ROBE'S RESTRICTED THREE-BODY PROBLEM REVISITED

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Abstract. Robe's restricted three-body problem is reanalyzed with a view to incorporate a new assumption, namely that the configuration of the fluid body is that described by an hydrostatic equilibrium figure (Roche's ellipsoid). In the concomitant gravitational field a full treatment of the buoyancy force is given. The pertinent equations of motion are derived, the linear stability of the equilibrium solution is studied and the connection between the effect of the buoyancy forces and a perturbation of the Coriolis force is pointed out.

Key words: Restricted three body problem, buoyancy forces.

1. Introduction

A new kind of restricted three-body problem that incorporates the effect of buoyancy forces was introduced by Robe in 1977. He regards one of the two principal bodies as a rigid spherical shell of mass m_1, filled with an homogeneous incompressible fluid of density ρ_1. The other one is a point mass m_2 located outside the shell. The particle of negligible mass (third body) has density ρ_3 and moves inside the shell under two influences: (a) the gravitational attraction of the principal bodies and (b) the buoyancy force of the fluid ρ_1. The Robe model may provide some insight into the problem of small oscillations of the earth's core in the gravitational field of the earth–moon system.

Robe considered two situations (1) that in which m_2 describes a circular orbit around the shell and (2) the case of elliptical orbits for m_2, assuming the shell empty (i.e. $\rho_1 = 0$) or the densities ρ_1 and ρ_3 to be equal. In both instances the center of the shell is an equilibrium point for the third body, which led him to study the conditions for its linear stability.

Shrivastava and Garain (1991) studied the effect of a small perturbation in the Coriolis and centrifugal forces on the location of the equilibrium point. They considered the circular case with equal densities ($\rho_1 = \rho_3$) and evaluated the concomitant shift in the location of the equilibrium point.

In deriving the expression for the buoyancy force **E**, both Robe (1977) and Shrivastava *et al.* (1991) assumed that the pressure field of the fluid ρ_1 has spherical symmetry around the center of the shell, in accordance with its assumed spherical shape. However, they took into account just one of the three components of the

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pressure field: *that due to the own gravitational field of the fluid ρ_1 itself.* The remaining two contributions are: (1) that originating in the attraction of m_2 and (2) that arising from the centrifugal force. These, in turn, give raise to additional components of the buoyancy force. (Notice that the component due to the attraction of m_2 lacks spherical symmetry.)

The purpose of the present effort is precisely that of studying the effect of these two contributions in connection with the dynamics of Robe's model. To this effect, we shall consider that the fluid body m_1 adopts the shape of an ellipsoidal figure of hydrostatic equilibrium, specifically, a Roche's ellipsoid (Chandrasekhar, 1987). By recourse to the analytical expressions that obtain for the pressure field in this situation, a full account of the effects of the buoyancy forces can be given.

The pertinent equations of motion will be discussed in Section 2, while the important case $\rho_1 = \rho_3$ is considered in Section 3. Section 4 deals with the stability of the equilibrium point. The relation between the buoyancy force acting on a small particle moving in a uniformly rotating fluid in hydrostatic equilibrium and perturbations in the Coriolis force is addressed in Section 5. Finally, some conclusions are drawn in Section 6.

2. Equations of Motion

In what follows we will take that the primary m_1 is described by a Roche ellipsoid. Since one deals here with an equilibrium figure, there is no need to assume the existence of a rigid outer shell. According to the framing of Roche's problem, we suppose that the primary m_1 describes a circular orbit around the secondary m_2 in such a way that the concomitant relative configuration remains unchanged.

In order to fix the notation let R stand for the distance between the centers of mass of the primaries, while Ω denotes the constant angular velocity of rotation about their common center of mass. Adopt a uniformly rotating coordinate system $Ox_1x_2x_3$, with origin at the center of mass of m_1, Ox_1 pointing towards m_2 and Ox_1x_2 being the orbital plane of m_2 around m_1.

Assume for the angular velocity Ω the 'Keplerian' value

$$\Omega^2 = \frac{G(m_1 + m_2)}{R^3} . \tag{1}$$

If in the Taylor expansion of the gravitational potential due to m_2, only terms up to second order in x_i are retained, the hydrodynamical equations in the rotating frame read (Chandrasekhar, 1987)

$$\begin{aligned} \rho_1\, \mathrm{d}u_i/\mathrm{d}t &= -\partial P/\partial x_i + \rho_1(\partial/\partial x_i) \\ &\quad \times \left[B + \frac{1}{2}\,\Omega^2(x_1^2 + x_2^2) + \mu\left(x_1^2 - \frac{1}{2}\,x_2^2 - \frac{1}{2}\,x_3^2\right)\right] \\ &\quad + 2\rho_1\Omega\varepsilon_{il3}u_l, \quad (i = 1, 2, 3) \end{aligned} \tag{2}$$

where u_i are the components of the fluid velocity field in the rotating frame, P is the pressure field, B stands for the gravitational potential due to the fluid mass, and μ is given by

$$\mu = Gm_2/R^3. \tag{3}$$

For hydrostatic equilibrium, Equations (2) can be recast in the simple fashion

$$\nabla\left\{B + \frac{1}{2}\,\Omega^2(x_1^2 + x_2^2) + \mu\left(x_1^2 - \frac{1}{2}\,x_2^2 - \frac{1}{2}\,x_3^2\right) - \frac{P}{\rho_1}\right\} = 0. \tag{4}$$

Roche's ellipsoids constitute solutions to Equation (4). They are ellipsoidal figures with semiaxes a_1, a_2 and a_3 parallel, respectively, to the coordinates Ox_1, Ox_2, and Ox_3. The semiaxes a_i verify the following relations (Chandrasekhar, 1987)

$$[(3+p)a_1^2 + a_3^2]\mu^* = 2[A_1a_1^2 - A_3a_3^2], \tag{5}$$

$$[pa_2^2 + a_3^2]\mu^* = 2[A_2a_2^2 - A_3a_3^2], \tag{6}$$

where

$$p = m_1/m_2, \tag{7}$$

and

$$\mu^* = \mu/(\pi G\rho_1), \tag{8}$$

while the quantities A_i are given by

$$A_i = a_1a_2a_3\int_0^\infty \frac{du}{\Delta(a_i^2 + u)}, \quad (i = 1, 2, 3), \tag{9}$$

with

$$\Delta^2 = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u). \tag{10}$$

With these definitions, it is plain from Equations (1) and (3) that

$$\Omega^2 = (1+p)\mu. \tag{11}$$

The potential B at an internal point x_i of the homogeneous ellipsoid is given by

$$B = \pi G\rho_1(I - A_1x_1^2 - A_2x_2^2 - A_3x_3^2), \tag{12}$$

where I stands for

$$I = a_1^2A_1 + a_2^2A_2 + a_3^2A_3. \tag{13}$$

Our interest lies in describing the motion of a small mass m_3 ($m_3 \ll m_{1,2}$) within the Roche ellipsoid. This small mass moves under the influence of three forces (per unit mass of the affected particle):

(1) The attraction of the fluid ρ_1

$$\text{grad}\, B, \tag{14}$$

(2) the gravitational field due to the point mass m_2

$$\text{grad}\left\{\mu R x_1 + \mu\left(x_1^2 - \frac{1}{2}\, x_2^2 - \frac{1}{2}\, x_3^2\right)\right\}, \tag{15}$$

where, for the sake of consistence with the above description of hydrostatic equilibrium, only terms up to second order have been kept, and

(3) the buoyancy force $\mathbf{E}$ arising in the fluid ρ_1.

In order to obtain the expression for the buoyancy force $\mathbf{E}$, we must consider the pressure field inside the fluid, assumed to be in hydrostatic equilibrium in the rotating reference frame. From elementary hydrostatics it is known that the buoyancy force acts on a small body of volume V according to

$$\mathbf{E}^* = -V\, \text{grad}\, P, \tag{16}$$

so that the buoyancy force per unit mass is

$$\mathbf{E} = \frac{\mathbf{E}^*}{V\rho_3}, \tag{17}$$

and, keeping Equation (4) in mind,

$$\mathbf{E} = -(\rho_1/\rho_3)\nabla\left\{B + \frac{1}{2}\,\Omega^2(x_1^2 + x_2^2) + \mu\left(x_1^2 - \frac{1}{2}\, x_2^2 - \frac{1}{2}\, x_3^2\right)\right\}. \tag{18}$$

The combined action of the forces given by (14), (15), and (18) upon our small particle can be expressed with the help of the potential

$$\begin{aligned} v = &\left\{B + \mu R x_1 + \mu\left(x_1^2 - \frac{1}{2}\, x_2^2 - \frac{1}{2}\, x_3^2\right)\right\} \\ &- (\rho_1/\rho_3)\left\{B + \frac{1}{2}\,\Omega^2(x_1^2 + x_2^2) + \mu\left(x_1^2 - \frac{1}{2}\, x_2^2 - \frac{1}{2}\, x_3^2\right)\right\}. \end{aligned} \tag{19}$$

Regarding the total mass of the primaries, $m_1 + m_2$, as the unit of mass, and selecting the units of time and length in such a way that $\Omega = 1$, and $R = 1$, the quantity μ of Equation (3) becomes numerically equal to the ratio $m_2/(m_1 + m_2)$ so that (cf. Equation (7))

$$\mu = (1 + p)^{-1}, \tag{20}$$

and Newton's gravitational constant equals unity. With this choice of units, the equations of motion for m_3, which are the focus of our concern, adopt the appearance

$$\mathrm{d}^2x_1/\mathrm{d}t^2 - 2\,\mathrm{d}x_2/\mathrm{d}t = \partial U/\partial x_1, \tag{21}$$

$$\mathrm{d}^2x_2/\mathrm{d}t^2 + 2\,\mathrm{d}x_1/\mathrm{d}t = \partial U/\partial x_2, \tag{22}$$

$$\mathrm{d}^2x_3/\mathrm{d}t^2 = \partial U/\partial x_3, \tag{23}$$

where

$$\begin{aligned} U &= v + \frac{1}{2}\,[(x_1-\mu)^2 + x_2^2] \\ &= (1-\rho_1/\rho_3)\left\{B + \frac{1}{2}\,(x_1^2 + x_2^2) + \mu\left(x_1^2 - \frac{1}{2}\,x_2^2 - \frac{1}{2}\,x_3^2\right) + \frac{\mu^2}{2}\right\}. \end{aligned} \tag{24}$$

The equations of motion (21)–(23) can be recast in the convenient fashion

$$\mathrm{d}^2x_1/\mathrm{d}t^2 - 2\,\mathrm{d}x_2/\mathrm{d}t = D[1 + 2\mu - C_1]x_1, \tag{25}$$

$$\mathrm{d}^2x_2/\mathrm{d}t^2 + 2\,\mathrm{d}x_1/\mathrm{d}t = D[1 - \mu - C_2]x_2, \tag{26}$$

$$\mathrm{d}^2x_3/\mathrm{d}t^2 = D[-\mu - C_3]x_3, \tag{27}$$

where

$$D = 1 - \frac{\rho_1}{\rho_3}, \tag{28}$$

and

$$\begin{aligned} C_i &= 2(\pi G \rho_1)A_i \\ &= 2(\mu/\mu^*)A_i, \quad i = 1,2,3. \end{aligned} \tag{29}$$

Notice that μ^* and the ratio a_2/a_1 are determined by Equations (5) and (6) as a function of the input data p (or μ) and a_3/a_1. These two ratios allow one to evaluate the A_i and *a posteriori* the C_i. Summing up, the whole dynamics of our problem is characterized by just three adimensional figures, namely, p, D, and a_3/a_1.

3. The Case $\rho_1 = \rho_3$

We consider here that special instance (discussed by Robe (1977) and Shrivastava *et al.* (1991)) in which the small particle and the fluid have equal densities. The concomitant equations of motion adopt the form

$$\mathrm{d}^2x_1/\mathrm{d}t^2 - 2\,\mathrm{d}x_2/\mathrm{d}t = 0, \tag{30a}$$

$$d^2x_2/dt^2 + 2\,dx_1/dt = 0, \tag{30b}$$

$$d^2x_3/dt^2 = 0, \tag{30c}$$

so that their integration yields

$$x_1 = \alpha\cos(2t+\gamma) + \beta, \tag{31a}$$

$$x_2 = -\alpha\sin(2t+\gamma) + \delta, \tag{31b}$$

$$x_3 = \varepsilon t + \chi, \tag{31c}$$

where α, β, δ, γ, ε, and χ are integration constants.

It is worthwhile to point out that all triplets (x_1, x_2, x_3) are equilibrium points of the equations of motion (30). This fact has a simple interpretation: as the fluid is assumed to be in hydrostatic equilibrium in the rotating frame, all the elements of the fluid remain at rest in that reference system. If the small particle has the same density as that of the fluid, it is indistinguishable from any of its elements. Thus, it will be in equilibrium everywhere. We see that in this instance the new terms in the buoyancy force have important effects in the behaviour of Robe's model. Notice that (1) neglecting the tidal deformation of the fluid body m_1 due to the other primary m_2 and (2) ignoring the components of the buoyancy force arising from, respectively, the gravitational attraction of m_2 and the appropriate centrifugal forces (Robe, 1977; Shrivastava *et al.*, 1991), has the effect of making the center of the fluid the *only* equilibrium point. It should be pointed out that, in Robe's original treatment, when there is no fluid inside the shell, i.e., when $\rho_1 = 0$, the center is indeed the only equilibrium point.

4. The Stability of the Equilibrium Point

It is easy to verify that the center of m_1 (of coordinates $(0, 0, 0)$ is an equilibrium point of the equations of motion (25)–(27). It is then of some importance to study its stability.

From Equation (27) one easily ascertains that the motion parallel to the x_3 axis is stable when the small particle is denser than the medium ($D > 0$). The remaining Equations (25) and (26) admit solutions of the form

$$x_1 = \xi\,e^{Lt}, \tag{32}$$

$$x_2 = \varphi\,e^{Lt}, \tag{33}$$

with L given by the biquadratic equation

$$\begin{aligned} L^4 &+ [4 - D(2+\mu - C_1 - C_2)]L^2 \\ &+ D^2(1+2\mu - C_1)(1-\mu - C_2) = 0. \end{aligned} \tag{34}$$

Setting $S = L^2$, we obtain

$$S_{1,2} = \frac{1}{2}\{D(Q_1 + Q_2) - 4 \pm \Delta^{1/2}\}, \tag{35}$$

where

$$Q_1 = 1 + 2\mu - C_1, \tag{36}$$

$$Q_2 = 1 - \mu - C_2, \tag{37}$$

and

$$\Delta = (Q_1 - Q_2)^2 D^2 - 8(Q_1 + Q_2)D + 16. \tag{38}$$

It is plain from Equation (35) that

$$S_1 + S_2 = D(Q_1 + Q_2) - 4, \tag{39}$$

and

$$S_1 S_2 = D^2 Q_1 Q_2. \tag{40}$$

The equilibrium situation is a stable one if S_1 and S_2 are real and negative. Therefore we must have

$$\Delta > 0, \tag{41}$$

$$S_1 + S_2 < 0, \tag{42}$$

and

$$S_1 S_2 > 0. \tag{43}$$

In what follows it is assumed that D is a positive number. This is tantamount to stating that the small particle is denser than the surrounding fluid. Since $C_i > 0$, it is easy to see that condition (42) is automatically fulfilled.

In order to satisfy condition (43), the quantities Q_1 and Q_2 must have the same sign. We will now show that, within the present generalization of Robe's restricted three-body problem, those quantities are both negative. From (5), (20) and (29) we have

$$\begin{aligned} C_1 &= \frac{1}{1 - (A_3/A_1)(a_3/a_1)^2}\left[\frac{3 + p + (a_3/a_1)^2}{1 + p}\right] \\ &= \frac{1}{1 - (A_3/A_1)(a_3/a_1)^2}\left[1 + 2\mu + \frac{(a_3/a_1)^2}{1 + p}\right] \\ &> 1 + 2\mu. \end{aligned} \tag{44}$$

In a similar way we have (cf. Equations (4), (20) and (29))

$$\begin{aligned} C_2 &= \frac{1}{1-(A_3/A_2)(a_3/a_2)^2}\left[\frac{p+(a_3/a_2)^2}{1+p}\right] \\ &= \frac{1}{1-(A_3/A_2)(a_3/a_2)^2}\left[1-\mu+\frac{(a_3/a_2)^2}{1+p}\right] \\ &> 1-\mu. \end{aligned} \tag{45}$$

From (44) and (45) it is clear that Q_1 and Q_2 are both negative. Thus, the stability conditions (42) and (43) are always fulfilled. This fact is independent of the value adopted by the small body's density ρ_3. Let us now consider the remaining stability condition (41). It is plain that we have $\Delta > 0$, since we assumed $D > 0$, and we have proved that $Q_{1,2} < 0$. Summing up, *in this formulation of Robe's restricted problem, the equilibrium point is always stable.*

5. Relations between the Buoyancy and the Coriolis Forces

Let us consider now the general situation of a small body of density ρ_2 that moves within a fluid mass of constant density ρ_1. The fluid is supposed to be in hydrostatic equilibrium in an uniformly rotating frame. If Ω, ϕ, and P denote, respectively, the angular velocity of rotation, the gravitational potential, and the pressure field, hydrostatic equilibrium entails (we assume the z axis parallel to the rotation axis and its origin fixed in an inertial system)

$$\nabla\left[\frac{1}{2}\,\Omega^2(x^2+y^2)+\phi-P/\rho_1\right]=0. \tag{46}$$

The forces (per unit mass) acting upon the small body are: (1) the gravitational attraction, given by

$$\mathbf{F}=\nabla\phi, \tag{47}$$

and (2) the buoyancy force, that reads

$$\mathbf{E}=-(\rho_1/\rho_2)\nabla\left[\frac{1}{2}\,\Omega^2(x^2+y^2)+\phi\right]. \tag{48}$$

The action of both forces upon the small particle can be accounted for by the total potential

$$v=\phi-(\rho_1/\rho_2)\left[\frac{1}{2}\,\Omega^2(x^2+y^2)+\phi\right]. \tag{49}$$

The pertinent equations of motion (in the rotating frame) adopt the appearance

$$\mathrm{d}^2x/\mathrm{d}t^2-2\Omega\,\mathrm{d}y/\mathrm{d}t=\partial U/\partial x, \tag{50}$$

$$d^2y/dt^2 + 2\Omega\, dx/dt = \partial U/\partial y, \tag{51}$$

$$d^2z/dt^2 = \partial U/\partial z, \tag{52}$$

where

$$U = v + \frac{1}{2}\,\Omega^2(x^2+y^2) = (1-\rho_1/\rho_2)\left[\frac{1}{2}\,\Omega^2(x^2+y^2)+\phi\right]. \tag{53}$$

By choosing a new (independent) variable τ given by

$$\tau = t(1-\rho_1/\rho_2)^{1/2}, \tag{54}$$

the equations of motion can be recast in the form

$$d^2x/d\tau^2 - 2\Omega(1-\rho_1/\rho_2)^{-1/2}\, dy/d\tau = \partial V/\partial x, \tag{55}$$

$$d^2y/d\tau^2 + 2\Omega(1-\rho_1/\rho_2)^{-1/2}\, dx/d\tau = \partial V/\partial y, \tag{56}$$

$$d^2z/d\tau^2 = \partial V/\partial z, \tag{57}$$

where

$$V = \frac{1}{2}\,\Omega^2(x^2+y^2)+\phi. \tag{58}$$

We see that V looks like the usual effective potential in the rotating frame. It takes into account just the gravitational forces accounted for by ϕ. Hence, Equations (55)–(57) are the familiar equations of motion in a rotating frame (*without buoyancy terms*), but with a perturbed Coriolis force. Therefore, the effect of the buoyancy forces might be thought as being *equivalent to a perturbation of the Coriolis force.* One could then speculate on possible connections between the present problem and the study of effects due to perturbations in the Coriolis force as described by Shrivastava and Garain (1991).

6. Conclusions

In the present work we have revisited Robe's restricted three-body problem under the assumption that the fluid body assumes the shape of the Roche ellipsoid. This has allowed for a full account of the buoyancy force, without neglecting any component.

The pertinent equations of motion where derived and special attention was paid to that important instance in which the density of the fluid equals that of the small particle. In this case, it was found that any point inside the fluid is an equilibrium one.

In the general case, the only equilibrium point is the ellipsoid's center. Its stability was analyzed and, under the assumption that the smaller particle is denser than the surrounding medium, it can be ascertained that the equilibrium is always stable.

A tentative connection between the effect of the buoyancy forces and a perturbation of the Coriolis force was pointed out.

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