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A Linear Theory of Gravitation (*).

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DESER and LAURENT (1) found recently a linear, nonlocal theory of gravitation which satisfies the experimental tests of general relativity. Their theory leads to a tensor field for the static point particle, identical to the linearized Schwarzschild solution, namely (2):

$$\left\{\begin{aligned} &\overset{s}{\psi}_{00} = \frac{m}{r}\,, \qquad \overset{s}{\psi}_{ij} = \frac{m x_i x_j}{r^3}\,, \qquad \overset{s}{\psi}_{0i} = 0\,,\\ &g_{\mu\nu} = \eta_{\mu\nu} + 2\varkappa \overset{s}{\psi}_{\mu\nu}\,,\\ &\eta_{\mu\nu} = (-1,\,1,\,1,\,1)\,. \end{aligned}\right. \tag{1}$$

In this paper we present an alternative theory which we think simpler and more transparent than D.L.'s theory, and, further, it leads to the same results. (Both theories coincide when the energy-momentum tensor is conserved.)

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(1) S. DESER and B. E. LAURENT: *Ann. of Phys.*, **50**, 76 (1968). Hereafter referred to as D-L.

(2) See, for example: P. G. BERGMANN: *Introduction to the Theory of Relativity* (Englewood Cliffs., N. J.), p. 203.

In order to simplify notations we introduce the following projection operators:

$$
(2)\qquad \begin{cases} P^2_{\mu\nu;\varrho\sigma} = \frac{1}{2}(P_{\mu\varrho}P_{\nu\sigma} + P_{\mu\sigma}P_{\nu\varrho}) - \frac{1}{3}P_{\mu\nu}P_{\varrho\sigma}\,, \\ P^1_{\mu\nu;\varrho\sigma} = \frac{1}{2}(\eta_{\mu\varrho}\partial_{\nu\sigma} + \eta_{\nu\sigma}\partial_{\mu\varrho} + \eta_{\nu\varrho}\partial_{\mu\sigma} + \eta_{\mu\sigma}\partial_{\nu\varrho} - 4\partial_{\mu\nu}\partial_{\varrho\sigma})\,, \\ P^0_{\mu\nu;\varrho\sigma} = \frac{1}{3}P_{\mu\nu}P_{\varrho\sigma}\,, \\ \bar{P}^0_{\mu\nu;\varrho\sigma} = \partial_{\alpha\beta}\partial_{\mu\nu}\,, \end{cases}
$$

where

$$
(3)\qquad P_{\alpha\beta} = \eta_{\alpha\beta} - \partial_{\alpha\beta} \qquad \text{and} \qquad \partial_{\alpha\beta} = \frac{\partial_\alpha\partial_\beta}{\Box}\,.
$$

These projectors are all symmetric in μ, ν and in ϱ, σ independently and also symmetric under the interchange $\mu\nu \rightleftarrows \varrho\sigma$.

P^2 is traceless and divergenceless, P^1 is traceless and has zero double divergence, P^0 is divergenceless and $\bar{P}^0$ satisfies

$$
\partial_{\alpha\beta}\bar{P}^0_{\beta\mu;\varrho\sigma} = \bar{P}^0_{\alpha\mu;\varrho\sigma}\,.
$$

Finally, they also satisfy

$$
P^i P^j = \delta^{ij}P^j\,, \qquad \sum P^i = 1\,.
$$

In the case of free massive fields they are respectively spin-two, spin-one and spin-zero projection operators. Thus we shall use this nomenclature although it is not correct for massless fields.

It should be noted that the subspace of « zero spin » is doubly degenerate and so it is possible to choose in general two projectors Q and $\bar{Q}$, depending on a continuous parameter,

$$
Q_{\mu\nu;\varrho\sigma} = Q_{\mu\nu}Q_{\varrho\sigma}\,, \qquad \bar{Q}_{\mu\nu;\varrho\sigma} = \bar{Q}_{\mu\nu}\bar{Q}_{\varrho\sigma}\,,
$$

where

$$
Q_{\alpha\beta} = \frac{\cos\theta}{\sqrt{3}}P_{\alpha\beta} - \sin\theta\,\partial_{\alpha\beta}\,,
$$

$$
\bar{Q}_{\alpha\beta} = \frac{\sin\theta}{\sqrt{3}}P_{\alpha\beta} + \cos\theta\,\partial_{\alpha\beta}\,.
$$

For $\theta = 0$, we get $Q = P$, $\bar{Q} = \bar{P}^0$. Another choice which will prove to be convenient is for

$$
\theta = \frac{2\pi}{3}\,, \qquad Q = P'\,, \qquad \bar{Q} = P''\,,
$$

$$
(4)\qquad \begin{cases} P'_{\mu\nu\varrho\sigma} = \frac{4}{3}P'_{\mu\nu}P'_{\varrho\sigma}\,, \\ P''_{\mu\nu\varrho\sigma} = \frac{1}{4}\eta_{\mu\nu}\eta_{\varrho\sigma}\,, \end{cases}
$$

with

$$
P'_{\mu\nu} = \frac{1}{4}\eta_{\mu\nu} - \partial_{\mu\nu}\,.
$$

It is easily shown that, for the solution (1), we have

(5)
$$P^2\psi^s = \begin{cases} \psi^{s(2)}_{00} = \dfrac{4}{3}\dfrac{m}{r}\,, \\[2ex] \psi^{s(2)}_{ij} = \dfrac{m}{3}\left(\dfrac{\delta_{ij}}{r} + \dfrac{x_i x_j}{r^3}\right), \\[2ex] \psi^{s(2)}_{0i} = 0\,, \end{cases}$$

(6)
$$P^1\psi^s = 0\,,$$

(7)
$$P'\psi^s = \begin{cases} \psi'^s_{00} = -\dfrac{m}{3r}\,, \\[2ex] \psi'^s_{ij} = \dfrac{m}{3r}\left(-\delta_{ij} + \dfrac{2x_i x_j}{r^2}\right), \\[2ex] \psi'^s_{0i} = 0\,, \end{cases}$$

$$P''\psi^s = 0\,.$$

It is interesting to note that the linearized Schwarzschild solution belongs to the space spanned by the projectors P^2 and P'; *i.e.*

(8)
$$(P^2 + P')\psi^s = \psi^s\,.$$

We now consider to theory of massless spin-2 particles. The Fierz-Pauli ([3]) equation for this field is

(9)
$$G_{\mu\nu}(\phi) = \Box\phi_{\mu\nu} - \phi^{\alpha}_{\mu,\alpha\nu} - \phi^{\alpha}_{\nu,\alpha\mu} + \phi^{\alpha}_{\alpha,\mu\nu} - \eta_{\mu\nu}(\Box\phi^{\alpha}_{\alpha} - \phi^{\alpha\beta}_{\ \ ,\alpha\beta}) = 0\,.$$

It can be written, in our notation,

(10)
$$G_{\mu\nu}(\phi) = \Box(P^2_{\mu\nu\varrho\sigma} - 2P^0_{\mu\nu\varrho\sigma})\phi^{\varrho\sigma} = 0\,,$$

or simply

$$\Box(P^2 - 2P^0)\phi = 0\,.$$

It should be noted that $P^2 - 2P^0$ is the only linear combination of the projection operators (cf. 2), which is divergenceless, gauge invariant ([4]) and whose d'Alembertian is nonsingular. (It does not contain $\Box^{-1}$.)

([3]) M. Fierz and W. Pauli: *Proc. Roy. Soc.*, **173**, 211 (1939). See also G. Wentzel: *Quantum Theory of Fields*, Chap. VI (New York, 1949).

([4]) Invariance relative to the transformation $\phi_{\mu\nu} \to \phi_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$.

This equation is derived from the Lagrangian

$$\left\{\begin{aligned} &\mathscr{L} = \tfrac{1}{2}\phi^{\mu,\nu}_{\ ,\alpha}\phi_{\mu\nu,}^{\ \ \ \alpha} - \tfrac{1}{2}\phi^{\mu}_{\mu,\alpha}\phi^{\nu,\alpha}_{\nu} - \phi^{\alpha\nu}_{\ ,\alpha}\phi^{\beta}_{\nu,\beta} + \phi^{\alpha\nu}_{\ ,\alpha}\phi^{\beta}_{\beta,\nu}\,, \\ &\mathscr{L} = \tfrac{1}{2}\phi(P^2 - 2P^0)\Box\phi + \text{divergences}\,. \end{aligned}\right. \tag{11}$$

If we impose on $\phi_{\mu\nu}$ the Hilbert gauge

$$\partial^\nu(\phi_{\nu\mu} - \tfrac{1}{2}\eta_{\mu\nu}\phi^\alpha_\alpha) = 0\,, \tag{12}$$

we obtain, by the usual procedure, a Hamiltonian density which, for a plane wave, takes the form

$$\mathscr{H} = \tfrac{1}{2}\omega^2\phi_{\mu\nu}\phi^{\mu\nu}\,. \tag{13}$$

Using again gauge (12), we obtain (when the third axis is taken along the direction of propagation)

$$\mathscr{H} = 2\omega^2(\phi_{12}\phi_{12} + \phi_{11}\phi_{11})\,, \tag{14}$$

which shows that we are dealing with massless spin-two particles, as we have then only two independent polarization states (in spite of having the projectors P^2 and P^0 in the equation of motion (10)).

We want to mantain Hilbert condition (12) even in the presence of interaction. We shall do that by using the Lagrangian-multiplier technique. To simplify notation, we introduce the new field variables

$$\begin{gathered} \overline{\phi}_{\mu\nu} = \phi_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\phi^\alpha_\alpha = [(1 - 2P'')\phi]_{\mu\nu}\,, \\ (1 - 2P'')^2 = 1 \qquad (\text{cf. } (4))\,, \end{gathered} \tag{15}$$

for which the Hilbert's gauge goes into the Lorentz's gauge

$$\partial^\nu\overline{\phi}_{\mu\nu} = 0\,. \tag{16}$$

The Lagrangian (11) expressed in terms of $\overline{\phi}_{\mu\nu}$ is

$$\mathscr{L} = \tfrac{1}{2}\overline{\phi}(1 - 2P'')(P^2 - 2P^0)(1 - 2P'')\Box\overline{\phi}\,,$$

which, taking into account (16) and adding the interaction and Lagrange-multiplier terms, reads now

$$\mathscr{L} = \tfrac{1}{2}\overline{\phi}\Box(1 - 2P'')\overline{\phi} + \lambda^\mu\partial^\nu\overline{\phi}_{\mu\nu} + \varkappa\overline{\phi}^{\mu\nu}T_{\mu\nu}\,. \tag{17}$$

(17) leads to the following equations:

$$\Box[(1 - 2P'')\overline{\phi}]_{\mu\nu} = -\varkappa T_{\mu\nu} + \tfrac{1}{2}(\partial_\mu\lambda_\nu + \partial_\nu\lambda_\mu)(=\Box\phi)\,, \tag{18}$$

$$\partial^\nu\overline{\phi}_{\mu\nu} = 0\,. \tag{19}$$

From (18), taking the trace,

$$\Box \bar{\phi}^{\alpha}_{\alpha} = \varkappa T_{\alpha}^{\ \alpha} - \partial_{\alpha} \lambda^{\alpha} ,$$

from which, replacing in (18), we obtain

$$\Box \bar{\phi}_{\mu\nu} = -\varkappa(T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} T^{\alpha}_{\alpha}) + \tfrac{1}{2}(\partial_{\mu}\lambda_{\nu} + \partial_{\nu}\lambda_{\mu} - \eta_{\mu\nu}\partial_{\alpha}\lambda^{\alpha}) . \tag{20}$$

Taking the divergence of this expression

$$\Box \lambda_{\mu} = 2\varkappa \partial^{\nu}(T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} T^{\alpha}_{\alpha}) ,$$

i.e.

$$\lambda_{\mu} = \frac{1}{\Box} 2\varkappa \partial^{\nu}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{\alpha}_{\alpha}\right) , \tag{21}$$

which replaced in (20) is easily seen to give

$$\Box \bar{\phi} = -\varkappa(P^2 - 2P^0)\, T . \tag{22}$$

For the case of the static point source, taking

$$\varkappa T_{00} = 8\pi m \delta(\boldsymbol{r}) , \qquad T_{\mu\nu} = 0 , \qquad \mu \text{ or } \nu \neq 0 , \tag{23}$$

the solution of (22) is

$$\bar{\phi}_{\mu\nu} = \bar{\phi}^{(2)}_{\mu\nu} + \bar{\phi}^{0}_{\mu\nu} , \tag{24}$$

where

$$\left\{ \begin{array}{lll} \bar{\phi}^{(2)}_{00} = \dfrac{4}{3}\dfrac{m}{r} , & \bar{\phi}^{(2)}_{ij} = \dfrac{m}{3}\left(\dfrac{\delta_{ij}}{r} + \dfrac{x_i x_j}{r^3}\right) , & \bar{\phi}^{(2)}_{0i} = 0 , \\ \bar{\phi}^{(0)}_{00} = -\dfrac{4}{3}\dfrac{m}{r} , & \bar{\phi}^{(0)}_{ij} = \dfrac{2m}{3}\left(\dfrac{\delta_{ij}}{r} + \dfrac{x_i x_j}{r^3}\right) , & \bar{\phi}^{(0)}_{0i} = 0 . \end{array} \right. \tag{25}$$

Comparing with Schwarzschild's eqs. (5) and (7) we see that

$$\bar{\phi}_{\mu\nu} - \overset{s}{\psi}_{\mu\nu} = \eta_{\mu\nu}\frac{m}{r} ,$$

i.e. they differ essentially by a scalar field.

This supports the addition of a scalar field to the Lagrangian (7)

$$\mathscr{L} = \tfrac{1}{2}\bar{\phi}\Box(1 - 2P'')\bar{\phi} + \lambda^{\mu}\partial^{\nu}\bar{\phi}_{\mu\nu} + \varkappa\bar{\phi}_{\mu\nu}T^{\mu\nu} + \tfrac{1}{4}\varphi\Box\varphi + \tfrac{1}{2}\varkappa\eta_{\mu\nu}\varphi T^{\mu\nu} , \tag{26}$$

which gives [5] besides eqs. (18) and (19) for the tensor field, the following equations for the scalar field:

$$\Box \varphi = - \varkappa T^{\alpha}_{\alpha} \; ; \tag{27}$$

taking into account (23), we have

$$\varphi = - \frac{2m}{r} \, . \tag{28}$$

But now the total metric (the coefficient of $T^{\mu\nu}$ in (26)) is

$$\psi_{\mu\nu} = \overline{\phi}_{\mu\nu} + \tfrac{1}{2} \eta_{\mu\nu} \varphi \, , \tag{29}$$

and now it is easy to see that, from (25) and (28),

$$\psi^{s}_{\mu\nu} = \overline{\phi}_{\mu\nu} + \tfrac{1}{2} \eta_{\mu\nu} \varphi = \psi_{\mu\nu} \, . \tag{30}$$

So, we have shown that in the static point particle case, our solution coincides with that of D-L and thus contains all the classical tests of general relativity.

This is not the case in general, as the D-L solution has trace identically zero; on the other hand, taking the $\Box$ of the trace of (29) and using (22) and (27) we obtain

$$\Box \psi^{\alpha}_{\alpha} = - 2 \varkappa \partial_{\alpha\beta} T^{\alpha\beta} \, , \tag{31}$$

which cannot be zero unless

$$\partial_{\alpha\beta} T^{\alpha\beta} = 0 \, .$$

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(5) A completely equivalent Lagrangian, without the introduction of a Lagrange multiplier λ and, of course, without a gauge condition, can be written in the following way:

$$\mathscr{L} = \tfrac{1}{2} \phi \Box (P^2 - 2P^0) \phi - \varkappa \phi (P^2 - 2P^0) T + \tfrac{1}{4} \varphi \Box \varphi + \tfrac{1}{2} \varkappa \varphi \, T^{\alpha}_{\alpha} \, .$$