Foundations of Physics Letters, Vol. 13, No. 2, 2000

ADDENDUM TO "TREATMENT OF GAMOW STATES USING TEMPERED ULTRADISTRIBUTIONS"[1]

A. L. De Paoli, M. A. Estevez, M. C. Rocca, and H. Vucetich

Departamento de Física, Fac. de Ciencias Exactas
Universidad Nacional de La Plata
C. C. 67 (1900) La Plata, Argentina

Received 26 January 2000

In Ref. [1] we showed that it is possible to extend analitically, and with the use of tempered ultradistributions, the pseudonorm defined by T. Berggren for Gamow states. In that reference we define this pseudonorm for all states determined by the zeros of the Jost function for any short range potential. However, the proof is not completely general due to the fact that the statement $h_l(0,r) = C\phi_l(0,r)$ is not true in all cases. In this addendum we give a new proof, general and independent of that statement.

Key words: tempered ultradistributions, Gamow states.

1. THE PSEUDONORM OF EIGENSTATES OF SHORT-RANGE POTENTIALS

According to Ref. [2], the Jost function is related to the regular and irregular solutions via

$$\mathcal{W}\left[f_l(k,r),\phi_l(k,r)\right] = \frac{f_l(k)}{k^l}, \tag{1}$$

where $\mathcal{W}[f,\phi]$ is the Wronskian of the two solutions.

---

[1] *This work was partially supported by Consejo Nacional de Investigaciones Científicas, Comisión de Investigaciones Científicas de la Pcia. de Buenos Aires, Argentina and by PMT-PICT0079 of ANPCYT (FONCYT), Argentina.*

Then the derivative $\dot{f}_l(k)$ of the Jost function with respect to the variable k satisfies

$$\mathcal{W}\left[\dot{f}_l(k,r), \phi_l(k,r)\right] + k^l \mathcal{W}\left[f_l(k,r), \dot{\phi}_l(k,r)\right] = \frac{\dot{f}_l(k)}{k^l} - \frac{l f_l(k)}{k^{l+1}}. \quad (2)$$

In particular, when k_0 is a zero of the Jost function, (2) takes the form

$$\dot{f}_l(k_0) = k_0^l \mathcal{W}\left[\dot{f}_l(k_0,r), \phi_l(k_0,r)\right] + k_0^l \mathcal{W}\left[f_l(k_0,r), \dot{\phi}_l(k_0,r)\right]. \quad (3)$$

The regular and irregular solutions are related at $k = k_0$ by the Jost function (Ref. [2]):

$$f_l(k_0,r) = C(k_0)\phi_l(k_0,r), \quad C(k_0) = \frac{-2ik_0^{l+1}}{f_l(-k_0)}. \quad (4)$$

Following the procedure of Ref. [2], we get from (3):

$$\dot{f}_l(k_0) = {k_0}^l \lim_{\beta\to\infty} \{\mathcal{W}\left[\dot{f}_l(k_0,\beta), \phi_l(k_0,\beta)\right] - 2k_0 C(k_0) \int_0^\beta {\phi_l}^2(k_0,r)\, dr\}. \quad (5)$$

From (5), we deduce immediately

$$\lim_{\beta\to\infty} \int_0^\beta {\phi_l}^2(k_0,r)\, dr$$

$$= -\lim_{\beta\to\infty} \frac{f_l(-k_0)}{4ik_0^{l+2}} \mathcal{W}\left[\dot{f}_l(k_0,\beta), \phi_l(k_0,\beta)\right] + \frac{\dot{f}_l(k_0) f_l(-k_0)}{4ik_0^{2l+2}}. \quad (6)$$

Now we want to show that: (i) the integral appearing in (6) can be defined as an ultradistribution in the variable k_0 and (ii) in the limit $\beta \to \infty$, as an ultradistribution in k_0, the Wronskian $\mathcal{W}$ vanishes. With this purpose and according to Ref. [2], we note that $k^l f_l(k,r) = h_l(k,r)$ is an entire analytic function of the variable k and therefore so is $k^{l+1} f_l(k,r)$. Hence $k^{l+1} \dot{f}_l(k,r) = g_l(k,r)$ is an entire analytic function of k. We can now write (5) in terms of $g_l(k,r)$ as

$$\dot{f}_l(k_0) = \lim_{\beta\to\infty} \left\{ \frac{\mathcal{W}\left[g_l(k_0,\beta), \phi_l(k_0,\beta)\right]}{k_0} - 2k_0^{l+1} C(k_0) \int_0^\beta {\phi_l}^2(k_0,r)\, dr \right\}. \quad (7)$$

But

$$\lim_{\beta\to\infty} \oint_{\Gamma} \frac{\mathcal{W}\left[g_l(k,\beta),\phi_l(k,\beta)\right]}{k}\phi(k)\,dk$$

$$= \lim_{\beta\to\infty} \mathcal{W}\left[g_l(0,\beta),\phi_l(0,\beta)\right]\phi(0), \tag{8}$$

where $\phi(k) \in h$ is an entire analytic test function and the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $Im(k) > \rho$, $\rho > 0$ and back from ∞ to $-\infty$ for $Im(k) < -\rho$, $-\rho < 0$ (Γ lies outside a horizontal band that contains the singularity in the origin). In terms of

$$k^{l+1}\dot{f}_l(k,r) = g_l(k,r), \quad k^l f_l(k,r) = h_l(k,r),$$

Eq. (2) transforms into

$$\mathcal{W}\left[g_l(k,r),\phi_l(k,r)\right] + k\mathcal{W}\left[h_l(k,r),\dot{\phi}_l(k,r)\right] = k f_l(k) - l f_l(k), \tag{9}$$

and, when $k = 0$,

$$\mathcal{W}\left[g_l(0,r),\phi_l(0,r)\right] = -l f_l(0). \tag{10}$$

Therefore,

$$\lim_{\beta\to\infty} \frac{\mathcal{W}\left[g_l(k,\beta),\phi_l(k,\beta)\right]}{k} = -\frac{l f_l(k)}{k} + P_l(k). \tag{11}$$

We select $P_l(k) = 0$, so that

$$\lim_{\beta\to\infty} \frac{\mathcal{W}\left[g_l(k,\beta),\phi_l(k,\beta)\right]}{k} = -\frac{l f_l(k)}{k}. \tag{12}$$

Then, when $k = k_0$ is a zero of the Jost function, one gets

$$\lim_{\beta\to\infty} \frac{\mathcal{W}\left[g_l(k_0,\beta),\phi_l(k_0,\beta)\right]}{k_0} = 0. \tag{13}$$

Alternatively, we can have

$$\lim_{\beta\to\infty} \frac{\mathcal{W}\left[g_l(k,\beta),\phi_l(k,\beta)\right]}{k} = -\frac{l f_l(0)}{k} + P_l(k). \tag{14}$$

In this case we select

$$P_l(k) = \frac{l[f_l(0) - f_l(k)]}{k} \tag{15}$$

and recover Eq. (13).

As a consequence of Eq. (13), Eq. (7) takes the form

$$\lim_{\beta \to \infty} \int_0^\beta {\phi_l}^2(k_0, r)\, dr = \frac{\dot{f}_l(k_0) f_l(-k_0)}{4ik_0^{2l+2}}, \tag{16}$$

where the limit is taken in the sense of ultradistributions. By definition the pseudonormalized state is

$$\psi_l(k_0, r) = \left[\frac{4ik_0^{2l+2}}{\dot{f}_l(k_0) f_l(k_0)}\right]^{1/2} \phi_l(k_0, r) \tag{17}$$

and can be thought of as a tempered ultradistribution in the variable k_0.

REFERENCES

1. A. L. De Paoli, M. Estevez, M. C. Rocca, and H. Vucetich, *Found. Phys. Lett.* **12**, 497 (1999).
2. R. G. Newton, *J. Math. Phys.* **1**, 319 (1960).