RESIDUES ON COMPLEX SPACES

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The purpose of this note is to present a theory of residues for meromorphic differential forms with arbitrary singularities. In this theory, the *residue* $\mathrm{Res}[\tilde{\omega}]$ of a meromorphic p-form $\tilde{\omega}$, defined on a complex manifold X of dimension n, is a $(2n-p-1)$-current on X, with support on the polar set Y of $\tilde{\omega}$.

More generally, we will assume that X is an n-dimensional paracompact complex analytic space, with structural sheaf $\mathcal{O}_X$, that Y is a closed subspace locally defined by one equation, and that $U = X - Y$ is dense in X and has only simple points.

To define $\mathrm{Res}[\tilde{\omega}]$, and the related current $V{*}[\tilde{\omega}]$, the *principal value* of $\tilde{\omega}$, we suppose that $\tilde{\omega}$ is representable as a quotient $\frac{\omega}{\phi}$ on an open set $W \subset X$, where ω is a holomorphic p-form regular on W, and $\phi \in \Gamma(W, \mathcal{O}_X)$ is an equation for Y on W. Then

$$\mathrm{Res}[\tfrac{\omega}{\phi}](\alpha) = \lim_{\delta \to 0} \int_{|\phi| = \delta} \frac{\omega \wedge \alpha}{\phi}$$

and

$$V{*}[\tfrac{\omega}{\phi}](\beta) = \lim_{\delta \to 0} \int_{|\phi| > \delta} \frac{\omega \wedge \beta}{\phi} ,$$

for all smooth forms α and β with compact supports defined on W, of degrees $2n-p-1$ and $2n-p$, respectively, and where the domains $\{|\phi| = \delta\}$ and $\{|\phi| > \delta\}$ are properly oriented.

These local definitions do not depend on the particular representation $\frac{\omega}{\phi}$ of $\tilde{\omega}$, and can be patched to globally defined currents $V{*}[\tilde{\omega}]$ and $\mathrm{Res}[\tilde{\omega}]$.

The proof of the existence of the above limits, and of their independence on the local representations of meromorphic forms, uses Hironaka's resolution of singularities, to reduce the general problem to the case where ϕ has only normal crossings.

This proof will be given in a forthcoming paper by David Lieberman and myself, that will also relate our notion of residue with the theory of Leray, Norguet and Dolbeault, and will include applications to Serre's duality and to the classical duality of a manifold.

In this note we only describe the relationship of $V*$ and Res with the classical cohomology and homology sequences associated with the couple (Y,X).

To this purpose, we consider the exact sequence

$$0 \to \Omega_X^{\cdot} \to \Omega_X^{\cdot}(*Y) \to Q_X^{\cdot} \to 0,$$

where $\Omega_X^{\cdot}$ and $\Omega_X^{\cdot}(*Y)$ are the complexes of holomorphic forms on X and of meromorphic forms on X with poles on Y, respectively, and $Q_X^{\cdot}$ is the quotient complex, and the exact sequence

$$0 \to {}'\mathcal{D}_{\cdot Y^\infty} \to {}'\mathcal{D}_{\cdot X} \to {}'\mathcal{D}_{\cdot X/Y^\infty} \to 0,$$

where ${}'\mathcal{D}_{\cdot X}$ is the complex of currents on X, ${}'\mathcal{D}_{\cdot Y^\infty}$ the subcomplex of the currents on X with support on Y, and ${}'\mathcal{D}_{\cdot X/Y^\infty}$ is the quotient complex.

One can prove that $V*$ and Res define homomorphisms of complexes

$$V* : \Omega_X^p(*Y) \to {}'\mathcal{D}_{2n-p,\, X/Y^\infty},$$

$$\text{Res}: \quad Q_X^{\cdot} \to {}'\mathcal{D}_{2n-p-1,\, Y^\infty},$$

which, together with the standard map

$$V: \Omega_X^p \to {}'\mathcal{D}_{2n-p,\, X}$$

constructed by integration, provide a commutative diagram of hypercohomology

$$
(1)\qquad
\begin{array}{ccccccc}
\to \mathbb{H}^{p}(X;\Omega^{\cdot}_{X}) & \to & \mathbb{H}^{p}(X;\Omega^{\cdot}_{X}(*Y)) & \to & \mathbb{H}^{p}(X;Q^{\cdot}_{X}) & \to & \\
\downarrow V & & \downarrow V* & & \downarrow \mathrm{Res} & & \\
\to H_{2n-p}\Gamma(X;{}'\mathcal{D}_{\cdot X}) & \to & H_{2n-p}\Gamma(X;{}'\mathcal{D}_{\cdot X/Y^{\infty}}) & \to & H_{2n-p-1}\Gamma(X;{}'\mathcal{D}_{\cdot Y^{\infty}}) & \to & .
\end{array}
$$

The hyperhomologies of the bottom line have been replaced by homologies of global sections; they are isomorphic, since the different sheaves of currents involved are acyclic.

Consider now the commutative diagram

$$
(2)\qquad
\begin{array}{ccccccc}
\longrightarrow H^{p}(X;\mathbb{C}) & \longrightarrow & H^{p}(U;\mathbb{C}) & \longrightarrow & H^{p+1}_{Y}(X;\mathbb{C}) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow H_{2n-p}(X;\mathbb{C}) & \longrightarrow & H_{2n-p}(U;\mathbb{C}) & \longrightarrow & H_{2n-p-1}(Y;\mathbb{C}) & \longrightarrow & ,
\end{array}
$$

where the top line is the classical sequence of cohomology with closed supports (and supports in Y) associated with the couple (Y,X), and where the bottom line is the exact sequence of Borel-Moore homology (closed supports) associated with (Y,X). The vertical homomorphisms are constructed by cap-product with the fundamental class of X.

The relationship between these diagrams is summarized in the following

Theorem. There is a canonical homomorphism from diagram (1) *to diagram* (2).

a) *At the V-level, we have a commutative diagram*

$$\begin{array}{ccccc}
H^p(X;\mathbb{C}) & \xrightarrow{e} & \mathbb{H}^p(X;\Omega^\cdot_X) & \xrightarrow{I} & H^p(X;\mathbb{C}) \\
\downarrow & & \downarrow V & & \downarrow \\
H_{2n-p}(X;\mathbb{C}) & \xrightarrow{I^*} & H_{2n-p}\Gamma(X;{}'\mathcal{D}_{\cdot X}) & \xrightarrow{e^*} & H_{2n-p}(X;\mathbb{C})
\end{array} ,$$

where e is an edge homomorphism, I is given by integration, and e^* and I^* are (essentially) their duals. No assumptions on the paracompact space X are needed at this level. $I \circ e$ and $e^* \circ I^*$ are isomorphisms.

b) *Suppose X is reduced and $U = X - Y$ is regular and dense in X. At the $V*$-level, we have the diagram*

$$\begin{array}{ccccc}
 & & \mathbb{H}^p(X;\Omega^\cdot_X(*Y)) & \xrightarrow{I} & H^p(U;\mathbb{C}) \\
 & & \downarrow V* & & \downarrow \cap \\
H_{2n-p}(U;\mathbb{C}) & \xrightarrow{I^*} & H_{2n-p}\Gamma(X;{}'\mathcal{D}_{\cdot X/Y^\infty}) & \xrightarrow{e^*} & H_{2n-p}(U;\mathbb{C})
\end{array} ;$$

I is an isomorphism (by Grothendieck's theorem), as are $\cap$ (Poincaré duality) and $e^* \circ I^*$.

c) *With the same hypothesis as in* b), *we have at the* Res-*level a diagram*

$$\begin{array}{ccccc}
H^p_Y(X;\mathbb{C}) & \xrightarrow{e} & \mathbb{H}^p(X;Q^\cdot_X) & \xrightarrow{u} & H^p_Y(X;\mathbb{C}) \\
\downarrow & & \downarrow \mathrm{Res} & & \downarrow \\
H_{2n-p-1}(Y;\mathbb{C}) & \xrightarrow{I^*} & H_{2n-p-1}\Gamma(X;{}'\mathcal{D}_{\cdot Y^\infty}) & \xrightarrow{e^*} & H_{2n-p-1}(Y;\mathbb{C})
\end{array}$$

where $u \circ e$ and $e^* \circ I^*$ are canonical identifications.

d) *In the case X and Y are manifolds, the homomorphism from* (1) *to* (2) *is an isomorphism.*

The relation between this definition of residue and that of Leray-Norguet is given by the

Theorem. *Let* $\tilde{\omega} = \frac{d\phi}{\phi} \wedge \psi$, *where* $\psi \in \Omega^{p-1}(X)$, $\phi \in \Gamma(X, 0_X)$ *and* X *is regular. Then*

$$\mathrm{Res}[\frac{d\phi}{\phi} \wedge \psi] = (-1)^{p+1}\, I[\phi_z^{-1}(0)] \wedge \psi \ ,$$

where $\phi_z^{-1}(0)$ *denotes the cycle inverse image of* 0 *by* ϕ *(with appropriate multiplicities),* $I[\phi_z^{-1}(0)]$ *is the integration current on* $\phi_z^{-1}(0)$, *and* $I[\phi_z^{-1}(0)] \wedge \psi$ *is the current* $\alpha \mapsto I[\phi_z^{-1}(0)](\psi \wedge \alpha)$.

When Y is a manifold, one can follow the map

$$H_Y^p(X;\mathbb{C}) \to H_{2n-p-1}(Y;\mathbb{C})$$

by the Poincaré duality isomorphism $H_{2n-p-1}(Y;\mathbb{C}) \to H^{p-1}(Y;\mathbb{C})$. In the conditions of the last theorem, if $\tilde{\omega}$ represents a cohomology class of $H_Y^p(X;\mathbb{C})$, then ψ represents the image of this class by the composed map:

$$H_Y^p(X;\mathbb{C}) \to H^{p-1}(Y;\mathbb{C}).$$

We want to remark that the local definitions of Res and $V*$ have been first proposed by L. Bungart in 1967. Our proof of their existence is an improvement of a proof also offered by Bungart (unpublished).

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