsymbol{\alpha}).$$

Now the proof will be completed by proving that $F \subset G(\mu) = X_\Psi(\boldsymbol{\alpha})$. So it should be shown that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} p_i \left| \frac{S_n(\Phi_i(x))}{n} - \alpha_i \right| = 0,$$

for any $x \in F$ and $\alpha_i = \int \Phi_i d\mu$. To establish this fact is used a technique similar to [6], which consists in splitting the interval $[0, n)$ in small subintervals to bound the statistical sums $S_n(\Phi_i(x)) = \sum_{j=0}^{n-1} \left(\Phi_i \left(f_1^j(x) \right) \right)$,

For $\Phi \in C(X_1)$ set

$$Var(\Phi, \epsilon.\, \mathbf{a}) := \max_{j=1,2,\ldots k} \sup_{d_i(\tau_{j-1}(x), \tau_{j-1}(y))<\varepsilon} \{|\Phi(x) - \Phi(y)|\}$$

.

Let us consider the sequences $\{n_\ell\}$. $\{h_\ell\}$ and $\{b_\ell\}$ used for the constructions of the sets D_ℓ and H_ℓ. Let $n, \ell \geq 1$ and $0 \leq p \leq N_{\ell+1}$, such that

$h_\ell + p(n_{\ell+1} + m_{\ell+1}) < n < h_\ell + (p+1)(n_{\ell+1} + m_{\ell+1})$. Then the interval $[0, n)$ can be partitioned as

$$[0, h_\ell) \cup \left[h_\ell, h_\ell + p(n_{\ell+1} + m_{\ell+1})\right) \cup \left[h_\ell, h_\ell + p(n_{\ell+1} + m_{\ell+1})\right) \cup \left[h_\ell + p(n_{\ell+1} + m_{\ell+1}), n\right).$$

and in turn the intervals $\left[h_\ell, h_\ell + p(n_{\ell+1} + m_{\ell+1})\right)$ are decomposed into intervals alternatively of lengths $n_{\ell+1}$ and $m_{\ell+1}$. Let $x \in F$, by [6], the statistical sums $S_n(\Phi_i(x))$ are partitioned in sums over small intervals and is obtained the bound for the "error"

$$\left|S_n\left(\Phi_j(x)\right) - n\alpha_j\right| \leq I_1(j) + I_2(j) + I_3(j) + I_4(j),$$

with

$$I_1(j) = \left|S_{h_\ell}\left(\Phi_j(x)\right) - h_\ell \alpha_j\right|$$

and

$$I_3(i) = \sum_{s=1}^{p} \left| S_{n_\ell+1}\left(\Phi_j\left(f_1^{h_\ell + c_s + m_{\ell+1}}(x)\right)\right) - n_{\ell+1}\alpha_j \right|,$$

where $c_s = (s-1)(n_{\ell+1} + m_{\ell+1})$, and the intervals $I_2(j)$, $I_4(j)$ satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\infty} p_j\ I_2(j) = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{\infty} p_j\ I_4(j) = 0.$$

Then to prove that $x \in X_\Psi(\boldsymbol{\alpha})$, should be justify that $\lim_{n\to\infty} \dfrac{1}{n} \sum_{j=1}^{\infty} p_j\ I_k(j) = 0$, $k = 1, 2, 3, 4$.

For any $x \in F$, there is a $\overline{x} \in H_\ell$ such that

$$d_{i,t_i(t_1)}(\tau_{i-1}(x), \tau_{i-1}(\overline{x}) < \varepsilon/2^{\ell+1}, i = 1, 2, \ldots, k$$

and if $1 \leq s \leq p$ then there is a point $x_s \in W_{\ell+1} \subset X_\Psi(\alpha, \delta_{\ell+1}, n_{\ell+1})$ such that

$$d_{i,t_i(n_{1+1})}(\tau_{i-1}(x_s), \tau_{i-1}\left(f_1^{v_s}(x)\right) < \varepsilon/2^{\ell+1}, i = 1, 2, \ldots, k, \text{ with } v_s = h_\ell + c_s + n_{\ell+1}.$$

From this, by [6], $I_3\left(i\right)$ can be bounded

$$I_3\left(j\right) \leq \sum_{s=1}^{p}\left|S_{n_{\ell}+1}\left(\Phi_j\left(f_1^{v_s}\left(x\right)\right)\right) - S_{n_{\ell}+1}\left(\Phi_j\left(x_s\right)\right)\right| + \sum_{s=1}^{p}\left|S_{n_{\ell}+1}\left(\Phi_j\left(f_1^{v_s}\left(x\right)\right)\right) - n_{\ell+1}\alpha_j\right|$$
$$\leq n_{\ell+1} Var\left(\Phi_j,\ \varepsilon/2^{\ell+1},\right) + n_{\ell+1}\delta_{\ell+1},$$

since $x_s \in W_{\ell+1} \subset X_{\Psi}(\alpha, \delta_{\ell+1}, n_{\ell+1})$. Therefore

$$\frac{1}{n}\sum_{j=1}^{\infty} p_j\ I_3\left(j\right) \leq \sum_{j=1}^{\infty} p_i\ Var\left(\Phi_j,\ \varepsilon/2^{\ell+1},\ \mathbf{a}\right) + \delta_{\ell+1}$$

and so, since $\ell \to \infty$ as $n \to \infty$, we have $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{\infty} p_j\ I_3\left(j\right) = 0$.

The idea to bound $I_1\left(j\right)$ is similar, we have

$$I_1\left(j\right) \leq \left|S_{h_{\ell}}\left(\Phi_j\left(x\right)\right) - \ S_{h_{\ell}}\left(\Phi_j\left(\overline{x}\right)\right)\right| + \left|\ S_{h_{\ell}}\left(\Phi_i\left(\overline{x}\right)\right) - h_{\ell}\alpha_j\right|$$
$$\leq h_{\ell} Var\left(\Phi_j,\ \varepsilon/2^{\ell+1},\right) + \max_{y\in H_{\ell}}\left|\ S_{h_{\ell}}\left(\Phi_i\left(y\right)\right) - h_{\ell}\alpha_j\right|.$$

That $\lim_{n\to\infty}\frac{1}{n}\max_{y\in H_{\ell}}\left|\ S_{h_{\ell}}\left(\Phi_j\left(y\right)\right) - h_{\ell}\alpha_j\right| = 0$, can be proved like in [6] and $Var\left(\Phi_j,\ \varepsilon/2^{\ell+1},\right) \to 0$ as $\ell \to \infty$ by the continuity of the maps Φ_i, we have that $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{\infty} p_j\ I_1\left(j\right) = 0$.

With this $F \subset G\left(\mu\right) = X_{\Psi}(\boldsymbol{\alpha})$ and so

$$\Lambda_{\Psi}^{\mathbf{a}}(\boldsymbol{\alpha}) \leq h^{\mathbf{a}}\left(\ F\right) \leq\ h^{\mathbf{a}}\left(G\left(\mu\right)\right).$$

□

Proposition 2.5 $\Lambda_{\Psi}^{\mathbf{a}}() \geq\ h_{\mu}^{\mathbf{a}}\left(f_1\right).$

Proof For a given $\gamma > 0$, , can be consider $\varepsilon > 0$ and $\delta > 0$ such that $\Lambda_{\Psi}^{\mathbf{a}}(\boldsymbol{\alpha}) + \gamma > \limsup_{n\to\infty}\frac{1}{n}\log N_n^{\mathbf{a}}\left(\alpha, \varepsilon, 5\delta\right)$. Recall that by the approximation theorem of Young, for any measure $\mu \in \mathcal{M}(X_1, f_1)$, here is an invariant measure ν such that $\nu = \sum_{\ell=1}^{t}\lambda_{\ell}\nu_{\ell}$, with ν_j ergodic, $\sum_{j=1}^{t}\lambda_j = 1$ and $D\left(\mu, \nu\right) < \delta$. Let $1 \leq \ell \leq t$, $N \geq 1$, set

$$Y_{\ell}(N) = \left\{x \in X_1 : \sum_{j=1}^{\infty} p_j\left|\frac{S_n\left(\Phi_j\left(x\right)\right)}{n} - \alpha_j\right| < \delta,\quad \text{for } n \geq N\right\}, \text{with } \alpha_j = \int\Phi_j d\mu.$$

We have $\nu_{\ell}\left(Y_{\ell}(N)\right) > 1 - \gamma$, $\ell = 1, \ldots, t$. By the proposition 2.2, for any $\varepsilon > 0$, there is an integer N_{ℓ} such that, for $n \geq N_{\ell}$

$$r_n^{\mathbf{a}}\left(\nu_{\ell}, 4\varepsilon, \gamma\right) > \exp\left(n\left(h_{\nu_{\ell}}^{\mathbf{a}}\left(f_1\right) - \gamma\right)\right).$$

Since $\nu_{\ell}\left(Y_{\ell}(N)\right) > 1 - \gamma$, the quantity $r_n^{\mathbf{a}}\left(\nu_{\ell}, 4\varepsilon, \gamma\right)$ series to count the minimal number of balls $B_{n,\varepsilon}^{\mathbf{a}}$ needed to cover $Y_{\ell}(N)$, and so this number is equal of greater than $\exp\left(n\left(h_{\nu_{\ell}}^{\mathbf{a}}\left(f_1\right) - \gamma\right)\right)$,

A set $E \subset X_1$ is $\mathbf{a}$,n, ε-separated if for any $x \neq y \in E$ holds $d_n^{\mathbf{a}}\left(x, y\right) = \max_{i=1,2,\ldots,k}\left\{d_{i,}\left(\tau_{i-1}\left(x\right), \tau_{i-1}\left(y\right)\right)\right\} > \varepsilon$. By $E_{n,\varepsilon}^{\mathbf{a}}$ is denoted a $\mathbf{a}$,n, ε-separated set contained in $Y_{\ell}(N)$ and with maximal cardinality . Let $n_{\ell} = \left[\lambda_{\ell}n\right]$, $\ell = 1, \ldots, t$, and such that

$n_\ell \geq \max\left\{N, N_{1,...,}N_\ell\right\}$ for N sufficiently large, For $x_\ell \in E^{\mathbf{a}}_{n_i,4\varepsilon} \subset Y_\ell/N)$, $\ell = 1, 2, \ldots, t$, there exists, by the **a**-specification property, a $m = m(\varepsilon)$ and a point $y = y(x_1, x_2, \ldots, x_t)$ such that:

$$d_{i,t_i(n_\ell)}\left(\tau_{i-1}\left(f_1^{a_s}(x_\ell)\right), \tau_{i-1}(y)\right) < \varepsilon,$$

where $a_1 = 0$, $a_s = (s-1)m + \sum_{r=1}^{t-1} n_s$. By the other hand $card E^{\mathbf{a}}_{n,4\varepsilon} \geq \exp\left(n\left(h^{\mathbf{a}}_{\nu_\ell}(f_1) - \gamma\right)\right)$, for any $n \geq N_\ell$.

Let $\overline{n} = a_t + n_t$. the following fact are valid:

(1) For each $x_\ell \in E^{\mathbf{a}}_{n_i,4\varepsilon}$, $i = 1, 2, \ldots, t$ the corresponding $y = y(x_1, x_2, \ldots, x_t)$ belongs to $X_\Psi(\alpha, 5\delta, \overline{n})$ for n sufficiently large.
(2) If $(x_1, x_2, \ldots, x_t) \neq (\overline{x_1}, \overline{x_2}, \ldots, \overline{x_t}) \in E^{\mathbf{a}}_{n_i,4\varepsilon}$, $\ell = 1, 2, \ldots, t$ then

$$d_{i,t_i(\overline{n})}\left(\tau_{i-1}(y), \tau_{i-1}(\overline{y})\right) > 2\varepsilon.$$

The proofs of these claims are similar, with slight differences, to that presented in [6], we display here the main aspects of the proofs.

To prove (1) it must be seen that

$$\sum_{j=1}^{\infty} p_j \left|\frac{S_{\overline{n}}\left(\Phi_j(y)\right)}{\overline{n}} - \int \Phi_j d\mu\right| < 5\delta,$$

for n sufficiently large. Let $\nu = \sum_{\ell=1}^{t} \lambda_\ell \nu_\ell$, with ν_ℓ ergodic, $\sum_{\ell=1}^{t} \lambda_\ell = 1$ and $D(\mu, \nu) < \delta$. Then

$$\left|\frac{S_{\overline{n}}\left(\Phi_j(y)\right)}{\overline{n}} - \int \Phi_j d\mu\right| \leq \left|\frac{S_{\overline{n}}\left(\Phi_j(y)\right)}{\overline{n}} - \sum_{\ell=1}^{t}\int \Phi_j d\nu_\ell\right| + \left|\sum_{\ell=1}^{t}\int \Phi_i d\nu_\ell - \int \Phi_i d\mu\right|$$
$$= \left|\frac{S_{\overline{n}}\left(\Phi_i(y)\right)}{\overline{n}} - \sum_{\ell=1}^{t}\int \Phi_i d\nu_\ell\right| + \left|\int \Phi_i d\nu - \int \Phi_i d\mu\right|.$$

Since $D(\mu, \nu) < \delta$ we have that

$$\sum_{j=1}^{\infty} p_j \left|\frac{S_{\overline{n}}\left(\Phi_j(y)\right)}{\overline{n}} - \int \Phi_i d\mu\right| \leq \sum_{j=1}^{\infty} p_j \left|\frac{S_{\overline{n}}\left(\Phi_i(y)\right)}{\overline{n}} - \sum_{\ell=1}^{t}\int \Phi_j d\nu_\ell\right| + \delta,$$

and so is needed to prove that

$$\sum_{j=1}^{\infty} p_j \left|\frac{S_{\overline{n}}\left(\Phi_j(y)\right)}{\overline{n}} - \sum_{\ell=1}^{t}\int \Phi_j d\nu_\ell\right| < 4\delta. \tag{2.17}$$

In [6] this is proved by doing

$$\left|\frac{S_{\overline{n}}\left(\Phi_j(y)\right)}{\overline{n}} - \sum_{\ell=1}^{t}\int \Phi_i d\nu_\ell\right| \leq S_1(j) + S_2(j) + S_3(j) + S_4(j),$$

where

$$S_1(j) = \sum_{\ell=1}^{t} \frac{[\lambda_\ell n]}{\overline{n}} \left|\frac{S[\lambda_\ell n]\left(\Phi_i\left(f_1^{a_\ell} y\right)\right)}{\overline{n}} - \frac{S[\lambda_\ell n]\left(\Phi_j\ (x_\ell)\right)}{\overline{n}}\right|,$$

$$S_2(j) = \sum_{\ell=1}^{t} \frac{[\lambda_\ell n]}{\overline{n}} \left| \frac{S[\lambda_\ell n]\left(\Phi_j\ (x_\ell)\right)}{\overline{n}} - \int \Phi_j d\nu_\ell \right|,$$

$$S_3(j) = \left| \sum_{\ell=1}^{t} \frac{[\lambda_\ell n]}{\overline{n}} - \lambda_\ell \int \left|\Phi_j\right| d\nu_\ell \right|,$$

and

$$S_4(j) = \frac{1}{\overline{n}} \sum_{\ell=2}^{t} \sum_{s=a_\ell-m}^{a_\ell-1} \left|\Phi_j\left(f_1^s y\right)\right|.$$

Thus it must be proved that $\sum_{j=1}^{\infty} p_j S_k(j) < \delta$, for $k = 1, 2, 3, 4$.

We have that $\overline{n} \to \infty$ as $n \to \infty$ and $\frac{[\lambda_\ell n]}{\overline{n}} \to \lambda_\ell$, as $n \to \infty$.

By the continuity of the each map Φ_j we have that $Var\left(\Phi_j, \epsilon. \mathbf{a}\right) \to 0$ as $\varepsilon \to 0$ and can be done $\sum_{j=1}^{\infty} p_j Var\left(\Phi_j, \epsilon. \mathbf{a}\right) < \delta$. Hence

$$\sum_{j=1}^{\infty} p_j \sum_{\ell=1}^{t} \frac{[\lambda_\ell n]}{\overline{n}} \left| \frac{S[\lambda_\ell n]\left(\Phi_j\left(f_1^{a_\ell} y\right)\right)}{\overline{n}} - \frac{S[\lambda_\ell n](\Phi_i\ (x_\ell))}{\overline{n}} \right| \leq \sum_{j=1}^{\infty} p_j \sum_{\ell=1}^{t} Var\left(\Phi_j, \epsilon. \mathbf{a}\right) \lambda_\ell < \delta.$$

This is due to $\sum_{j=1}^{\infty} p_j = 1$ and that each x_ℓ satisfies $d_{i,t_i(n)}\left(\tau_{i-1}\left(\left(f_1^{a_s}(x_\ell)\right)\right), \tau_{i-1}(y)\right) < \varepsilon$.

For the second sum

$$\sum_{j=1}^{\infty} p_j \sum_{\ell=1}^{t} \frac{[\lambda_\ell n]}{\overline{n}} \left| \frac{S[\lambda_\ell n](\Phi_i\ (x_\ell))}{\overline{n}} - \int \Phi_i d\nu_\ell \right| \leq \delta \sum_{\ell=1}^{t} \lambda_\ell = \delta,$$

because $x_\ell \in E^{\mathbf{a}}_{n_i,4\varepsilon} \subset Y_\ell/N)$.

Using $\| \Phi_j \| \leq 1$ is proved, like in [6] that $\sum_{j=1}^{\infty} p_j S_3(j) < \delta$ and $\sum_{j=1}^{\infty} p_j S_4(j) < \delta$.

For the proof of 2) let $x_t \neq \overline{x_t}$, we have, for $i = 1, 2, \ldots, k$,

$$\begin{aligned} &d_{i,t_i(n_1)}(\tau_{i-1}(x_\ell), \tau_{i-1}(\overline{x_t}) \\ &\quad \leq d_{i,t_i(n_1)}(\tau_{i-1}(x_\ell), \tau_{i-1}\left(f_1^{va_\ell}(y)\right) + d_{i,t_i(n_1)}\left(\tau_{i-1}\left(f_1^{va_\ell}(y)\right), \tau_{i-1}\left(f_1^{va_\ell}(\overline{y})\right)\right) \\ &\qquad + d_{i,t_i(n_1)}(\tau_{i-1}\left(f_1^{va_\ell}(\overline{y})\right), \tau_{i-1}(\overline{x_t})). \end{aligned}$$

Then, since $x_t, \overline{x_t} \in E^{\mathbf{a}}_{n_i,4\varepsilon}$ we get

$$\begin{aligned} &d_{i,t_i(n_1)}\left(\tau_{i-1}\left(f_1^{va_\ell}(y)\right), \tau_{i-1}\left(f_1^{va_\ell}(\overline{y})\right)\right) \\ &\quad \geq d_{i,t_i(n_1)}\left(\tau_{i-1}\left(f_1^{va_\ell}(\overline{y})\right), \tau_{i-1}(\overline{x_t})\right) - 2\varepsilon > 4\varepsilon - 2\varepsilon = 2\varepsilon. \end{aligned}$$

But

$$d_{i,t_i(\overline{n})}(\tau_{i-1}\left(f_1^{va_\ell}(y)\right), \tau_{i-1}\left(f_1^{va_\ell}(\overline{y})\right)) < d_{i,t_i(\overline{n})}(\tau_{i-1}(y), \tau_{i-1}(\overline{y})),$$

so that

$$d_{i,t_i(\overline{n})}(\tau_{i-1}(y), \tau_{i-1}(\overline{y})) > 2\varepsilon.$$

Thus, by the fact (2) the number of points $y = y(x_1, x_2, \ldots, x_t)$, obtained from each $(x_1, x_2, \ldots, x_t) \in E^{\mathbf{a}}_{n_{1i},4\varepsilon} \times \cdots \times E^{\mathbf{a}}_{n_t,4\varepsilon}$ does not exceed the minimal number of balls $B^{\mathbf{a}}_{\overline{n},\varepsilon}$ needed to cover $X_\Psi(\alpha, 5\delta, \overline{n})$. Therefore

$$\prod_{\ell=1}^{t} card E^{\mathbf{a}}_{n_{\ell i},4\varepsilon} \leq N^{\mathbf{a}}_{\overline{n}}(\alpha, \varepsilon, 5\delta),$$

and so

$$N^{\mathbf{a}}_{\overline{n}}(\alpha, \varepsilon, 5\delta) \geq \prod_{\ell=1}^{t} \exp\left([\lambda_\ell n]\left(h^{\mathbf{a}}_{\nu_\ell}(f_1) - \gamma\right)\right) = \exp\left(\sum_{\ell=1}^{t}\left([\lambda_\ell n]\left(h^{\mathbf{a}}_{\nu_\ell}(f_1) - \gamma\right)\right)\right),$$

for $\gamma > 0$. Recall that

$$\overline{n} \to \infty \text{ and } \frac{[\lambda_\ell n]}{\overline{n}} \to \lambda_\ell, \text{ as } n \to \infty,$$

then, taking $\limsup_{n\to\infty} \lim_{\varepsilon\to 0} \lim_{\delta\to 0}$, get

$$\Lambda^{\mathbf{a}}_\Psi(\boldsymbol{\alpha}) \geq h^{\mathbf{a}}_\mu(f_1) - \gamma.$$

Since γ is arbitrary proposition is proved. □

We have, by Propositions 2.4 and 2.5 that $h^{\mathbf{a}}_\mu(f_1) \leq h^{\mathbf{a}}(G(\mu))$, for any $\mu \in \mathcal{M}(X_1, f_1)$, and as we pointed out the opposite is valid by [17], therefore the proof theorem 1.2 is completed.

3 Proof of the Theorem 1.1

Firstly we state the following weighted version of Bowen lemma:

Theorem 3.1 [17] *Let* (X_i, d_i, f_i), $i = 1, 2, \ldots, k$, *dynamical systems, let* $B^{\mathbf{a}}(t) = \left\{x \in X_1 : \text{there is a } \mu \in V(x) \text{ such that } h^{\mathbf{a}}_\mu(f_1) \leq t\right\}$, *recall that* $V(x)$ *denotes the set of weak limits of the sequence of measures* $\left\{\mathcal{E}_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f_1^i(x)}\right\}$.

Proof Recall the **a**-weighted multifractal decomposition, for a finite sequence of dynamical systems (X_i, d_i, f_i) and maps $\Phi_1, \Phi_2 \ldots, \Phi_k \in C\left(X_1^r\right)$, is defined as

$$K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha} = \left\{x \in X_1 : \lim_{n\to\infty} V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n, x) = \alpha\right\}.$$

where

$$V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n, x) = \sum_{j=1}^{k} \frac{1}{\left(s_j(n)\right)^r} V_{\Phi_j}\left(s_j(n), x\right).$$

with $s_j(n) = \left\lfloor (a_1, \ldots, a_j)\, n\right\rfloor$.

By the Stone–Weirstrass theorem, for any Φ_j there exits a map $\widetilde{\Phi_j}$ of the form $\widetilde{\Phi_j} = \sum_\ell \varphi^{(1)}_{\ell,j} \otimes \cdots \otimes \varphi^{(r)}_{\ell,j}$, $j = 1, 2, \ldots k$, and such that for any $\varepsilon > 0$ holds $\left\|\Phi_j - \widetilde{\Phi_j}\right\|_\infty < \varepsilon$.

Thus

$$V^{\mathbf{a}}_{\{\widetilde{\Phi_j},\widetilde{\Phi_j}\ldots,\widetilde{\Phi_k}\}}(n,x)=\sum_{\ell}\prod_{i=1}^{r}\sum_{j=1}^{k}\frac{S_{s_j(n)}\left(\varphi^{(i)}_{\ell,j}\right)(x)}{s_j(n)}$$

and by [1] we have

$$\lim_{n\to\infty}V^{\mathbf{a}}_{\{\widetilde{\Phi_j},\widetilde{\Phi_j}\ldots,\widetilde{\Phi_k}\}}(n,x)=\sum_{j=1}^{k}\int\widetilde{\Phi}_j d\mu^{\otimes r},$$

for any $\mu\in V(x)$.

We shall see that

(i) $K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha}\subset B^{\mathbf{a}}\left(\sup_{\mu\in\mathcal{M}(\alpha,\{\Phi_1,\Phi_2\ldots,\Phi_k\})}h^{\mathbf{a}}_{\mu}(f_1)\right)$, where

$$\mathcal{M}(\alpha,\{\Phi_1,\Phi_2\ldots,\Phi_k\})=\left\{\mu\in\mathcal{M}(X_1,f_1):\sum_{j=1}^{k}\int\Phi_j d\mu^{\otimes r}=\alpha\right\}.$$

.

(ii) $G(\mu)\subset K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha}$., $\mu\in\mathcal{M}(\alpha,\{\Phi_1,\Phi_2\ldots,\Phi_k\})$

Once these claims be proved, we will have, by the weighted Bowen lemma

$$h^{\mathbf{a}}(K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha})\le\sup_{\mu\in\mathcal{M}(\alpha,\{\Phi_1,\Phi_2\ldots,\Phi_k\})}h^{\mathbf{a}}_{\mu}(f_1), \tag{3.1}$$

and by the saturation property

$$h^{\mathbf{a}}(G(\mu))=h^{\mathbf{a}}_{\mu}(f_1)\le h^{\mathbf{a}}(K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha}).$$

Then the variational principle for weighted V-statistics would be established.

To prove (i) let $x\in K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha}$, let $\mu\in V(x)$ so that there is a sequence of integers $\{n_k\}$ such that $\mathcal{E}_{n_k}(x)$ weakly converges to μ. We have

$$\begin{aligned}\sum_{j=1}^{k}\int\Phi_j d\mu^{\otimes r}-\alpha&=\sum_{j=1}^{k}\int\Phi_j d\mu^{\otimes r}-\sum_{j=1}^{k}\int\widetilde{\Phi_j}d\mu^{\otimes r}+\sum_{j=1}^{k}\int\widetilde{\Phi_j}d\mu^{\otimes r}-V^{\mathbf{a}}_{\{\widetilde{\Phi_j},\widetilde{\Phi_j}\ldots,\widetilde{\Phi_k}\}}(n_k,x)\\&\quad+V^{\mathbf{a}}_{\{\widetilde{\Phi_j},\widetilde{\Phi_j}\ldots,\widetilde{\Phi_k}\}}(n_k,x)-V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n_k,x)+V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n_k,x)-\alpha,\end{aligned}$$

hence

$$\begin{aligned}\left|\sum_{j=1}^{k}\int\Phi_j d\mu^{\otimes r}-\alpha\right|&\le\sum_{j=1}^{k}\left|\int\Phi_j d\mu^{\otimes r}-\int\widetilde{\Phi_j}d\mu^{\otimes r}\right|+\left|\sum_{j=1}^{k}\int\widetilde{\Phi_j}d\mu^{\otimes r}-V^{\mathbf{a}}_{\{\widetilde{\Phi_j},\widetilde{\Phi_j}\ldots,\widetilde{\Phi_k}\}}(n_k,x)\right|\\&\quad+\left|V^{\mathbf{a}}_{\{\widetilde{\Phi_j},\widetilde{\Phi_j}\ldots,\widetilde{\Phi_k}\}}(n_k,x)-V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n_k,x)\right|+\left|V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n_k,x)-\alpha\right|.\end{aligned}$$

thus, for $\varepsilon>0$ and n_k enough large and since $x\in K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha}$, we obtain

$$\left|\sum_{j=1}^{k}\int\Phi_j d\mu^{\otimes r}-\alpha\right|\le ki\epsilon+\varepsilon+\varepsilon+\varepsilon\ ,$$

and since ε is arbitrary this leads to $\mu\in\mathcal{M}(\alpha,\{\Phi_1,\Phi_2\ldots,\Phi_k\})$. From this we have that if $x\in K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha}$ then there is a $\mu\in V(x)$ and such that

$$h^{\mathbf{a}}_{\mu}(f_1)\le\sup_{\mu\in\mathcal{M}(\alpha,\{\Phi_1,\Phi_2\ldots,\Phi_k\})}h^{\mathbf{a}}_{\mu}(f_1).$$

Therefore

$$K^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\},\alpha} \subset B^{\mathbf{a}}\left(\sup_{\mu\in\mathcal{M}(\alpha,\{\Phi_1,\Phi_2\ldots,\Phi_k\})} h^{\mathbf{a}}_{\mu}(f_1)\right)$$

and (i) is proved.

For (ii) let $x \in G(\mu)$,with $\mu \in \mathcal{M}(\alpha, \{\Phi_1, \Phi_2 \ldots, \Phi_k)\}$, as we have seen,

$$\lim_{n\to\infty} V^{\mathbf{a}}_{\{\widetilde{\Phi}_j,\widetilde{\Phi}_j\ldots,\widetilde{\Phi}_k\}}(n,x) = \sum_{j=1}^{k}\int_{X_1^r} \widetilde{\Phi}_j d\mu^{\otimes r} \tag{3.2}$$

thus we have, for $\varepsilon > 0$,

$$\begin{aligned}
&\left|\lim_{n\to\infty} V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n,x) - \sum_{j=1}^{k}\int_{X_1^r}, \Phi_j d\mu^{\otimes r}\right| \\
&\le \left|\lim_{n\to\infty} V^{\mathbf{a}}_{\{\widetilde{\Phi}_j,\widetilde{\Phi}_j\ldots,\widetilde{\Phi}_k\}}(n,x) - \lim_{n\to\infty} V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n,x)\right| \\
&\quad + \left|\lim_{n\to\infty} V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n,x) - \sum_{j=1}^{k}\int_{X_1^r} \widetilde{\Phi}_j d\mu^{\otimes r}\right| \\
&\quad + \left|\sum_{j=1}^{k}\int_{X_1^r} \widetilde{\Phi}_j d\mu^{\otimes r} - \sum_{j=1}^{k}\int_{X_1^r} \Phi_j d\mu^{\otimes r}\right| < 2\varepsilon.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} V^{\mathbf{a}}_{\{\Phi_1,\Phi_2\ldots,\Phi_k\}}(n,x) = \sum_{j=1}^{k}\int_{X_1^r}, \Phi_j d\mu^{\otimes r} = \alpha,$$

because $\mu \in \mathcal{M}(\alpha, \{\Phi_1, \Phi_2 \ldots, \Phi_k)\}$.

With the proof of (i) and (ii) concludes the demonstration of the theorem 1.2. □

Let us consider as an example the case of Bernoulli schemes, let $(X_i, \sigma_i, \Omega_i)$, $j = 1, 2, \ldots k$, be a finite family with X_i the set of infinite sequences in symbols of the alphabet Ω_i, i.e. $X_i = \left\{x^{(i)} = \left(x^{(i)}\right)_1 \left(x^{(i)}\right)_2 \ldots,\ \left(x^{(i)}\right)_j \in \Omega_i,\ j = 1, 2, \ldots\right\}$, and σ_i ; $X_i \to X_i$ the shift map. Let $\Phi_1, \Phi_2 \ldots, \Phi_k \in C\left(X_1^r\right)$, we consider the special case of that any Φ_i depends on the first m coordinates of each variable. The case $k = 1$, was presented in [5] Let $\mu \in \mathcal{M}(\alpha, \{\Phi_1, \Phi_2 \ldots, \Phi_k)\}$, so $\sum_{i=1}^{k}\int_{X_1^r} \Phi_i d\mu^{\otimes r}$ depends on the values of μ on cylinders $\mathcal{C}_m$ of length m. In a similar way to [5], can be seen that the supreme is attained on a Markov measure, which for $m = 1$ is a Bernoulli measure $\mu_{\mathbf{p}}$, associated to a probability vector $\mathbf{p}$. Let $x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)} \in X_1$, and consider the particular of any Φ_i of the form

$\Phi_i = \Phi_i\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)}\right) = \varphi_1^{(i)}\left(\left(x_1^{(i)}\right)_1\right)\varphi_2^{(i)}\left(\left(x_2^{(i)}\right)_1\right)\ldots\varphi_r^{(i)}\left(\left(x_r^{(i)}\right)_1\right), = 1, 2, \ldots k$, therefore if $\mu_{\mathbf{p}}$ is the maximizing Bernoulli measure, for probability vector $\mathbf{p}$, then we have

$$\mu_{\mathbf{p}}^{\otimes r}\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)}\right) = \mu_{\mathbf{p}}\left(\left(x_1^{(i)}\right)_1\right)\mu_{\mathbf{p}}\left(\left(x_2^{(i)}\right)_1\right)\ldots\mu_{\mathbf{p}}\left(\left(x_r^{(i)}\right)_1\right) \tag{3.3}$$

and so, if

$$S(\mathbf{p}) = \sum_{i=1}^{k} \int_{X_1^r} \Phi_i\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)}\right) d\mu_{\mathbf{p}}^{\otimes r}\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)}\right)$$
$$= \sum_{i=1}^{k} \int_{X_1^r} \Phi_i\left(x_1^{(i)}, x_2^{(i)}, \ldots, x_r^{(i)}\right) d\mu_{\mathbf{p}}\left(\left(x_1^{(i)}\right)_1\right) \mu_{\mathbf{p}}\left(\left(x_2^{(i)}\right)_1\right) \ldots \mu_{\mathbf{p}}\left(\left(x_r^{(i)}\right)_1\right) \tag{3.4}$$

then for a probability vector $\mathbf{p} = (p_0, p_1, \ldots, p_{t-1})$, with $t = card\Omega_1$, is

$$S(\mathbf{p}) = \sum_{i=1}^{k} \prod_{h=1}^{r} \sum_{s=0}^{t-1} \varphi_h^{(i)}(s), p_s. \tag{3.5}$$

Therefore, the entropy must be maximized with respect to probability vectors $\mathbf{p}$ and

$$h^{\mathbf{a}}(K^{\mathbf{a}}_{\{\Phi_1, \Phi_2 \ldots, \Phi_k\}, \alpha}) = \max_{\mathbf{p}: S(\mathbf{p}) = \alpha} \sum_{i=1}^{k} h_{(\tau_{i-1})_*(\mu_{\mathbf{p}})}(\sigma_i). \tag{3.6}$$

For more general shifts, i.e. symbolic spaces of sequences with not all sequences allowed, the condition of **a**-specification is expressed as follows (see for instance Ref. [1]):

Let $(X_1, \sigma_1, \Omega_1), \ldots, (X_k, \sigma_k, \Omega_k)$ be shifts on alphabets $\Omega_1, \ldots, \Omega_k$. The sequences of length n on X_1 (words) allowed by the system (admissible sequences) is denoted by $\mathcal{L}_n(X_1)$ so that the language on X_1 is $\mathcal{L}(X_1) = \bigcup_{n \geq 1} \mathcal{L}_n(X_1)$. The metric considered is

$$d_n^{\mathbf{a}}(x, y) = \max_{i=1,..,k} \left\{ \frac{|\tau_i(x) \wedge \tau_i(y)|}{a_1 + \cdots + a_i} \right\},$$

where

$$|u \wedge v| = \begin{cases} 0, & \text{if } u_1 \neq v_1 \\ \max\left\{n : u_j = v_j \text{for } 1 \leq j \leq n\right\} & \text{if } u_1 = v_1 \end{cases}$$

We say that the shift X satisfies specification if there exists $s \leq M$ (for some integer M) such that, for any two words x and y that are admissible in X, there is a word w of length s such that

$$\tau_i(x)\,\tau_i(w)\,\tau_i(y) \in \mathcal{L}(X_i) \quad \text{for any } i = 1, \ldots, k,$$

the maximizing measure being Markov.

Let $s_i \in (0, 1)$, $i = 1, \ldots, k$, the so-called Manneville–Pomeau maps, are interval maps

$$g_{s_i} : [0, 1] \to [0, 1] : x \to x + x^{1+s_i} \ mod\ 1.$$

Let $f_i(x) = g_{s_i}(x)$ $(i = 1, \ldots, k)$ then following Takens and Verbitskiy [14] can be seen that the sequence $([0, 1], f_1), \ldots, ([0, 1], f_k)$ is conjugate to a sequence of full shifts that satisfy weighted specification. If f_1 is expansive and

$$\varphi(x) = -log\left|f_1'(x)\right|,$$

then there exists a unique absolutely continous f_1-invariant measure which is an equilibrium state for the potential $\varphi(x)$.

In a similar way the logistic sequence $f_i(x) = \alpha_i x(1 - x)$ $(i = 1, \ldots, k)$ stisfy weighted specification for parameters $\alpha_1, \ldots, \alpha_k$ within a set of positive Lebesgue measures.

We finish these examples with the called β-shifts, say the sequence $f_i(x) = \beta_i x - [\beta_i x]$ $(i = 1, \ldots, k)$ with $[\cdot]$ the integer part of $\cdot$, $\beta_i > 1$ and the functions $f_i(x)$ defined from $[0, 1)$ into $[0, 1)$. By the classification of Li and Wu [11], there exist adequate sets of parameters $\beta_1, \ldots, \beta_k$ such that the sequences $([0, 1), f_i)$ $(i = 1, \ldots, k)$ satisfy weighted specification. However, since the β_i-shifts are not continuous in $[0, 1)$, the variational theorem is not applicable to this kind of sequences.

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