Theoretical and Mathematical Physics, **131**(1): 527–542 (2002)

EVALUATION OF CASIMIR ENERGIES THROUGH SPECTRAL FUNCTIONS

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We consider applications of elliptic differential operators and their associated spectral functions in quantum field theory problems. The role of zeta functions and traces of heat kernels in the regularization of Casimir energies is emphasized, and the renormalization procedure is discussed with simple examples.

1. Zero-point energy in field quantization

We introduce the concept of the Casimir energy in this section, using two different regularization methods to evaluate it in a simple example. The two methods are shown to be consistent after renormalization [1]–[4].

1.1. The free massive neutral scalar field in the unbounded Minkowski space–time. To introduce the problem of the vacuum (ground state) energy in quantum field theory, it is convenient to start by reviewing the simple example of a free massive neutral scalar field in the unbounded Minkowski space–time. The classical Lagrangian density is then given by

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2,$$

where m is the mass of the field ϕ. The Euler–Lagrange formalism implies the classical Klein–Gordon equation of motion

$$(\partial^\mu\partial_\mu + m^2)\phi = 0.$$

The momentum canonically conjugate to the field is defined by

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi,$$

and the classical Hamiltonian is therefore given by

$$H = \int d^3x\,\frac{1}{2}\bigl[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\bigr].$$

The transition to the quantum theory is achieved by replacing the functions ϕ and π with operator-valued functions, replacing the Poisson brackets with commutators, and imposing the equal-time commutation relations

$$[\hat\phi(t,\vec{x}),\hat\pi(t,\vec{x'})] = i\delta^3(\vec{x}-\vec{x'}) \tag{1}$$

with the right-hand sides of the other commutators vanishing.

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 131, No. 1, pp. 98–117, April, 2002.

In the momentum representation with $k^\mu = (k^0, \vec{k}) = (\omega_{\vec{k}}, \vec{k})$ and $\omega_{\vec{k}} = \omega_{-\vec{k}} = \sqrt{\vec{k}^2 + m^2} > 0$, both fields can be expanded as

$$\hat{\phi}(t, \vec{x}) = \int d\tilde{k}\, [\hat{a}(\vec{k})e^{-ikx} + \hat{a}^\dagger(\vec{k})e^{ikx}],$$

$$\hat{\pi}(t, \vec{x}) = -i \int d\tilde{k}\, [\hat{a}(\vec{k})e^{-ikx} - \hat{a}^\dagger(\vec{k})e^{ikx}] = \partial_0 \hat{\phi}(t, \vec{x}),$$

where

$$d\tilde{k} = \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}$$

is the Lorentz-invariant measure. We note that the $\omega_{\vec{k}}$ are the square roots of the eigenvalues of the operator $\Delta + m^2$.

Quantization condition (1) then becomes

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k'})] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k'}).$$

with the right-hand side of all other commutators vanishing. The Hamiltonian operator becomes

$$\widehat{H} = \frac{1}{2} \int d\tilde{k}\, \omega_{\vec{k}} [\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + \hat{a}(\vec{k})\hat{a}^\dagger(\vec{k})] =$$

$$= \frac{1}{2} \int d\tilde{k}\, \omega_{\vec{k}} [2\hat{a}^\dagger(\vec{k})\hat{a}(\vec{k}) + (2\pi)^3 2\omega_{\vec{k}} \delta^3(0)].$$

The vacuum state is defined by

$$\hat{a}(\vec{k})|0\rangle = 0, \qquad \langle 0|0\rangle = 1, \qquad \langle 0|\widehat{H}|0\rangle = \frac{1}{2} \int d^3k\, \omega_{\vec{k}} \delta^3(0).$$

The mathematically meaningless factor $\delta^3(0)$ in the formulas introduced appears because of the infinite size of the system. To give meaning to the last expression, the system considered is enclosed in a cube of volume V with periodic boundary conditions imposed on the field; the infinite-volume limit is then taken with the result

$$\langle 0|\widehat{H}|0\rangle = \frac{V}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{\vec{k}^2 + m^2},$$

where V is the (infinite) space volume.

This shows why the zero-point energy per unit volume diverges in the case of a free scalar field: the divergence is due to the sum of the zero-point energies of an infinite number of oscillators. For the free theory in the unbounded Minkowski space, this vacuum expectation value can be defined to be equal to zero by introducing the normal ordering prescription. But the question then naturally arises about the value of the vacuum energy in the presence of a background field and/or in the case where the quantized field occupies a bounded spatial region and therefore depends on the boundary conditions. The Casimir energy is the vacuum energy evaluated under these additional conditions [2].

1.2. Massless scalar field subject to the Dirichlet boundary conditions in one spatial direction. We suppose that the field is confined between two parallel plates separated by the distance a in the x direction. The field satisfies the boundary conditions on the plates,

$$\phi(t, 0, y, z) = \phi(t, a, y, z) = 0.$$

In this case, the negative and positive frequency components of the field are proportional to $\sin k_n x$, where $k_n = n\pi/a$, $n = 1, 2, \ldots$, and $\omega_n = (k_n^2 + k_y^2 + k_z^2)^{1/2}$. The vacuum energy per unit area of the plates is therefore given by

$$\frac{E_V}{A} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk_y\, dk_z}{(2\pi)^2} \sum_{n=1}^{\infty} \left(\left(\frac{n\pi}{a}\right)^2 + k_y^2 + k_z^2\right)^{1/2}.$$

As anticipated, this vacuum energy per unit transversal area is divergent: both the series and the integral diverge. As usual, to interpret this expression, we must regularize it, isolate the divergences, and then renormalize (whenever possible) the classical energy in accordance with physical considerations.

The first method that we use is the zeta-function regularization [5], [6]. It is based on the analyticity properties of the zeta function of an operator, which in this case is minus the Laplacian. A formal definition of the spectral function known as the zeta function is given in the next section.

We define[1]

$$\frac{E_V}{A} = \frac{\mu}{2}\int_{-\infty}^{\infty} \frac{dk_y\, dk_z}{(2\pi)^2} \sum_{n=1}^{\infty} \left(\left(\frac{n\pi}{a\mu}\right)^2 + \left(\frac{k_y}{\mu}\right)^2 + \left(\frac{k_z}{\mu}\right)^2\right)^{-s/2}\Bigg|_{s=-1}, \tag{2}$$

where s is a complex variable with $\mathrm{Re}(s)$ sufficiently large to guarantee convergence. This expression defines an analytic function of s in this region. The vacuum energy is then defined via the analytic extension of this function to $s = -1$. The parameter μ with the dimension of mass is introduced to make the quantity in the sum in (2) dimensionless; it must disappear from any physically meaningful result.

Using the representation [7]

$$z^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} dt\, t^{s-1} e^{-zt}, \quad \mathrm{Re}(s) > 0, \quad \mathrm{Re}(z) > 0, \tag{3}$$

we can now rewrite Eq. (2) as

$$\frac{E_V}{A} = \frac{\mu}{2}\int_{-\infty}^{\infty} \frac{dk_y\, dk_z}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{\Gamma(s/2)}\int_0^{\infty} dt\, t^{s/2-1} \exp\left\{\left(\left(\frac{n\pi}{a\mu}\right)^2 + \left(\frac{k_y}{\mu}\right)^2 + \left(\frac{k_z}{\mu}\right)^2\right)t\right\}\Bigg|_{s=-1}.$$

For sufficiently large $\mathrm{Re}(s)$, the sum and the integral can be transposed with the result

$$\frac{E_V}{A} = \frac{\mu}{2}\sum_{n=1}^{\infty} \frac{1}{\Gamma(s/2)}\int_0^{\infty} dt\, t^{s/2-1} e^{-(n\pi/(a\mu))^2 t} \int_{-\infty}^{\infty} \frac{dk_y\, dk_z}{(2\pi)^2} \exp\left\{-\left[\left(\frac{k_y}{\mu}\right)^2 + \left(\frac{k_z}{\mu}\right)^2\right]t\right\}\Bigg|_{s=-1}.$$

Both Gaussian integrals can now be evaluated, and Eq. (3) can be used again to obtain

$$\frac{E_V}{A} = \frac{\mu^3}{8\pi}\frac{\Gamma(s/2-1)}{\Gamma(s/2)}\sum_{n=1}^{\infty}\left(\frac{n\pi}{a\mu}\right)^{-s+2}\Bigg|_{s=-1} = \frac{\mu^3}{4\pi(s-2)}\left(\frac{a\mu}{\pi}\right)^{s-2}\zeta_{\mathrm{R}}(s-2)\Bigg|_{s=-1},$$

where we use the definition of the Riemann zeta function

$$\zeta_{\mathrm{R}}(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

[1]For simplicity, we keep the same symbol for the renormalized quantity.

This series defines an analytic function for $\mathrm{Re}(s) > 1$, and its analytic extension to the entire s plane has only a simple pole at $s = 1$. In particular, its value at $s = -3$ is $1/120$.

Our final result for the vacuum energy is therefore given by

$$\frac{E_V}{A} = -\frac{\pi^2}{1440a^3}. \tag{4}$$

In this simple case, the zeta-function regularization gives a finite result and no further renormalization is needed.

We now compare result (4) to the one given by another regularization method, the exponential cutoff. As we see in what follows, it is based on using another spectral function known as the trace of the heat kernel. We define

$$\frac{E_V}{A} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dk_y\,dk_z}{(2\pi)^2}\sum_{n=1}^{\infty}\left(\left(\frac{n\pi}{a\mu}\right)^2 + \left(\frac{k_y}{\mu}\right)^2 + \left(\frac{k_z}{\mu}\right)^2\right)^{1/2} \times$$
$$\times \exp\left\{-\left(\left(\frac{n\pi}{a\mu}\right)^2 + \left(\frac{k_y}{\mu}\right)^2 + \left(\frac{k_z}{\mu}\right)^2\right)^{1/2} t\right\}\bigg|_{t=0}.$$

The exponential is introduced to ensure convergence, thus allowing the interchange of the sum and the integral. Again, the parameter μ with the dimension of mass is arbitrary. The previous equation can also be written as

$$\frac{E_V}{A} = -\frac{\mu}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{dk_y\,dk_z}{(2\pi)^2}\sum_{n=1}^{\infty}\exp\left\{-\left(\left(\frac{n\pi}{a\mu}\right)^2 + \left(\frac{k_y}{\mu}\right)^2 + \left(\frac{k_z}{\mu}\right)^2\right)^{1/2} t\right\}\bigg|_{t=0} =$$
$$= -\frac{\mu}{4\pi}\frac{d}{dt}\int_0^{\infty}dk\,k\sum_{n=1}^{\infty}\exp\left\{-\left(\left(\frac{n\pi}{a\mu}\right)^2 + \left(\frac{k}{\mu}\right)^2\right)^{1/2} t\right\}\bigg|_{t=0}$$

or, after interchanging the sum and the integral and changing variables,

$$\frac{E_V}{A} = -\frac{\mu^3}{2}\frac{d}{dt}\sum_{n=1}^{\infty}\int_{(n\pi/(a\mu))^2}^{\infty}dk\,e^{-k^{1/2}t}\bigg|_{t=0}.$$

The sum can now be evaluated using the Euler–McLaurin formula

$$\sum_{n=1}^{\infty}f(n) = -\frac{1}{2}f(0) + \int_0^{\infty}f(x)\,dx - \sum_{k=1}^{\infty}\frac{1}{(2k)!}B_{2k}f^{(2k-1)}(0). \tag{5}$$

It can be shown that the final result for the vacuum energy in this regularization scheme is given by

$$\frac{E_V}{A} = \frac{3a\mu^4}{2\pi^2t^4}\bigg|_{t=0} - \frac{\mu^3}{4\pi t^3}\bigg|_{t=0} - \frac{\pi^2}{1440a^3}. \tag{6}$$

We note that two divergences remain in Eq. (6) in the form of poles. The first divergent term in (6) is the vacuum energy in the entire space (it comes from the integral in Eq. (5)). The second divergence is due to the mode $n = 0$ (the first term in the right-hand side of the same equation) and is therefore caused by the boundary conditions. Both divergences can be eliminated by the condition that $E_V/A \to 0$ as $a \to \infty$. This can in fact be understood as a renormalization of the classical energy, which, in view of the geometry of our problem, is given by

$$E_{\text{class}} = paA + \sigma A,$$

where p is the pressure and σ is the surface tension. The remaining finite part in (6) then agrees with the result of the zeta regularization given by Eq. (4).

We have considered a simple example (there is no mass and the boundaries are flat). In what follows, we study the connection between these two regularization methods and discuss the renormalization of the Casimir energy in more general cases.

2. Elliptic differential operators and boundary problems: Spectral functions

In the previous section, we evaluated zero-point energies using two different regularization methods: the zeta function and the exponential cutoff. They are based on using certain functions of the spectrum of a given differential operator, called spectral functions. In this section, we discuss the conditions under which such functions can be defined and consider some of their useful properties [8]–[12].

2.1. Differential operators on boundaryless compact manifolds. We let M be a compact boundaryless manifold of dimension n and E be a complex vector bundle over M. A *partial differential operator of order* m acting on sections of E can be written in local coordinates as

$$A = \sum_{|\alpha|\le m} a_\alpha(x) D_x^\alpha, \qquad D_x^\alpha = \prod_{j=1}^{n} \left(-i\frac{\partial}{\partial x_j}\right)^{\alpha_j}, \qquad |\alpha| = \sum_{j=1}^{n} \alpha_j.$$

The coefficients $a_\alpha(x)$ are $q\times q$ matrices in general.

As an example, we consider the operator

$$-\frac{d^2}{dx^2} + x\frac{d}{dx} + 1.$$

In this case $n = 1$ and $m = 2$ and therefore $j = 1$ and $|\alpha| = \alpha_1 \le 2$. The operator can be written in the compact form

$$\sum_{\alpha_1\le 2} a_{\alpha_1}(x)(-i)^{\alpha_1}\left(\frac{d}{dx_1}\right)^{\alpha_1}.$$

The coefficient $a_0(x) = 1$ corresponds to the index value $\alpha_1 = 0$, $a_1(x) = ix$ corresponds to $\alpha_1 = 1$, and $a_2(x) = 1$ corresponds to $\alpha_1 = 2$.

The *symbol of an operator* A is defined as

$$\sigma(A) = \sigma(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha.$$

It is an mth-degree polynomial in the dual variable ξ obtained by formally replacing D_x^α with the monomial ξ^α. In terms of the symbol, the action of an operator on functions in its domain can be written as

$$Af(x) = \int d\xi\, e^{ix\xi}\sigma(x,\xi)\tilde f(\xi),$$

where $\tilde f(\xi)$ is the Fourier transform of $f(x)$.

For the operator in the previous example, $\sigma(x,\xi) = \xi^2 + ix\xi + 1$. It can be easily shown that

$$\left(-\frac{d^2}{dx^2} + x\frac{d}{dx} + 1\right)f(x) = \int d\xi\, e^{ix\xi}(\xi^2 + ix\xi + 1)\tilde f(\xi).$$

The *principal symbol* is the highest-order part of the symbol. It is a homogeneous mth-degree polynomial in ξ,

$$\sigma_m(A) = \sigma_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

In our example, $\sigma_2(x, \xi) = \xi^2$.

A differential operator is said to be *elliptic* if its principal symbol is invertible for $|\xi| = 1$ (it has no zero eigenvalues for $|\xi| = 1$ or, equivalently, $\det \sigma_m(x, \xi) \neq 0$ for $|\xi| = 1$). The ellipticity is obvious in our example.

Given an operator A, its resolvent is the operator $(A - \lambda I)^{-1}$. The ray $\mathcal{K} = \{\arg \lambda = \theta\}$ in the complex plane λ is called *the minimum growth ray of the resolvent* if there are no eigenvalues of the principal symbol on this ray, i.e., the equation

$$\sigma_m(x, \xi)u = \lambda u$$

has only the trivial solution for $\lambda \in \mathcal{K}$. It can be proved that the L^2 norm of the resolvent is $O(|\lambda|^{-1})$ along such a ray. In our example, the problem $\xi^2 u = \lambda u$ has only the trivial solution for any $\lambda \neq \xi^2$. Because ξ is real, any ray in $\mathbb{C} - \mathbb{R}_+$ is a minimum growth ray.

2.2. Complex powers of a differential operator. Given an elliptic differential operator A with the minimum growth ray $\mathcal{K}$ and $\mathrm{Re}(s) > 0$, we define

$$A^{-s} = \frac{i}{2\pi} \int_\Gamma \lambda^{-s}(A - \lambda I)^{-1}\, d\lambda,$$

where Γ is a curve starting at ∞, coming along the ray $\mathcal{K}$ to a small circle at the origin, and returning to ∞ along the ray (we note that this curve encloses the eigenvalues of the principal symbol in the clockwise direction).

To describe the operator A^{-s}, we can construct an approximation $B(\lambda)$ to the resolvent $(A - \lambda I)^{-1}$, known as the parametrix, which reproduces the behavior of the resolvent as $\lambda \to \infty$ along the minimum growth ray [8]. The parametrix is constructed by considering λ as a part of the principal symbol of the operator A, as

$$\sigma(B) \sim \sum_{j=0}^{\infty} b_{-m-j}(x, \xi, \lambda)$$

under the condition

$$\sigma(B(A - \lambda I)) = I.$$

The coefficients b_{-m-j} are known as the Seeley coefficients. It can be verified (using the formula for the composition of symbols of operators) that they satisfy the set of algebraic equations

$$b_{-m}(a_m - \lambda) = 1,$$

$$b_{-m-l}(a_m - \lambda) + \sum (\partial_\xi^\alpha b_{-m-j})(D_x^\alpha a_{m-k}) = 0, \quad l > 0.$$

Here, the sum must be taken over all $k + j + |\alpha| = l$ and $j < l$. The a_{m-k} are the symbols of the differential operator A of different orders. With these coefficients, we obtain an approximation to the symbol of A^{-s},

$$\sigma(A^{-s}) \sim \sum_{j=0}^{\infty} \frac{i}{2\pi} \int_\Gamma \lambda^{-s} b_{-m-j}(x, \xi, \lambda)\, d\lambda.$$

Starting from this expression, it can be shown that the kernel $K_{-s}(x, y)$ of A^{-s} is continuous for $\mathrm{Re}(ms) > n$. For $x \neq y$, it extends to an entire function of s. For $x = y$, it extends to a meromorphic function, whose only singularities are simple poles at $s = (n - j)/m$, $j = 0, 1 \ldots$. Each pole is due to a particular term in the previous expression, and the residues are thus determined by the integrals of the Seeley coefficients along Γ.

2.3. The zeta function: Relation to the eigenvalues. For a given elliptic operator A, the first spectral function that we consider is its zeta function defined as the trace

$$\zeta(A,s) = \operatorname{tr} A^{-s}.$$

The analyticity properties of the zeta function are derived from those of the kernel $K_{-s}(x,x)$. The residues at the poles of $\zeta(A,s)$ are the integrals over the manifold M of the residues corresponding to $K_{-s}(x,x)$ and are therefore determined by the Seeley coefficients. Whenever an operator has a complete orthogonal set of eigenfunctions, its zeta function can be expressed in terms of the corresponding eigenvalues.

We now suppose that the bundle has a smooth Hermitian inner product and M is endowed with a smooth volume element dv. If the operator A is normal with respect to these structures (i.e., $A^\dagger A = AA^\dagger$), then it has a complete orthonormal set of eigenfunctions $A\phi_k = \lambda_k \phi_k$, and we can write

$$K_{-s}(x,y)\,dv_y = \sum_k \lambda_k^{-s} \phi_k(x)\phi_k^\dagger(y)\,dv_y.$$

Setting $x = y$ and integrating over M, we then obtain

$$\zeta(A,s) = \operatorname{tr} A^{-s} = \sum_k \lambda_k^{-s}. \tag{7}$$

As an example, we consider the operator

$$A = -\frac{d^2}{dx^2} + P$$

on the unit circle, where P is the operator of projection onto zero modes. The eigenvalues of A are given by $\lambda_n = n^2$ for $n = \pm 1, \pm 2, \dots$ and $\lambda_0 = 1$ for the zero mode. Therefore,

$$\zeta(A,s) = 2\sum_{n=1}^{\infty} (n^2)^{-s} + 1 = 2\zeta_{\mathrm{R}}(2s) + 1.$$

This Riemann zeta function is known to be analytic for $\operatorname{Re}(2s) > 1$, i.e., $\operatorname{Re}(s) > 1/2 = n/m$. Its analytic extension has a simple pole at $2s = 1$.

2.4. The heat kernel and its trace. If all eigenvalues of the principal symbol lie in the region S_0: $-\pi/2 + \varepsilon < \arg\lambda < \pi/2 - \varepsilon$, then the spectrum of A lies in the sector S_α: $-\pi/2 + \varepsilon < \arg(\lambda + \alpha) < \pi/2 - \varepsilon$ for some $\alpha > 0$, and we can define

$$e^{-At} = \frac{i}{2\pi} \int_\Gamma e^{-\lambda t}(A - \lambda I)^{-1}\,d\lambda, \quad t > 0,$$

where Γ is the boundary of S_α. It can be shown that this is the fundamental solution of the heat equation $Au + \partial u/\partial t = 0$ with the initial condition $u(x,0) = \delta(x)$. The operator e^{-At} is therefore called the heat kernel of the operator A.

The approximation to the resolvent $B(\lambda)$ then allows taking the limit as $t \to +0$ in the previous integral, and we thus obtain an asymptotic expansion for e^{-At} in increasing (in general, noninteger) powers of t. The coefficients in this expansion are also determined by the Seeley coefficients.

As before, if A has a complete set of eigenfunctions, then the kernel of e^{-At} can be written as

$$K(t,x,y) = \sum_k e^{-\lambda_k t} \phi_k(x) \phi_k^\dagger(y).$$

The trace

$$h(A,t) = \operatorname{tr} e^{-At} = \sum_k e^{-\lambda_k t} \tag{8}$$

is the the second spectral function that we use in what follows.

There is a very close relation between the zeta function of an operator and the trace of its heat kernel. In fact, it follows from Eqs. (7) and (8) that

$$\zeta(A,s) = \sum_k \lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dz \sum_k e^{-\lambda_k z} z^{s-1} = \frac{1}{\Gamma(s)} \int_0^\infty dz\, h(A,z) z^{s-1}, \tag{9}$$

i.e., the two spectral functions are related through the so-called Mellin transformation.

2.5. Elliptic boundary systems. Up to this point, we have considered boundaryless manifolds. We now consider how the concepts introduced above extend to manifolds with boundaries. We let M be a compact manifold of dimension n with a smooth boundary ∂M. In each local coordinate system, we let $x = (x_1, \ldots, x_{n-1})$ be the coordinates on ∂M. We let $t \in \mathbb{R}$ be the interior normal to the boundary. Therefore, $(x,t) \in \mathbb{R}^n$. We let $\mathbb{R}^n_+$ be the half-space $t \geq 0$. In $\mathbb{R}^n_+$, we consider the mth-order differential operator

$$A = \sum_{j=0}^{m} A_j(x,t) D_t^{m-j}, \qquad D_t = -i \frac{\partial}{\partial t},$$

where A_j is a differential operator of an order not greater than j on $\mathbb{R}^{n-1}$. With (ξ, τ) denoting the symbolic variable corresponding to (x,t), we have

$$\sigma(A) = \sum_j \sigma(A_j)(x,t,\xi) \tau^{m-j}.$$

The principal symbol is

$$\sigma_m = \sum_j \sigma_j(A_j)(x,t,\xi) \tau^{m-j}.$$

Moreover, we define the partial principal symbol by

$$\sigma'_m = \sum_j \sigma_j(A_j)(x,0,\xi) D_t^{m-j}.$$

We now suppose that near the boundary, we have certain given operators (defining the boundary conditions)

$$B_j = \sum_{k=1}^{m} B_{jk} D_t^{m-k}, \quad 1 \leq j \leq \frac{mq}{2},$$

where the B_{jk} are a system of differential operators ($1 \times q$ matrices) acting on $\mathbb{R}^{n-1}$. We concentrate on the case where these boundary operators are merely multiplicative. Then

$$\sigma(B_j) = \sum_{k=1}^{m} \sigma(B_{jk}) \tau^{m-k}, \qquad \sigma'(B_j) = \sum_{k=1}^{m} \sigma(B_{jk}) D_t^{m-k}.$$

A collection of operators $A, B_1, \dots, B_{mq/2}$ constitute an *elliptic boundary system* if A is elliptic and if for arbitrary $g = (g_1, \dots, g_{mq/2})$, $x \in \mathbb{R}^{n-1}$, and $\xi \neq 0$, $\xi \in \mathbb{R}^{n-1}$, there is a unique solution to the problem

$$\sigma'_m(A)(x, \xi, D_t)u = 0, \quad t > 0,$$

$$\lim_{t\to\infty} u(t) = 0,$$

$$\sigma'(B_j)(x, \xi, D_t)u = g_j \quad \text{for } t = 0,\ j = 1, \dots, \frac{mq}{2}.$$

These conditions are also known as the Lopatinski–Shapiro conditions [12]. Whenever they hold, an operator A_B can be defined as the operator A acting on functions u such that $B_j u = 0$.

A collection $A, B_1, \dots, B_{mq/2}$ constitutes a *strongly elliptic boundary system* in a cone $\mathcal{K} \subset \mathbb{C}$ including the origin if

a. for $(\xi, \tau) \neq (0, 0)$, $\sigma_m(A)$ has no eigenvalues in $\mathcal{K}$ and

b. for each x and each $(\xi, \lambda) \neq (0, 0)$ with $\lambda \in \mathcal{K}$, the boundary problem

$$\sigma'_m(A)(x, \xi, D_t)u = \lambda u,$$

$$\lim_{t\to\infty} u(t) = 0,$$

$$\sigma'(B_j)(x, \xi, D_t)u = g_j \quad \text{for } t = 0,\ j = 1, \dots, \frac{mq}{2},$$

has a unique solution.

We note that this reduces to the Lopatinski–Shapiro condition for $\lambda = 0$. The cone $\mathcal{K}$ is known as the Agmon cone [11].

When the strong ellipticity condition holds, an approximation to the resolvent $(A_B - \lambda)^{-1}$ can be found [9], and we can use it to obtain

$$(A_B)^{-s} = \frac{i}{2\pi} \int_\Gamma \lambda^{-s} (A_B - \lambda I)^{-1}\, d\lambda,$$

where Γ is a curve in the cone where $(A_B - \lambda I)^{-1}$ exists. The coefficients in the expansion of the parametrix must now satisfy not only the condition that the operator $A_B - \lambda$ be invertible but also the boundary conditions. In addition to the volume Seeley coefficients b_{-m-j}, there then exist new boundary coefficients d_{m-j}, and their determination leads to a set of differential (not algebraic) equations.

The conditions that determine the pole structure of $K_{-s}(x, y)$ are similar to those in the boundaryless case, but the residues at the poles are now given by volume integrals of the coefficients b plus boundary integrals of the coefficients d. As before, the zeta function can be defined as the trace of the $-s$th power. In particular, if the operator A_B has a complete set of eigenfunctions, then expansions (7) hold and

$$h(A_B, t) = \sum_k e^{-\lambda_k t}. \tag{10}$$

Spectral functions (7) and (10) are again related through the Mellin transformation.

As an example, we consider the Laplacian on the boundary of a cylinder with the Dirichlet boundary conditions. We let $A = -\partial_x^2 - \partial_y^2$ act on functions $\varphi(x, y)$ such that $\varphi(0, y) = \varphi(1, y) = 0$, which are periodic in the y direction. The boundary then corresponds to $x = 0$, $x = 1$. The boundary operator is merely multiplicative, $B = 1$. We let (τ, ξ) be the Fourier variables associated with (x, y). Then

$$\begin{aligned} &\sigma(A) = \sigma_2(A) = \xi^2 + \tau^2, \\ &\sigma(B) = \sigma_0(B) = \sigma'_0(B) = 1. \end{aligned} \tag{11}$$

It can then be shown that the differential operator A is elliptic. In fact, $\sigma_2(A) = \xi^2 + \tau^2 \neq 0$ for $(\tau, \xi) \neq (0,0)$ because $\tau, \xi \in \mathbb{R}$.

It can also be shown that the boundary problem is elliptic in the weak (Lopatinski–Shapiro) sense. We consider the boundary at $x = 0$. Then x is the variable normal to the boundary. For $\xi \neq 0$, the differential equation

$$\sigma_2'(A)\, u = (-\partial_x^2 + \xi^2)\, u = 0$$

has a solution of the form

$$u = Ce^{|\xi|x} + De^{-|\xi|x}.$$

The condition that this solution vanish as $x \to \infty$ requires $C = 0$. For the remaining part, the problem $u(0, \xi) = g$ has a unique solution $D = g$ for arbitrary g, which shows that weak ellipticity holds at $x = 0$. It can be similarly shown to hold at $x = 1$.

The differential operator A has an Agmon cone. Indeed, the equation

$$\sigma_2\, u = (\tau^2 + \xi^2)\, u = \lambda\, u$$

with $(\tau, \xi) \neq (0,0)$ has nontrivial solutions only for $\lambda = \tau^2 + \xi^2 \in \mathbb{R}_+$. The differential operator therefore has the Agmon cone $\mathcal{K} = \mathbb{C} - \mathbb{R}_+$.

It also follows that the boundary problem is strongly elliptic in our example (has an Agmon cone). Again, we consider the boundary at $x = 0$. For $(\xi, \lambda) \neq (0,0)$, the differential equation

$$\sigma_2'(A)u = (-\partial_x^2 + \xi^2)u = \lambda u$$

has a solution of the form

$$u = Ce^{\sqrt{\xi^2 - \lambda^2}x} + De^{-\sqrt{\xi^2 - \lambda^2}x}.$$

The condition that it vanish as $x \to \infty$ requires $C = 0$, i.e., the problem $u(0, \xi, \lambda) = g$ has a unique solution $D = g$ for arbitrary g. Together with the fact that the differential operator A has an Agmon cone, this shows that the boundary problem is strongly elliptic in $\mathcal{K} = \mathbb{C} - \mathbb{R}_+$.

3. Comparison of the zeta and exponential regularizations of Casimir energies

In this section, we show that both the zeta and the exponential regularizations, in general, give divergent results for the Casimir energy. A general relation between the two results is established, and the existence of a unique, physically meaningful result after renormalization is discussed [13], [14].

3.1. The general result. In Sec. 1, we studied a simple example of evaluating the Casimir energy: a massless scalar field between "conducting" plates. We obtained a finite result for the vacuum energy in the zeta regularization and pole divergences in the exponential cutoff regularization. Moreover, these divergences showed a dependence on the distance between the plates consistent with the classical action and differed from the dependence of the finite part on the distance between the plates. The energy could therefore be renormalized away.

We now face several questions: Is renormalization still possible in a more general case? Does the zeta function always give a finite result? Are the divergences in the exponential regularization always poles? What is the relation between the results given by the two regularizations in a more general case?

To answer these questions, we use the general results in Sec. 2 regarding the structure of the zeta function and the trace of the heat kernel, as applied to second-order operators. We recall that in the scalar case, for instance, the vacuum energy is given by

$$E_V = \frac{1}{2}\sum_n \omega_n,$$

where ω_n are the zero-point energies. We consider a scalar field in a $(d{+}1)$-dimensional space–time, where d is the dimension of the compact spatial manifold, with or without a smooth boundary. We set $\omega_n = \lambda_n^{1/2}$, where the λ_n satisfy the associated boundary problem

$$D_B\varphi_n = \eta_n\varphi_n \tag{12}$$

with

$$\eta_n = \begin{cases} \lambda_n & \text{for } D_B = D, \\ 0 & \text{for } D_B = B. \end{cases}$$

Here, D is a second-order differential operator on M, and B is the operator defining the boundary conditions.

The zeta-regularized [5], [6] vacuum energy is defined as

$$E_\zeta \equiv \frac{\mu}{2}\sum_n\left(\frac{\lambda_n}{\mu^2}\right)^{-s/2}\Bigg|_{s=-1} = \frac{\mu}{2}\zeta\left(\frac{s}{2},\frac{D_B}{\mu^2}\right)\Bigg|_{s=-1} = \frac{\mu}{2}\,\mathrm{tr}\left(\frac{D_B}{\mu^2}\right)^{-s/2}\Bigg|_{s=-1}, \tag{13}$$

and the cutoff-regularized expression is given by

$$E_{\exp} \equiv \frac{\mu}{2}\sum_n\frac{\lambda_n^{1/2}}{\mu}\exp\left\{-t\frac{\lambda_n^{1/2}}{\mu}\right\}\Bigg|_{t=0} = -\frac{\mu}{2}\frac{d}{dt}\left(h\left(t,\frac{D_B^{1/2}}{\mu}\right)\right)\Bigg|_{t=0}, \tag{14}$$

where

$$h\left(t,\frac{D_B^{1/2}}{\mu}\right) = \sum_n\exp\left\{-t\frac{\lambda_n^{1/2}}{\mu}\right\} = \mathrm{tr}\left(\exp\left\{-\frac{t}{\mu}D_B^{1/2}\right\}\right). \tag{15}$$

We recall that μ is an arbitrary parameter with the dimensions of mass in both cases.

To study the relation between the two regularizations, we use the following known result.

Lemma 1 [10]**.** *Let D be a second-order differential operator acting on a smooth compact d-dimensional manifold M, and let B be the differential operator defining boundary conditions on ∂M. If boundary problem* (12) *is strongly elliptic with respect to $\mathbb{C}-\mathbb{R}_+$, then*

a. *the function*

$$\mu^{-s}\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2},\frac{D_B}{\mu^2}\right)$$

is analytic for $\mathrm{Re}(s) > d$ and extends analytically to a meromorphic function with the singularity structure

$$\mu^{-s}\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2},\frac{D_B}{\mu^2}\right) = \sum_{j=0}^{N}\frac{2a_j}{s+j-d} + r_N\left(\frac{s}{2}\right),$$

where $r_N(s/2)$ is analytic for $\mathrm{Re}(s) > d-N-1$ and the coefficients a_j are determined by the integrated Seeley coefficients, and

b. *for each real c_1 and c_2 and any $\delta < \theta_0$,*

$$\left|\mu^{-s}\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2},\frac{D_B}{\mu^2}\right)\right| \le C(c_1,c_2,\delta)e^{-\delta|\operatorname{Im} s/2|}, \quad \left|\operatorname{Im}\frac{s}{2}\right| \ge 1, \quad c_1 \le \operatorname{Re}\frac{s}{2} \le c_2.$$

This lemma clearly shows that the vacuum energy evaluated through the zeta regularization as given by (13) involves a pole singularity whenever $a_{d+1} \neq 0$. To study exponential cutoff–regularized expression (14), we prove the following lemma.

Lemma 2. *Under the hypothesis of Lemma 1, the function*

$$\frac{dh(t, D_B^{1/2}/\mu)}{dt} = \frac{d\,\mathrm{tr}\big(\exp\{-tD_B^{1/2}/\mu\}\big)}{dt} = \sum_n -\frac{\lambda_n^{1/2}}{\mu}\exp\Big\{-t\frac{\lambda_n^{1/2}}{\mu}\Big\}$$

has the asymptotic expansion

$$\begin{aligned}
\frac{dh(t, D_B^{1/2}/\mu)}{dt} = {} & \sum_{k=0}^{d}(-k)\frac{1}{2\mu}\frac{\Gamma((k+1)/2)}{\Gamma(1/2)}a_{d-k}\left(\frac{t}{2\mu}\right)^{-k-1} + \\
& + \sum_{k=1}^{K}(-k)\frac{1}{2\mu}\frac{\Gamma(-k+1/2)}{\Gamma(1/2)}2a_{d+2k}\left(\frac{t}{2\mu}\right)^{2k-1} + \\
& + \sum_{k=0}^{K}(2k+1)\frac{1}{2\mu}\frac{(-1)^k}{\Gamma(1/2)\Gamma(k+1)}\left(\frac{t}{2\mu}\right)^{2k}\Bigg[r_{d+2k+1}\left(-k-\frac{1}{2}\right) + \\
& + a_{d+2k+1}\left(\Psi(1) + \sum_{l=0}^{k-1}\frac{1}{k-l}\right) + \sum_{j=0}^{d+2k}\frac{2a_j}{j-d-2k-1} + \\
& + 2a_{d+2k+1}\left((2k+1)\log\left(\frac{t}{2\mu}\right) - \frac{1}{2k+1}\right)\Bigg] + \rho_K(t)
\end{aligned} \tag{16}$$

as $t \to 0$, *where*

$$\rho_K = O\left(\left(\frac{t}{2\mu}\right)^{2K+1+\varepsilon}\right), \quad 0 < \varepsilon < 1.$$

We note that for $t \to 0$, this asymptotic expansion contains not only poles (in the first sum in (16)) but also logarithmic divergences (for $k = 0$ in the last sum). In fact, it involves poles of the orders $d+1, d, \dots, -1$ with the coefficients determined by a_{d-k}, $k = 0, \dots, d$. The coefficient of the logarithm is determined by a_{d+1}.

Proof of Lemma 2. The proof is only sketched here (see [13] for the details). We first note that

$$\Gamma(s)\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) = \int_0^\infty t^{s-1}h\left(t, \frac{D_B^{1/2}}{\mu}\right)dt$$

is the Mellin transform of $h\big(t, D_B^{1/2}/\mu\big)$. This can also be written as

$$\begin{aligned}
\Gamma(s)\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right) &= \frac{\Gamma(s)}{\Gamma(s/2)}\left[\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right)\right] = \\
&= \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s+1}{2}\right)\left[\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right)\right].
\end{aligned} \tag{17}$$

From Lemma 1 and the known singularity structure of $\Gamma((s+1)/2)$, it follows that (17) is analytic for $\mathrm{Re}(s) > d$ and

$$h\left(t, \frac{D_B}{\mu^2}\right) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{t}{\mu}\right)^{-s}\frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s+1}{2}\right)\left[\mu^{-s}\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right)\right], \tag{18}$$

where the integration path is such that $c > d$. From this expression, we obtain

$$\frac{dh(t, D_B/\mu^2)}{dt} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\, \frac{(-s)}{\mu}\left(\frac{t}{\mu}\right)^{-s-1}\frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s+1}{2}\right)\left[\mu^{-s}\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right)\right], \tag{19}$$

where the integral is performed along the same contour as before.

Using Lemma 1 together with

$$\Gamma\left(\frac{s+1}{2}\right) = O\left(\exp\left\{\left(-\frac{\pi}{2}+\epsilon\right)\left|\operatorname{Im}\frac{s}{2}\right|\right\}\right)$$

for any $\epsilon > 0$, it is now possible to obtain an asymptotic expansion for $d\,h(t, D_B/\mu^2)/dt$ by moving the integration path in (18) through the poles of

$$\Gamma\left(\frac{s+1}{2}\right)\left[\mu^{-s}\Gamma\left(\frac{s}{2}\right)\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right)\right].$$

These poles are located at $s = d - j$. For $s = d - j = k \geq 0$, $j \leq d$, they are simple poles, and their contribution to the Cauchy integral is given by

$$\frac{-k}{2\mu}\frac{\Gamma((k+1)/2)}{\Gamma(1/2)}a_{d-k}\left(\frac{t}{2\mu}\right)^{-k-1}, \quad k = 0, 1, \ldots, d.$$

For $s = d - j = -2k$, $k = 1, 2, \ldots$, they are also simple poles, and their contribution is

$$-k\frac{1}{2\mu}\frac{\Gamma(-k+1/2)}{\Gamma(1/2)}2a_{d+2k}\left(\frac{t}{2\mu}\right)^{2k-1}.$$

For $s = d - j = -(2k+1)$, $k = 0, 1, \ldots$, they are simple and double poles with the contribution

$$\frac{2k+1}{2\mu}\frac{(-1)^k}{\Gamma(1/2)\Gamma(k)}\left(\frac{t}{2\mu}\right)^{2k}\left[r_{d+2k+1}\left(-k-\frac{1}{2}\right) + \sum_{j=0}^{d+2k}\frac{2a_j}{j-d-2k-1}\right] \times$$
$$\times\frac{2k+1}{2\mu}\frac{(-1)^k}{\Gamma(1/2)\Gamma(k+1)}\left(\frac{t}{2\mu}\right)^{2k}a_{d+2k+1}\left[2\log\left(\frac{t}{2\mu}\right) - 2 + \Psi(1) + \sum_{l=0}^{k-1}\frac{1}{k-l}\right],$$

where the sum over l must be taken for $k > 0$. Therefore, moving the integration contour in (18) until the singularity at $s = -(2K+1)$ is included, we have

$$\begin{aligned}\frac{dh(t, D_B/\mu^2)}{dt} = &\sum_{k=0}^{d}(-k)\frac{1}{2\mu}\frac{\Gamma((k+1)/2)}{\Gamma(1/2)}a_{d-k}\left(\frac{t}{2\mu}\right)^{-k-1} + \\ &+\sum_{k=1}^{K}(-k)\frac{1}{2\mu}\frac{\Gamma(-k+1/2)}{\Gamma(1/2)}2a_{d+2k}\left(\frac{t}{2\mu}\right)^{2k-1} + \\ &+\sum_{k=0}^{K}(2k+1)\frac{1}{2\mu}\frac{(-1)^k}{\Gamma(1/2)\Gamma(k+1)}\left(\frac{t}{2\mu}\right)^{2k}\left[r_{d+2k+1}\left(-k-\frac{1}{2}\right) + \right. \\ &+ a_{d+2k+1}\left(\Psi(1) + \sum_{l=0}^{k-1}\frac{1}{k-l}\right) + \sum_{j=0}^{d+2k}\frac{2a_j}{j-d-2k-1} + \\ &\left. + 2a_{d+2k+1}\left((2k+1)\log\left(\frac{t}{2\mu}\right) - \frac{1}{2k+1}\right)\right] + \rho_K(t).\end{aligned} \tag{20}$$

The remainder $\rho_K(t)$ is given by the same integral as (19), but with $c < -2(K+1)$. Lemma 1b and the estimate for $|\Gamma((s+1)/2)|$ already discussed imply that this remainder is $O\big(|t/(2\mu)|^{2K+1+\varepsilon}\big)$, which completes the proof.

Evaluated at $t = 0$, this asymptotic expansion gives exponentially regularized vacuum energy (14) as

$$\begin{aligned} E_{\exp} &= -\frac{\mu}{2}\frac{dh(t, D_B/\mu^2)}{dt}\bigg|_{t=0} \\ &= -\frac{1}{2}\sum_{k=1}^{d}(-k)\frac{\Gamma((k+1)/2)}{2^{-k}\Gamma(1/2)}a_{d-k}\left(\frac{t}{\mu}\right)^{-k-1}\bigg|_{t=0} - \\ &\quad -\frac{1}{4\Gamma(1/2)}\left[r_{d+1}\left(-\frac{1}{2}\right) + a_{d+1}(\Psi(1)-2) + 2\sum_{j=0}^{d}\frac{a_j}{j-d-1}\right] + \frac{1}{2}\frac{a_{d+1}}{\Gamma(1/2)}\log\left(\frac{t}{2\mu}\right). \end{aligned} \tag{21}$$

Returning to Lemma 1, the zeta-regularized vacuum energy is given by

$$\begin{aligned} E_\zeta = \frac{\mu}{2}\zeta\left(\frac{s}{2}, \frac{D_B}{\mu^2}\right)\bigg|_{s=-1} &= \frac{1}{2\Gamma(-1/2)}\sum_{j=0}^{d}\frac{2a_j}{j-d-1} + \\ &\quad + \frac{1}{2\Gamma(-1/2)}r_{d+1}\left(-\frac{1}{2}\right) + \frac{\mu^{s+1}}{\Gamma(s/2)}\frac{a_{d+1}}{s+1}\bigg|_{s=-1} = \\ &= -\frac{1}{2\Gamma(1/2)}\sum_{j=0}^{d}\frac{a_j}{j-d-1} - \frac{1}{4\Gamma(1/2)}r_{d+1}\left(-\frac{1}{2}\right) + \\ &\quad + \frac{1}{2\Gamma(1/2)}a_{d+1}\left(\frac{\Psi(1)}{2} + 1 - \log(2\mu)\right) - \frac{1}{2\Gamma(1/2)}\frac{a_{d+1}}{s+1}\bigg|_{s=-1}. \end{aligned} \tag{22}$$

We can conclude from (21) and (22) that both regularization methods give divergent results in general. Two cases must be distinguished.

Case 1. If $a_{d+1} = 0$, the zeta regularization gives a finite result, which coincides with the minimal finite part in the exponential regularization. This last method yields poles of the orders $2, 3, \ldots, d+1$. The residue at the pole of the order $k+1$ is given by $\Gamma(k+1)$ times the residue of $(\mu/2)\zeta(s/2, D_B/\mu^2)$ at $s = k$, $k = 1, \ldots, d$.

Case 2. In the general case ($a_{d+1} \neq 0$), the exponential regularization, in addition to the poles, involves a logarithmic divergence whose coefficient is minus the residue of $(\mu/2)\zeta(s/2, D_B/\mu^2)$ at $s = -1$. As a consequence, the difference between the minimal finite parts in the exponential regularization and in the zeta regularization is

$$-\frac{1}{2}\frac{a_{d+1}}{\sqrt{\pi}}\Psi(1) = \frac{1}{2}\frac{a_{d+1}}{\sqrt{\pi}}\gamma, \tag{23}$$

where γ is the Euler–Mascheroni constant. Both schemes show a logarithmic dependence on μ (as discussed in [15] for the zeta case). If the difference between the two regularization results consists of renormalizable terms, then a physical interpretation is possible, and all the dependence on μ disappears.

All these results are also valid in the case of boundaryless manifolds.

3.2. The massive scalar field in 1+1 dimensions. We can now return to our example of the massless scalar field in $d + 1 = 4$ and verify that it falls into case 1 in Sec. 3.1.

As a simple example of case 2, we consider a massive scalar field in $d+1=2$ dimensions satisfying periodic boundary conditions in the spatial direction, $\varphi(t,L)=\varphi(t,0)$. It is easy to see that

$$\omega_n = \left[m^2 + \left(\frac{2n\pi}{L}\right)^2\right]^{1/2}, \quad n \in \mathbb{Z}.$$

Using the zeta regularization, we then obtain

$$E_\zeta = \frac{\mu}{2}\sum_{n=-\infty}^{\infty}\left[\left(\frac{2n\pi}{L\mu}\right)^2 + \left(\frac{m}{\mu}\right)^2\right]^{-s/2}\Bigg|_{s=-1} =$$

$$= \frac{\mu^{s+1}}{2}\left(\frac{2\pi}{L}\right)^{-s}\sum_{n=-\infty}^{\infty}\left[n^2 + \left(\frac{mL}{2\pi}\right)^2\right]^{-s/2}\Bigg|_{s=-1}.$$

Using representation (3), this can be rewritten as

$$\frac{\mu^{s+1}}{2}\left(\frac{2\pi}{L}\right)^{-s}\sum_{n=-\infty}^{\infty}\frac{1}{\Gamma(s/2)}\int_0^\infty dt\, t^{s/2-1}\exp\left\{-\left(n^2+\left(\frac{mL}{2\pi}\right)^2\right)t\right\}\Bigg|_{s=-1} =$$

$$= \frac{\mu^{s+1}}{2}\left(\frac{2\pi}{L}\right)^{-s}\frac{1}{\Gamma(s/2)}\int_0^\infty dt\, t^{\frac{s}{2}-1}\exp\left\{-\left(\frac{mL}{2\pi}\right)^2 t\right\}\Theta\left(0,\frac{t}{\pi}\right)\Bigg|_{s=-1},$$

where

$$\Theta(x,y) = \sum_{n=-\infty}^{\infty} e^{2\pi x n - \pi y n^2}$$

is the Jacobi theta function, which has the useful inversion property

$$\Theta(x,y) = \frac{1}{\sqrt{y}}e^{\pi x^2/y}\Theta\left(\frac{x}{iy},\frac{1}{y}\right).$$

Using this property, we can show that

$$E_\zeta = \frac{\mu^{s+1}}{2}\left(\frac{2\pi}{L}\right)^{-s}\frac{1}{\Gamma(s/2)}\Bigg[\left(\frac{2\pi}{mL}\right)^{s-1}\Gamma\left(\frac{s-1}{2}\right) +$$

$$+ 2\int_0^\infty dt\, t^{(s-1)/2-1}\exp\left\{-\left(\frac{mL}{2\pi}\right)^2 t\right\}\exp\left\{-\frac{n^2\pi^2}{t}\right\}\Bigg]\Bigg|_{s=-1}.$$

Here, we can see how the pole at $s=-1$ arises: it comes from the pole of the gamma function. After performing the integration and expanding around $s=-1$, we obtain

$$E_\zeta^{(1)} = -\frac{m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}K_1(nmL) + \frac{m^2L}{4\pi}\left[\frac{1}{s+1}\Bigg|_{s=-1} - \log\left(\frac{m}{2\mu}\right) - \frac{1}{2}\right]. \quad (24)$$

The series in (24) converges because of the behavior of the modified Bessel function K_1 as $n\to\infty$.

The exponential cutoff regularization gives

$$E_{\text{exp}}^{(1)} = -\frac{\mu}{2}\frac{d}{dt}\left(\sum_{n=-\infty}^{\infty}\exp\left\{-t\left(\left(\frac{2n\pi}{L\mu}\right)^2 + \left(\frac{m}{\mu}\right)^2\right)^{1/2}\right\}\right)\Bigg|_{t=0}.$$

This series can be rewritten using the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{p=-\infty}^{\infty} c_p,$$

where

$$c_p = \int_{-\infty}^{\infty} dx\, e^{2\pi i p x} f(x).$$

After differentiating and setting $t = 0$, we obtain

$$E_{\rm exp}^{(1)} = -\frac{m}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}K_1(nmL) + \frac{m^2 L}{4\pi}\left[-\log t - \log\left(\frac{m}{2\mu}\right) + 2\left(\frac{mt}{\mu}\right)^{-2} - \gamma - \frac{1}{2}\right]\Bigg|_{t=0}. \tag{25}$$

This result is also divergent and, as predicted, involves divergences in the form of poles and logarithms. Comparing the coefficients in this expression with the coefficients in Eq. (24) also shows a complete agreement with our general result.

Both results admit a renormalization under the condition $E \to 0$ as $R \to \infty$ or, equivalently, $E \to 0$ as $m \to \infty$ (as proposed in [16]), thus yielding a physically meaningful result with no dependence on μ.

An open question is what happens in more complicated geometries, in particular, for curved boundaries.

Acknowledgments. The author thanks C. G. Beneventano, M. De Francia, and H. Falomir for the many useful comments and suggestions concerning the presentation in this paper and also thanks D. V. Vassilevich and the organizers of the IFSAP for making participation in the school possible. A very special acknowledgment goes to Yuri and Victor Novozhilov and to Valery Marachevsky for their warm hospitality.

This work was supported in part by ANPCyT (PICT'97 00039), CONICET (PIP 0459), and UNLP (Proy. 11-X230), Argentina.

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