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Intersection Graphs and the Clique Operator

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Abstract. Let P be a class of finite families of finite sets that satisfy a property P. We call ΩP the class of intersection graphs of families in P and $CliqueP$ the class of graphs whose family of cliques is in P. We prove that a graph G is in ΩP if and only if there is a family of complete sets of G which covers all edges of G and whose dual family is in P. This result generalizes that of Gavril for circular-arc graphs and conduces those of Fulkerson-Gross, Gavril and Monma-Wei for interval graphs, chordal graphs, UV, DV and RDV graphs. Moreover, it leads to the characterization of Helly-graphs and dually chordal graphs as classes of intersection graphs. We prove that if P is closed under reductions, then $CliqueP = \Omega(P^* \cap H)$ ($P^* =$ Class of dual families of families in P). We find sufficient conditions for the Clique Operator, K, to map ΩP into ΩP^*. These results generalize several known results for particular classes of intersection graphs. Furthermore, they lead to the Roberts-Spencer characterization for the image of K and the Bandelt-Prisner result on K-fixed classes.

1. Introduction

Let P be a class of finite families of finite sets that satisfy a property $\mathscr{P}$. We will call ΩP the class of intersection graphs of families in P and $CliqueP$ the class of graphs whose family of cliques is in P. Please see other definitions in Section 2.

Several classes of graphs are defined as collections of intersection graphs of families which have a particular property $\mathscr{P}$. This is the case of interval graphs class [5, 12], proper interval graphs class [20], UV-graphs class, DV-graphs class, RDV-graphs class, [8, 9, 19], circular-arc graphs class [10], etc.

Other classes of graphs can be defined as collections of those graphs whose family of cliques has a given property, as are the proper interval graphs (whose cliques are intervals of a total order on its vertex set [20]), the ACI-graphs [17] (whose cliques are intervals of an acyclic order on its vertex set), the Dually-chordal graphs [3] (arba-graphs [2] or expanded-tree graphs [22] or TCG-graphs [14, 15, 16], whose cliques are subtrees in a tree), the RET-graphs [23] (whose cliques are directed paths in a rooted tree) and the Helly-graphs [1] (whose family of cliques has the Helly property).

On the other hand, chordal graphs, originally defined by those graphs which contain no chordless cycle of length exceeding three, can be defined as intersection graphs of subtrees in trees [7] and as graphs whose family of cliques is an α-acyclic family [3].

Furthermore, in many classes of intersection graphs, if a graph G is or is not a member can be tested by a property of the dual family of its cliques (Fulkerson-Gross Theorem for interval graphs [6], Tree-clique Theorem for chordal- UV-DV-RDV graphs [7, 8, 9, 19]).

We ask the following questions: Is it possible to generalize these results for a generic class of intersection graphs? When is it possible to define a class of intersection graphs by properties of its cliques?

In Section 3 we try to solve these problems. We obtain a general result: A graph G is in ΩP if and only if there is a family of complete-sets of G which covers all edges of G and its dual family is in P.

Moreover, if any family in P has the Helly property, then every family of complete-sets, fulfilling the above conditions, contains the family of cliques of G. Finally we will find sufficient conditions to prove that $CliqueP = \Omega P^*$ ($P^* =$ Class of dual families of families in P).

The clique graph $K(G)$ of a graph G is the intersection graph of its family of cliques. Many authors have studied the behavior of the Clique Operator in several classes of intersection graphs. For example in [18] Hedman proves that K maps the class of interval graphs in the class of proper interval graphs and in [3, 22] it is proved that K maps the class of chordal graphs in the class of dually-chordal graphs. Is it possible that a relation exists between these results?

In section 4 we will prove that, under some conditions, the Clique Operator transforms ΩP^* into ΩP. These results and the characterization of ΩP obtained in section 3 conduce to the Roberts-Spencer's characterization of the image of K. Moreover the Bandelt-Prisner's result about K-fixed classes can be obtained as an immediate corollary.

2. Definitions

All graphs considered in this paper are finite undirected graphs without loops or multiple edges. Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively. If u and v are adjacent we write $uv \in E(G)$. A set $L \subseteq V(G)$ is a *complete* of G if $uv \in E(G)$ for any two distinct elements $u, v \in L$. If in addition there is no complete of G which properly contains L (i.e. L is maximal with respect to the inclusion), it is a *clique* of G. By $\mathscr{C}(G)$ we denote the family of cliques of G.

Let $\mathscr{F} = (F_i)_{i \in I}$ be a finite family of finite sets, its dual family $\mathscr{F}^*$ is the family $\{\mathscr{F}(x)\}_{x \in X}$ where $X = \bigcup_{i \in I} F_i$ and $\mathscr{F}(x) = \{i \in I, x \in F_i\}$. We say that a graph G is the *intersection graph* of $(F_i)_{i \in I}$ if $V(G) = I$, and two vertices i and j are adjacent if and only if $F_i \cap F_j \neq \emptyset$.

The $2-$*section* of $\mathscr{F}$, denoted by $\mathscr{F}_2$, is the graph with $V(\mathscr{F}_2) = \bigcup_{i \in I} F_i$ and two vertices x and y are adjacent if and only if there exists $i \in I$ such that $x, y \in F_i$.

By P we will denote the class of all families $\mathscr{F}$ that satisfy a property $\mathscr{P}$. We will say that a property $\mathscr{Q}$ is the dual property of $\mathscr{P}$ when the families that satisfy $\mathscr{Q}$

are the dual families of those that satisfy $\mathscr{P}$. In this case, by P^* we will denote the class Q.

We adopt the following notation for some properties.

$\mathscr{H}$: Helly property, i.e., if $J \subseteq I$ and $F_i \cap F_j \neq \emptyset$ for all $i, j \in J$ then $\bigcap_{i \in J} F_i \neq \emptyset$.
$\mathscr{C}$: Conformal, i.e., every clique of $\mathscr{F}_2$ is a set of $\mathscr{F}$. (Dual property of $\mathscr{H}$).
$\mathscr{L}_1$: Reduced, i.e., for all $i, j \in I$; $F_i \subseteq F_j$ implies $i = j$.
$\mathscr{L}_2$: There exists a total order $\leq$ on $\bigcup_{i \in I} F_i$ such that for all $i \in I$, F_i is an interval with respect to $\leq$.
$\mathscr{L}_3$: $\mathscr{L}_1$ and $\mathscr{L}_2$.
$\mathscr{L}_4$: There exists a tree T such that $V(T) = I$, which satisfies:
For all $i, j \in I$ if k is a vertex belonging to the path of T from i to j, then $F_i \cap F_j \subseteq F_k$. (Dual property of $\mathscr{L}_5$).
$\mathscr{L}_5$: There is a tree T such that $V(T) = \bigcup_{i \in I} F_i$ and each F_i induces a subtree of T.
$\mathscr{L}_6$: There is a tree T such that $V(T) = \bigcup_{i \in I} F_i$ and each F_i induces a path of T.
$\mathscr{L}_7$: There is a directed tree T such that $V(T) = \bigcup_{i \in I} F_i$ and each F_i induces a directed path of T.
$\mathscr{L}_8$: There is a rooted directed tree T such that $V(T) = \bigcup_{i \in I} F_i$ and each F_i induces a directed path of T.
$\mathscr{L}_9$: There exists a circular order α on $\bigcup_{i \in I} F_i$ such that for all $i \in I$, F_i is an interval (arc) with respect to α.

Through this notation we have:

$\Omega L_2 = IG$ (Class of interval graphs).
$\Omega L_3 = Clique L_3 = PIG$ (Class of proper interval graphs).
$\Omega L_4 = Clique L_5 = DUALLYCHORDAL$ (Class of dually-chordal graphs) [3].
$\Omega(C \cap H) = Clique\ H = HELLY$ (Class of clique-Helly graphs) [1].
$\Omega L_5 = Clique L_4 = CHORDAL$ (Class of chordal graphs).
$\Omega L_6 = UV$ (Class of UV-graphs).
$\Omega L_7 = DV$ (Class of DV-graphs).
$\Omega L_8 = RDV$ (Class of RDV-graphs).
$\Omega L_9 = CAG$ (Class of circular-arc graphs).
$Clique L_6 = ACI$ (Class of ACI-graphs).
$Clique L_7 = RET$ (Class of rooted expanded tree-graphs).

We will say that a class P is *closed under reductions* if the family of maximal sets (with respect to the inclusion) of any family of P is also in P and a class P is *closed under augmentations* if for every family $(F_i)_{i \in I}$ in P, $(F_i \cup \{i\})_{i \in I}$ is also in P.

3. A Characterization of Intersection Graphs

Before we give the characterization, let us recall a couple of results.

Lemma 3.1. *If $\mathscr{L}$ is a family of complete-sets of G which covers all edges of G (i.e., if $vu \in E(G)$, then $\{u, v\}$ is contained in some element of $\mathscr{L}$) then G is the intersection graph of the family $\mathscr{L}^* = \{\mathscr{L}(v)\}_{v \in V(G)}$.*

Proof. From the assumption on the family $\mathscr{L}$ it follows: $uv \in E(G)$ if and only if there is $L \in \mathscr{L}$ such that $u, v \in L$, if and only if $\mathscr{L}(u) \cap \mathscr{L}(v) \neq \emptyset$. Then G is the intersection graph of $\{\mathscr{L}(v)\}_{v \in V(G)}$. □

Lemma 3.2. *If G is the intersection graph of a family $\mathscr{F}$ then its dual family $\mathscr{F}^*$ is a family of complete-sets of G which covers all edges of G.*

Proof. Let $\mathscr{F} = (F_v)_{v \in V(G)}$, and let $x \in \bigcup_{v \in V(G)} F_v$. If $u, v \in \mathscr{F}(x)$ then $x \in F_u \cap F_v$ and since G is the intersection graph of $\mathscr{F}$, we have that $uv \in E(G)$. Therefore for all $x \in \bigcup_{v \in V(G)} F_v$, $\mathscr{F}(x)$ is a complete of G. On the other hand, if $uv \in E(G)$ then $F_u \cap F_v \neq \emptyset$ and $u, v \in \mathscr{F}(x)$ for all $x \in F_u \cap F_v$. Thus the dual family of $\mathscr{F}$ covers all edges of G. □

Theorem 3.3. *$G \in \Omega P$ if and only if there is a family of complete-sets, $\mathscr{L}$, of G such that*:

(1) *$\mathscr{L}$ covers all edges of G.*
(2) $\mathscr{L}^* = \{\mathscr{L}(v)\}_{v \in V(G)} \in P$.

Proof. Let $\mathscr{F}$ be a family in P such that G is its intersection graph and let $\mathscr{L} = \mathscr{F}^*$ be its dual family. By Lemma 3.2 we have that $\mathscr{L}$ is a family of complete-sets of G which covers all edges of G. Moreover the dual family $\mathscr{L}^*$ of $\mathscr{L} = \mathscr{F}^*$ is $\mathscr{F}$. Because $\mathscr{F} \in P$, $\mathscr{L}$ fulfills the conditions of the theorem. Conversely, suppose that $\mathscr{L}$ is a family of complete-sets of G such that $\mathscr{L}$ covers all edges of G. By Lemma 3.1, G is the intersection graph of $\mathscr{L}^* = \{\mathscr{L}(v)\}_{v \in V(G)}$; in addition $\{\mathscr{L}(v)\}_{v \in V(G)} \in P$, thus $G \in \Omega P$. □

This theorem generalizes one of Gavril's Theorem [10] for circular-arc graphs. Observe that $CAG = \Omega L_9$ and not every family in L_9 has the Helly property. However, in the case that property $\mathscr{P}$ implies the Helly property $\mathscr{H}$ we obtain the following result.

Corollary 3.4. *If $P \subseteq H$ and $G \in \Omega P$ then every family of complete-sets fulfilling the conditions of Theorem 3.3 contains $\mathscr{C}(G)$.*

Proof. Let $\mathscr{L} = (L_i)_{i \in I}$ be a family of complete-sets of G fulfilling the conditions of Theorem 3.3. Let R be a clique of G; since G is the intersection graph of $\mathscr{L}^* = \{\mathscr{L}(v)\}_{v \in V(G)}$ (see Lemma 3.1), for every pair u, v in R there is some $L_j \in \mathscr{L}$ containing the edge uv, hence $L_j \in \mathscr{L}(u) \cap \mathscr{L}(v) \neq \emptyset$. But $\{\mathscr{L}(v)\}_{v \in V(G)} \in P$ and $P \subseteq H$, thus this family has the Helly property and there is a complete $L_i \in \mathscr{L}$ such that $L_i \in \bigcap_{v \in R} \mathscr{L}(v)$. Therefore, for all v in R, $L_i \in \mathscr{L}(v)$ thus $v \in L_i$ and, by the maximality of R, $L_i = R$. Hence we have that every clique of G is a member of the family $\mathscr{L}$. □

Since G is the intersection graph of the dual family $\mathscr{C}^* = \{\mathscr{C}(v)\}_{v \in V(G)}$ of $\mathscr{C}(G)$, if $\{\mathscr{C}(v)\}_{v \in V(G)}$ is in P we obtain that $G \in \Omega P$. Thus, from Corollary 3.4 we can give sufficient conditions to characterize the elements in ΩP in terms of the dual family of its cliques.

Theorem 3.5. *If $P \subseteq H$ and P^* is closed under reductions then $G \in \Omega P$ if and only if $\mathscr{C}^*(G) = \{\mathscr{C}(v)\}_{v \in V(G)} \in P$.*

Proof. If $G \in \Omega P$ then there is a family $\mathscr{L}$ fulfilling the conditions of Theorem 3.3 and by Corollary 3.4, $\mathscr{L}$ contains $\mathscr{C}(G)$. Because $\mathscr{C}(G)$ is the family of maximal sets of $\mathscr{L}$ and P^* is closed under reductions $\mathscr{C}(G) \in P^*$ thus the result follows. □

Lemma 3.6. *If $P = L_2, L_4, L_5, L_6, L_7$ or L_8, P^* is closed under reductions.*

Proof. Case. $P = L_5$: Let $(F_i)_{i \in I} \in L_5$ then there is a tree T such that $V(T) = \bigcup_{i \in I} F_i$ and each F_i induces a subtree of T . Suppose that there are two vertices $x, y \in V(T)$ such that $\mathscr{F}(x) \subseteq \mathscr{F}(y)$, if z is the next vertex to x on the only path in T from x to y, then $\mathscr{F}(x) \subseteq \mathscr{F}(z) \subseteq \mathscr{F}(y)$. By coalescing x and z and eliminating the edge between them, we obtain a new tree T' from T. Since all F_i which contains x contains z, we have that $\mathscr{F}' = (F_i - \{x\})_{i \in I} \in L_5$ and $\{\mathscr{F}'(v)\}_{v \in V(T')} \in {L_5}^*$. In this manner we can obtain that the family of maximal sets of $\{\mathscr{F}(v)\}_{v \in V(T)}$ is in ${L_5}^*$.

Cases L_2, L_6, L_7 and L_8 are similar.
Case. $P = L_4$ is trivial because ${L_4}^* = L_5$ which is closed under reductions. □

This result shows that Theorem 3.5 generalizes Fulkerson-Gross's Theorem [6], which characterizes interval graphs, Gavril's results [7, 8, 9] for chordal graphs, *UV*-graphs, and *RDV*-graphs, and the Clique-tree Theorem [19] for *DV*-graphs and other families.

Another form of Theorem 3.5, interchanging P by P^* is: If $P^* \subseteq H$ and P is closed under reductions then $\Omega P^* = CliqueP$. Observe that $P^* \subseteq H$ is equivalent to $P \subseteq C$ because $\mathscr{H}$ and $\mathscr{C}$ are dual properties. Thus, by taking $P \cap C$, which is closed under reductions because P and C have this property, the result will be a characterization of $Clique(P \cap C)$. But the family of cliques of any graph is conformal, then $CliqueP = Clique(P \cap C)$ and we obtain a characterization of *CliqueP* as a class of intersection graphs of Helly families, as follows.

Corollary 3.7. *If P is closed under reductions then $\Omega(P^* \cap H) = CliqueP$.*

Proof. Since $(P \cap C)^* = P^* \cap H$, the result follows. □

In this way we obtain the already known result for $HELLY$, by taking $P = H$, and a new result for $DUALLYCHORDAL$, by taking $P = L_5$.

Corollary 3.8. $HELLY = \Omega(C \cap H)$ [1].

$DUALLYCHORDAL = \Omega(L_4 \cap H)$.

4. The Clique Operator

The results obtained in the previous section permit to study the behavior of the Clique Operator in some particular classes of intersection graphs.

Theorem 4.1. *If $P \subseteq C$ and P is closed under reductions then $K(\Omega P^*) \subseteq \Omega P$.*

Proof. By Theorem 3.5, if $G \in \Omega P^*$ then $\mathscr{C}^*(G) = \{\mathscr{C}(v)\}_{v \in V(G)} \in P^*$ and $\mathscr{C}(G) \in P$. Therefrom $K(G)$, the intersection graph of the family $\mathscr{C}(G)$, is in ΩP. □

For the converse situation, we will need that any graph of ΩP can be represented by a reduced and conformal family.

Theorem 4.2. *If $\Omega P = \Omega(P \cap C \cap L_1)$ then $\Omega P \subseteq K(\Omega P^*)$.*

Proof. Let $G \in \Omega P$. By Theorem 3.3 there is a family $\mathscr{L}$ of complete-sets of G such that:

(1) $\mathscr{L}$ covers all edges of G.
(2) $\mathscr{L}^* = \{\mathscr{L}(v)\}_{v \in V(G)} \in P \cap C \cap L_1$.

Let $H = \mathscr{L}^*_2$. Since $\mathscr{L}^*$ is a conformal reduced family $\mathscr{C}(H) = \mathscr{L}^*$. Then $K(H) = G$ because $K(H)$ is the intersection graph of the family of cliques of H and G is the intersection graph of $\mathscr{L}^* = \mathscr{C}(H)$ (see Lemma 3.1). On the other hand, H is the intersection graph of $\{\mathscr{C}(v)\}_{v \in V(H)} = \mathscr{C}^*(H) = \mathscr{L} \in P^*$ thus $H \in \Omega P$. □

In the following corollaries we will see how these results can be used in particular classes.

Corollary 4.3. [18]

(1) $K(PIG) = PIG$.
(2) $K(IG) = PIG$.

Proof. (1) $L_3 \subseteq C$ and L_3 is closed under reductions, hence $\Omega L_3{}^* = CliqueL_3$, furthermore $K(\Omega L_3{}^*) \subseteq \Omega L_3$, i.e., $K(PIG) \subseteq PIG$. On the other hand, since $L_3 \subseteq L_1$, the conditions of Theorem 4.2 are true, and the other inclusion follows. □

(2) Since $L_2 \subseteq H$, $L_2{}^* \subseteq C$ and $L_2{}^*$ is closed under reductions, we obtain that $K(\Omega L_2) \subseteq \Omega L_2{}^*$. In addition $L_2 \subseteq C$ and L_2 is closed under reductions then $\Omega L_2{}^* = CliqueL_2$ and this last class is $CliqueL_3$. Therefrom $K(IG) \subseteq PIG$ and by (1) we obtain $K(IG) = PIG$. □

Corollary 4.4. (1) [3] [22] *$K(CHORDAL) = DUALLYCHORDAL$.*

(2) [17] *$K(DV) = ACI$ and $K(ACI) = DV$.*
(3) [23] *$K(RDV) = RET$ and $K(RET) = RDV$.*

Proof. (1) We will prove the conditions of Theorems 4.1 and 4.2 for $P = L_4$. By Lemma 3.6 $L_5{}^* = L_4$ is closed under reductions. Since every family in L_5 has the Helly property [4], $L_4 \subseteq C$. Then $\Omega L_4 = \Omega(L_4 \cap C)$. On the other hand, if G is the intersection graph of a family $(F_v)_{v \in V(G)} \in L_4 \cap C$ then also is of $(F_v \cup \{v\})_{v \in V(G)}$ and it is easy to see that L_4 and C are closed under augmentations, thus this family is belonging to $L_4 \cap C$ too. Therefore $\Omega L_4 = \Omega(L_4 \cap C \cap L_1)$.

(2) Since $L_7 \subseteq H$, $L_7 \subseteq C$, $\Omega L_7 = \Omega(L_7 \cap L_1)$ and, analogously to the proof for L_4, $\Omega L_7{}^* = \Omega(L_7{}^* \cap L_1)$. Therefrom the result follows.

(3) Analogously to the above by taking $P = L_8$. □

Observe the difference between (1) and (2)–(3):

$K(DUALLYCHORDAL) \neq CHORDAL$ because not every chordal graph is in the image of K.

But if we take $P \cap C \cap H$ in Theorems 4.1 and 4.2 we obtain the following result for classes P closed under augmentations.

Theorem 4.5. *If P is closed under reductions and augmentations then*

$$K(CliqueP \cap HELLY) = \Omega(P \cap C \cap H).$$

Proof. Since P, C and H are closed under reductions and augmentations we have that $P \cap C \cap H$ has these properties too. Thus $\Omega(P \cap C \cap H) = \Omega(P \cap C \cap H \cap L_1)$.

Then the hypotheses of Theorems 4.1 and 4.2 are true for $P \cap C \cap H$ thus $\Omega(P \cap C \cap H) = K(\Omega(P \cap C \cap H)^*) = K(\Omega(P \cap H)^* \cap H)$ because $C^* = H$.

On the other hand, by Corollary 3.7 for $P \cap H$, $\Omega(P \cap H)^* \cap H) = Clique(P \cap H)$ and this is equal to $CliqueP \cap HELLY$ because $HELLY$ is the class of Clique-Helly graphs.

Hence $\Omega(P \cap C \cap H) = K(CliqueP \cap HELLY)$. □

Corollary 4.6. *$K(CHORDAL \cap HELLY) = DUALLYCHORDAL$ and $K(DUALLY$-$CHORDAL) = CHORDAL \cap HELLY$.*

Proof. If we take $P = L_4$, since $L_4 \subseteq C$, L_4 is closed under reductions and augmentations we obtain

$K(CliqueL_4 \cap HELLY) = \Omega(L_4 \cap H)$, i.e., $K(CHORDAL \cap HELLY) = DUALLY$-$CHORDAL$ (see Corollary 3.8).

The other equality follows by taking $P = L_5$, since L_5 is closed under reductions and augmentations we obtain

$K(CliqueL_5 \cap HELLY) = \Omega(L_5 \cap C \cap H) = \Omega((L_4 \cap H)^* \cap H)$. Then, by Corollary 3.7, $\Omega((L_4 \cap H)^* \cap H) = Clique(L_4 \cap H)$.

Hence $K(DUALLYCHORDAL) = CHORDAL \cap HELLY$. □

In addition, the Roberts-Spencer's results for the image of K can be obtained as a corollary considering $\mathcal{G}$ as the class of all graphs.

Corollary 4.7. [21] $K(\mathcal{G}) = \Omega C$

Proof. As we said earlier, the family of cliques of any graph is conformal, thus $\mathcal{G} = CliqueC$. Since C is closed under reductions, we have that $CliqueC = \Omega C^*$ and $K(CliqueC) \subseteq \Omega C$. The other inclusion follows because $\Omega C = \Omega(C \cap L_1)$ (see proof for L_4 in Corollary 4.4). Finally the characterization of intersection graphs (Theorem 3.3) conduces to Roberts-Spencer's Theorem. □

Another result that can be obtained as a corollary is that of Bandelt-Prisner about K-fixed classes (a class of graphs $\mathcal{R}$ is *K-fixed* when $K(\mathcal{R}) = \mathcal{R}$).

Corollary 4.8. [1] *If $P = P^*$, $P \subseteq C$, P is closed under reductions and augmentations then $\Omega P = CliqueP$ is K-fixed.*

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