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Smoothed Wigner-distributions, decoherence, and the temperature-dependence of the classical-quantum frontier

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Abstract. We discuss the strange behavior at $T = 0$ of the phase-space Wigner distribution of the harmonic oscillator with the help of the purely quantal concepts of participation ratio of a mixed state and decoherence parameter. We also show that Wigner distribution-smoothing yields interesting insights, specially in the pathological instance in which the Wigner-Fano factor diverges. The associated decoherence parameter also sheds some light on temperature-dependence of the classical-quantum frontier.

1 Introduction

Quantal phase-space distributions constitute a subject of permanent interest, with applications in statistical mechanics, quantum chemistry, quantum optics, and classical optics. Also for signal analysis in diverse fields such as electrical engineering, seismology, biology, speech processing, and engine design [1]. Obligatory reference in this respect it to be made to the celebrated Wigner quasi-probability distribution $D_w(x,p)$ (also called the Wigner function or the Wigner-Ville distribution), a special type of quasi-probability distribution that was introduced by Wigner [2] to study quantum corrections to classical statistical mechanics. In trying to approximate it in some fashion one is prone to fall into the semiclassical domain. In turn, the quantum-classical transition presents alluring challenges and open problems (see, for instance, [3–10]). We will here see that some *temperature*-dependent details concerning the route towards such a limit are illuminated via quantal phase-space distributions à la Wigner.

Wigner's goal was to supplant the wave-function that appears in Schröedinger's equation with a probability distribution D_w in phase space. This D_w should function as a generating function for all spatial autocorrelation functions of a given quantum-mechanical wave-function $\psi(x)$. Thus, in the map between real phase-space functions and Hermitian operators introduced by Weyl [11], D_w maps on the quantum density matrix [11]. One speaks of the Weyl-Wigner transform of the density matrix. In 1949 Moyal [12], who had also re-derived it independently, recognized D_w as the quantum moment-generating functional, i.e., as the basis of an elegant encoding of all quantum expectation values, and hence quantum mechanics in phase space (*Weyl quantization*). Wigner's is the most elaborate phase-space formulation of quantum mechanics [2,13,14].

A rival phase space distribution is the one developed by Husimi (see more details below) [15–17]. Although both the Wigner and the Husimi distributions carry complete information regarding a quantum state, they exhibit different features. The Wigner function displays large oscillations and may adopt negative values which make it a quasi-distribution rather than a classical probability density. On a compact phase space of area A it is able to reveal fine structures on a sub-Planck scale of order $\hbar^2/A$ [18], structures that can be traced to quantum interferences from distant localized objects [18–21], that in turn enhance the state's sensitivity to perturbations [18–21]. Instead, the Husimi distribution is known to be a Gaussian smearing of the Wigner function on an area of size $\hbar$ that washes out the negative part and hence it is suitable as a probability density [22]. However, such smoothing may hide significant important attributes or aspects of the Wigner function [23]. Summing up, while the Wigner function exhibits high resolution, it is not free of long range quantum interferences. The Husimi distribution washes out quantum interferences at the price of hiding important semiclassical structures [23].

1.1 Pathological instances

It is known that there are some pathological cases for which the Weyl-Wigner procedure, that maps Hermitian

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operators $\hat{A}$ to phase-space functions $A_w(x,p)$, does not give the the correct correspondence between classical and quantum operators [24]. Among these circumstances one encounters an important operator, the square of the Hamiltonian [24], in which the Wigner function yields wrong results [24]. Husimi distributions are free from such defects. Smoothing may thus, in some special situations, "improve" on Wigner's descriptive capability. The smoothing or smearing process is the focus of attention in the present work, with reference to the so-called "intermediate" smoothing ideas advanced in reference [21]. We will illustrate our procedures with reference to the harmonic oscillator (HO) instance. This is such an important system that HO insights usually have a wide impact, as the HO constitutes much more than a mere example. Nowadays it is of particular interest for the dynamics of bosonic or fermionic atoms contained in magnetic traps [25–27] as well as for any system that exhibits an equidistant level spacing in the vicinity of the ground state, like nuclei or Luttinger liquids. Among many other examples one may mention that it is possible to describe relevant quantum effects in some bio-systems by approximating a group of proteins and its environment by a set of coupled harmonic oscillators [28].

We will employ information theory tools for our endeavor and illustrate our proceedings with reference to several quantal quantities, noise-signal factors in particular. It will be shown that a moderate amount of "smoothing" does improve on the phase-space distribution's predictive power. We pass now to discuss our main information-theoretic measure.

2 Fisher information

This is our information-theory tool. The last years have viewed a great deal of effort revolving around physical applications of Fisher's information measure (FIM) [29,30]. FIM is the source of a powerful variational principle, the extreme physical information one, that yields most of the canonical Lagrangians of theoretical physics [29,30], characterizing also in quite a proper fashion an "arrow of time", alternative to the one associated with Boltzmann's entropy [31,32]. The classical Fisher information associated with translations of a one-dimensional observable x with corresponding probability density $\rho(x)$ is [33]

$$I_x = \int dx\, \rho(x) \left(\frac{\partial \ln \rho(x)}{\partial x} \right)^2, \tag{1}$$

which obeys the so-called Cramer-Rao inequality

$$(\Delta x)^2 \geq I_x^{-1} \tag{2}$$

involving the variance of the stochastic variable x [33]

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \int dx\, \rho(x)\, x^2 - \left(\int dx\, \rho(x)\, x \right)^2. \tag{3}$$

In particular, for the Wigner distribution function of the harmonic oscillator $A_w(z)$ [2], the associated Fisher information measure reads [34]

$$I_w = \frac{1}{4} \int \frac{d^2 z}{\pi}\, A_w(z) \left(\frac{\partial \ln A_w(z)}{\partial |z|} \right)^2, \tag{4}$$

and, integrating over phase space, we find for the HO

$$I_w = 2 \tanh(\beta \hbar \omega / 2). \tag{5}$$

Note that $0 \leq I_w \leq 2$.

3 Bi-dimensional Gaussian smoothing

The simpler one-dimensional setting is discussed in the Appendix. Consider here the two bi-dimensional phase-space variables (x,p) and (X,P). We have

$$|z|^2 \equiv z^2 = \frac{1}{2\hbar} \left(m\omega x^2 + \frac{p^2}{m\omega} \right) = \frac{x^2}{4\sigma_x^2} + \frac{p^2}{4\sigma_p^2}, \tag{6}$$

and a similar expression for the Z-instance (we must replace x, p, and z by X, P, and Z, respectively). Introduce now the normalized kernel

$$G(z-Z) = \frac{z^{-1}}{\xi}\, e^{-(1/\xi)(z-Z)^2}, \tag{7}$$

where ξ a real non-negative parameter and the phase-space HO-Wigner function $A_w(z) = I_w e^{-I_w |z|^2}$ [2,34], with

$$\int \frac{d^2 z}{\pi}\, G(z-Z) = 1; \quad \int \frac{d^2 z}{\pi}\, A_w(z) = 1. \tag{8}$$

Our objective now it to smooth $A_w(z)$ and get a new phase-space $A_\xi(Z)$

$$A_\xi(Z) = \int \frac{d^2 z}{\pi}\, G(z-Z)\, A_w(z). \tag{9}$$

Setting $b = 1/\xi$ we then have

$$A_\xi(Z) = b I_w \int \frac{d^2 z}{\pi}\, z^{-1}\, e^{-b(z-Z)^2 - I_w z^2}. \tag{10}$$

It is now of help to introduce auxiliary quantities, namely,

$$\mu = b I_w = I_w/\xi; \quad \gamma = I_w + b, \tag{11}$$

and

$$A_\xi(Z) = \mu \int \frac{d^2 z}{\pi}\, z^{-1}\, e^{\Delta}, \tag{12}$$

with

$$\begin{aligned} \Delta &= -b\, [\xi\gamma\, z^2 + Z^2 - 2zZ] \\ &= -\gamma\, [(z-A)^2 + B], \end{aligned} \tag{13}$$

from which we identify

$$A = Z\, b/\gamma, \\ B = Z^2\,(b/\gamma)(1 - b/\gamma), \tag{14}$$

entailing

$$A_\xi(Z) = bI_w\, e^{-\frac{I_w}{\xi I_w+1}\, Z^2} \int \frac{d^2 z}{\pi}\, z^{-1}\, e^{-\gamma(z-A)^2}, \tag{15}$$

i.e.,

$$A_\xi(Z) = \frac{I_w}{\xi I_w + 1}\, e^{-\frac{I_w}{\xi I_w+1}\, Z^2}. \tag{16}$$

For true smearing we need $I_w/(\xi I_w + 1) < I_w$ that implies

$$\xi \geq 0 \tag{17}$$

(remember that $0 \leq I_w \leq 2$).

4 Husimi smoothing

As we saw, the Husimi distribution is known to be a Gaussian smearing of the Wigner function on an area of size $\hbar$ that washes out the negative part and hence it is suitable as a probability density [36]. For an HO of frequency ω at the temperature T the Fisher measure associated to the Husimi distribution reads [35]

$$I_h = 1 - e^{-\beta\hbar\omega} \equiv 1 - e^{-\hbar\omega/T} \equiv 1 - e^{-d}, \tag{18}$$

where the Husimi distribution acquires the form [35]

$$A_{Husimi} = I_h e^{-I_h\,|z|^2}. \tag{19}$$

Comparing now I_w with I_h a little algebra yields the relation

$$I_h = I_w\, \frac{1 + e^{-d}}{2}, \\ I_w = 2\frac{1 - e^{-d}}{1 + e^{-d}}, \tag{20}$$

so that, enforcing the equality

$$I_h = \frac{I_w}{\xi I_w + 1}, \tag{21}$$

one easily determines the smoothing factor ξ that leads from the Husimi distribution to the Wigner one, namely,

$$\xi = 1/2. \tag{22}$$

We see then that for $\xi = 0$ we have $I_\xi = I_w$ while for $\xi = 1/2$ one has $I_\xi = I_h$. The Fisher measure for smearing-parameter ξ is in general, according to (4)

$$I_\xi = \frac{I_w}{\xi I_w + 1}, \tag{23}$$

and steadily diminishes as ξ grows from 0 to 1/2, with $0 \leq I_\xi \leq 2$. Note in particular that for $\xi = 1$ one has

$$I_{\xi=1} = \frac{I_w}{I_w + 1}; \quad 0 \leq I_{\xi=1} \leq 2/3. \tag{24}$$

5 Participation ratio

By recourse to the concept of participation ratio $\mathcal{R}$ of a density operator $\hat\rho$ one is able to ascertain just how many quantum pure states enter it [37],

$$\mathcal{R} = \frac{1}{\mathrm{Tr}(\hat\rho^2)}; \quad 1 \leq \mathcal{R} \leq \infty. \tag{25}$$

The phase-space "equivalent-notion" is derived replacing traces by integrals over the phase-space and effecting the analogous calculation, using now the HO-A_ξ-distribution (16). Thus, one establishes the correspondence

$$\hat\rho \Leftrightarrow A_\xi; \quad \mathrm{Tr} \Leftrightarrow \int d^2 z/\pi. \tag{26}$$

As shown in references [38,39] one has

$$\mathcal{R}_\xi = \frac{1}{\int \frac{d^2 z}{\pi}\, A_\xi^2}, \tag{27}$$

which turns out to yield

$$\mathcal{R}_\xi = \frac{2}{I_\xi}. \tag{28}$$

It follows that

$$\mathcal{R}_\xi\, I_\xi = 2, \tag{29}$$

revealing a new (as far as we know) information-number of states complementarity relation. Since $0 \leq I_\xi \leq 2$, then $\mathcal{R}_\xi \geq 1$. The inverse Fisher's information measure directly yields half the number of states entering the density operator, giving the Fisher information the meaning of half the number of states inverse. The participation ratio for pure sates (namely, unity) is recovered at $\xi = T = 0$. For $T = 0$ and $\xi = 1$, $\xi = 1/2$ (Husimi instance) one has, respectively,

$$\mathcal{R}_1 = 3; \quad \mathcal{R}_{1/2} = 2, \tag{30}$$

indicating that three (two) pure states enter the associated density operator. For $\xi > 0$, as it grows, it also systematically augments the number of pure states "attributed" to the phase-space distribution A_ξ.

6 Decoherence parameter

Decoherence is that interesting process whereby the quantum mechanical state of any macroscopic system becomes rapidly correlated with that of its environment in such a manner that no measurement on the system alone (without a simultaneous measurement of the complete state of the environment) can exhibit any interference between two quantum states of the system. Decoherence is a rather exciting phenomenon and a subject of widespread attention [40,41]. However, it is difficult to provide a quantitative definition of it. All pertinent attempts always depend on the relevant experimental configuration and on the authors' taste [42–44]. An important related quantity is the

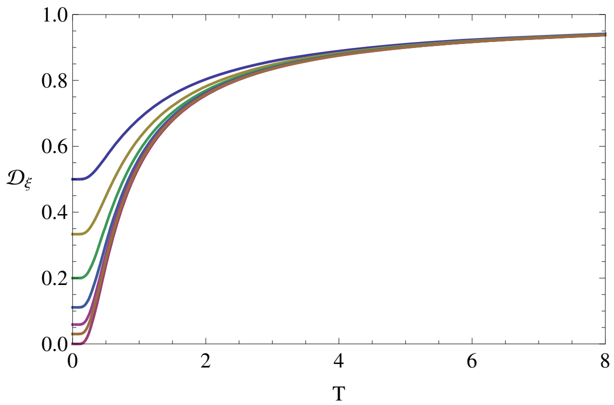

Fig. 1. (Color online) Decoherence parameter for various smoothing-values ξ, as a function of T in units of $\hbar\omega$. The uppermost curve corresponds to $\xi = 1/2$ and the lowest curve to $\xi = 0$. The intermediate values of ξ correspond to $1/4, 1/8, 1/16, 1/32$, and $1/64$, respectively.

square of the density matrix, in whose terms one can define a decoherence parameter $\mathcal{D}$ [45,46], ranging between 0 and 1,

$$\mathcal{D} = 1 - \frac{\mathrm{Tr}(\hat{\rho}^2)}{(\mathrm{Tr}\hat{\rho})^2}, \tag{31}$$

which we can also easily evaluate in I_ξ terms, i.e.,

$$\mathcal{D}_\xi = 1 - \frac{I_\xi}{2} = 1 - \frac{1}{\mathcal{R}_\xi}, \tag{32}$$

that makes it explicit a novel link between participation ratio and decoherence parameter. As one can expect, the larger the number of states entering a mixed state, the larger its decoherence parameter.

Note in Figure 1 that for $T \geq 4$ (in $\hbar\omega$-units), the smoothing parameter ξ plays no role at all, while that role is prominent at $T = 0$. This entails that one may speak of a "temperature-dependent" quantum-classical transition, of which ξ is a signature.

7 Thermal Wigner uncertainties for the HO

The literature on thermal uncertainties-relations is quite extensive. See, for instance [47–49] and references therein. We can immediately ascertain that the A_ξ – mean values vanish for any Gaussian distribution [38]

$$\langle x \rangle = \langle p \rangle = \langle z \rangle = 0. \tag{33}$$

Additionally, for Husimi distributions A_h one has [38]

$$\langle x^2 \rangle_{A_h} = \frac{2\sigma_x^2}{1 - e^{-\beta\hbar\omega}}; \quad \langle p^2 \rangle_{A_h} = \frac{2\sigma_p^2}{1 - e^{-\beta\hbar\omega}}. \tag{34}$$

Thus,

$$\Delta_{A_h} x \Delta_{A_h} p = \frac{\hbar}{I_h} \tag{35}$$

whose limit $T \to 0$ is

$$(\Delta_{A_h} x \Delta_{A_h} p)_{T=0} = \hbar, \tag{36}$$

in agreement with the "smearing" property of the Husimi distribution that has been alluded to above. Thermal A_ξ uncertainties for the HO constitute our objective now. We immediately find

$$(\Delta_\xi x)^2 = \langle x^2 \rangle_\xi = \int \frac{d^2 z}{\pi}\, x^2\, A_\xi(z) = \frac{\sigma_x^2}{2 I_\xi}. \tag{37}$$

In a similar vein

$$(\Delta_\xi p)^2 = \langle p^2 \rangle_\xi = \frac{\sigma_p^2}{2 I_\xi}, \tag{38}$$

which entails

$$\Delta_\xi \equiv \Delta_\xi x\, \Delta_\xi p = \frac{\hbar}{I_\xi} = \frac{\hbar}{2}\mathcal{R}_\xi, \tag{39}$$

that is the celebrated Heisenberg uncertainty relation, in the $\xi = T = 0$ – limit, recovered as the result of having incorporated $\hbar^2$-effects into the semiclassical description. Notice the interesting role played by the participation ratio. It controls the way in which the thermal uncertainty grows with T as temperature grows. Additionally,

$$\langle |z|^2 \rangle_\xi = \int \frac{d^2 z}{\pi}\, |z|^2\, A_\xi(z) = \frac{1}{I_\xi}, \tag{40}$$

which entails

$$(\Delta_\xi z)^2 I_\xi = 1, \tag{41}$$

i.e., the Cramer-Rao relation is always saturated (not only at $T = 0$), irrespective of the smoothing-parameter's value.

8 Fano factor

In statistics, the Fano factor [50] is a measure of the dispersion of a probability distribution, defined as

$$\mathcal{F} = \frac{\sigma_U^2}{\mu_U}, \tag{42}$$

where σ_U^2 is the variance and μ_U is the mean of a random process in some time-window U. The Fano factor can be viewed as a kind of noise-to-signal ratio, being a measure of the reliability with which the random variable can be estimated from a time window that on average contains several random events. For a Poisson process, the variance in the count equals the mean count, so that $\mathcal{F} = 1$. If the time window is chosen to be infinity, the Fano factor is similar to the variance-to-mean ratio which in statistics is also known as the "index of dispersion", dispersion index or coefficient of dispersion. This is a normalized measure of the dispersion of a probability distribution. In other words, it tells us whether a set of observed occurrences are clustered or dispersed compared to a standard statistical model. As just stated, the Poisson distribution has $\mathcal{F}=1$, but both the geometric and the negative binomial distributions have $\mathcal{F}>1$, while in the binomial instance we find $\mathcal{F}<1$. A constant "random variable" has $\mathcal{F}=0$. The Fano factor turns out to be a convenient noise-indicator of a non-classical field. In the case of a photon distribution (N photons) it reads [51]

$$\mathcal{F}\equiv\sigma=\frac{(\Delta\hat{N})^2}{\langle\hat{N}\rangle}, \tag{43}$$

and is in this sense related to the so-called Mandel parameter Q [52]

$$Q=\frac{(\Delta\hat{N})^2}{\langle\hat{N}\rangle}-1\equiv\mathcal{F}-1, \tag{44}$$

In consonance with the above remarks, for $\mathcal{F}<1$ ($Q\leq 0$), emitted light is referred to as sub-Poissonian since it has photo-count noise smaller than that of coherent (ideal laser) light with the same intensity $\mathcal{F}=1$ ($Q=0$), whereas for $\mathcal{F}>1$, ($Q>0$) the light is called super-Poissonian, exhibiting photo-count noise higher than the coherent-light noise. Of course, one wishes to minimize the Fano factor.

For a coherent state the Mandel parameter vanishes, i.e., $Q=0$ and $\mathcal{F}=1$. It is important to note that a field in a coherent state is considered to be the closest possible one to a classical field, since it saturates the Heisenberg uncertainty relation and has the same uncertainty in each quadrature component. Therefore, $Q=0$ or $\mathcal{F}=1$ define a boundary between a classical and a quantum field. It is clear then that both Q and $\mathcal{F}$ can function as indicators on non-classicality. Indeed, for a thermal state one has $Q>0$ and $F>1$, corresponding to a photon distribution broader than the Poissonian. For $Q<0$, ($\mathcal{F}<1$) the photon distribution becomes narrower than that of a Poisson-PDF and the corresponding state is non-classical. The most elementary examples of non-classical states are number states. Since they are eigenstates of the photon number operator $\hat{N}$ the fluctuations in $\hat{N}$ vanish and the Mandel parameter reads $Q=-1$ ($\mathcal{F}=0$) [53]. We heavily rely here from on Fano-results reported in reference [54]. From the study reported there on the expectation values of $\hat{H}$ and $\hat{N}=(\hat{H}-\hbar\omega/2)/(\hbar\omega)$ we immediately find, evaluating mean values with $A_\xi(z)$

$$\langle\hat{H}\rangle_\xi=\frac{\hbar\omega}{I_\xi};\ \ \langle\hat{N}\rangle_\xi=\frac{1}{I_\xi}-\frac{1}{2}, \tag{45}$$

and, for the "Wigner-delinquent" operator H^2,

$$\langle\hat{H}^2\rangle_\xi=2\langle\hat{H}\rangle_\xi^2;\ \ \langle\hat{N}^2\rangle_\xi=\frac{\langle\hat{H}^2\rangle_\xi}{(\hbar\omega)^2}-\frac{\langle\hat{H}\rangle_\xi}{\hbar\omega}+\frac{1}{4}, \tag{46}$$

$$\langle(\Delta\hat{H})^2\rangle_\xi=\langle\hat{H}\rangle_\xi^2;\ \ \langle(\Delta\hat{N})^2\rangle_\xi=\frac{1}{I_\xi^2}, \tag{47}$$

and, finally, for the ξ-Fano factor $\mathcal{F}_\xi$:

$$\mathcal{F}_\xi=\frac{2}{I_\xi\,(2-I_\xi)}, \tag{48}$$

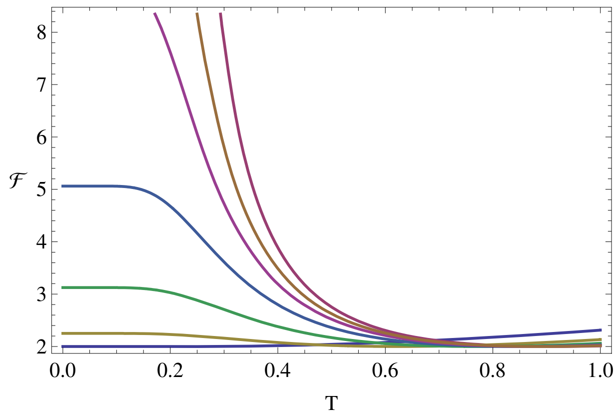

Fig. 2. (Color online) Fano factor as a function of T in units of $\hbar\omega$ for several (decreasing from the bottom up) values of ξ. We use the same values of Figure 1. The lowest curve corresponds to $\xi=1/2$.

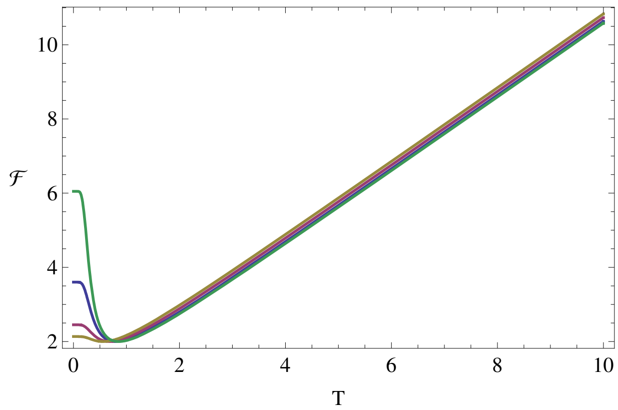

Fig. 3. (Color online) Fano factor as a function of T for several small ξ-values, namely, 0.05, 0.1, 0.2, and 0.3. The uppermost curve corresponds to $\xi=0.05$.

an important result! Notice that $I_\xi=I_w$ only in the particular instance $\xi=0$, that diverges at $T=0$ because there $I_w=2$ (Cf. Fig. 2). However, for all $\xi>0$ one immediately realizes that such $T=0-$divergence disappears. Consequently, the Fano factor will diverge just for the Wigner case and the smallest amount of smearing or smoothing prevents it. In other words, the smallest admixture of other states with the ground-state (see the participation ratio Section above) impedes the divergence. Our plots for $\mathcal{F}_\xi$ vs. ξ illustrates such an effect (Cf. Fig. 3). It is easy to analytically ascertain that, at $T=0$, the minimum $\mathcal{F}$-value is attained for $\xi=\xi_{min}=1/2$ (the Husimi-instance), where we get $\mathcal{F}_{min}=2$, well within the classical super-Poisson realm. We see now the Husimi distribution in a different light then, as the best possible "smoothed-Wigner distribution" at very low temperatures. Figure 4 depicts the Fano factor's behavior, at $T=0$, as a function of ξ. For $T\geq 0$, the associated $\xi_{min}(T)=(I_w-1)/I_w=1-\coth(\beta\hbar\omega/2)/2$. For $T>T_{crit}=0.910239$ there is no longer a $\mathcal{F}_{min}$-value.

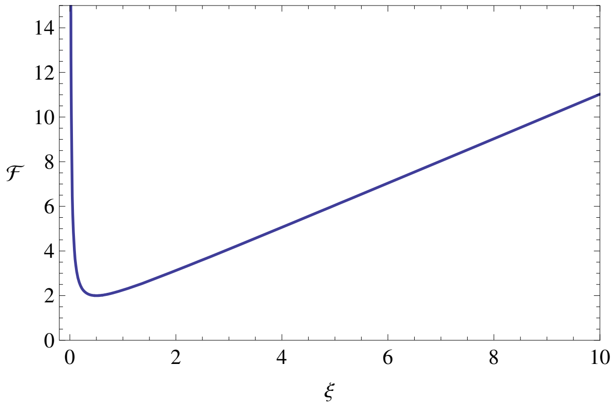

Fig. 4. (Color online) Fano factor as a function of ξ at $T = 0$. A clear minimum at $\xi = 1/2$ can be appreciated

8.1 Reasons for $\mathcal{F}$'s divergence at $T = 0$

We can easily understand the strange $T = 0$ – behavior of the Fano factor in terms of the participation ratio $\mathcal{R}$ discussed above. As explained previously, as $T \rightarrow 0$, the system tends to be found in the ground state with a participation ratio equal unity. One easily sees that

$$\mathcal{F} = (2\langle\hat{N}\rangle_\xi + 1)(\langle\hat{N}\rangle_\xi + 1/2)/2\langle\hat{N}\rangle_\xi. \tag{49}$$

Clearly, $\langle\hat{N}\rangle_\xi = 0$ for the zero-photon ground state, and for $\mathcal{R} = 1$ this is the only occupied state. Thus, $\mathcal{F}$ necessarily diverges. Such divergence is, indeed, the signature of having reached the quantum limit.

9 Conclusions

In this effort we have found or shown that

– The Fisher information measure of a smoothed (by the amount ξ) Wigner distribution can be cast as a function of Wigner-Fisher's one I_w in the fashion $I_\xi = I_w/(\xi\, I_w + 1)$.
– The complementarity information-participation ratio relation
$$\mathcal{R}_\xi\, I_\xi = 2. \tag{50}$$
Thus, the inverse Fisher's information-measure I_ξ directly yields half the number of states entering the density operator.
– The Fano factor expression as a function of Fisher's measure I_ξ and the smoothing parameter ξ reads
$$\mathcal{F}_\xi = \frac{2}{I_\xi\,(2 - I_\xi)}, \tag{51}$$
so that it diverges at $T = 0$ for $\xi = 0$. However, for all $\xi > 0$ such $T = 0$ – divergence disappears: the Fano factor will diverge just for the Wigner case and the smallest amount of smearing or smoothing prevents it.
– The Husimi distribution is now seen in a new way, namely, as the best possible smoothed Wigner distribution at very low temperatures.
– The Cramer-Rao relation is always saturated (not only at $T = 0$), irrespective of the smoothing-parameter's value, as seen in Section 7.
– With reference to the classical-quantum transition we have encountered that the smoothing factor becomes a signature of its temperature's dependence.

We conclude by underlining our main findings:

– While the Wigner function exhibits high resolution, it is not free of difficulties, in particular, long range quantum interferences.
– The Husimi distribution washes out quantum interferences at the price of hiding important semiclassical structures [23].
– With reference to the pathological Wigner-instance of the operator Hamiltonian-squared [24], the process of smoothing is of help.
– More specifically, we are speaking of Wigner function's smoothing intermediate between that provided by the Husimi function and zero-smearing. Smoothing is able to wash-out quantum interferences in such a way as to eliminate the $T = 0$ – divergence of the Fano factor.
– This divergence is to be properly understood as the signature of having reached the quantum-classical frontier.
– We saw also that the Husimi function provides the best possible Wigner-smoothing at very low temperatures.
– The participation ratio turns out to be a signature of the T-dependence of the quantum-classical transition as well.

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Appendix A: One-dimensional smoothing

We discuss now the concept of smoothing (or "smearing") in a simple setting in order to familiarize ourselves with the subject. Define the normalized kernel $(0 \leq \xi \leq 1)$

$$g(x - y) = \frac{1}{\sqrt{\xi\pi}}\, e^{-(x-y)^2/\xi},$$

and proceed to "smooth" ("smear") the Gaussian

$$f(x) = \frac{1}{\sqrt{\Upsilon\pi}}\, e^{-x^2/\Upsilon}. \tag{A.1}$$

To do this we must integrate the quantity

$$G = g(x - y)f(x) = \frac{1}{\pi\sqrt{\xi\Upsilon}}\, e^{-\frac{1}{\xi}(x^2 - 2yx + y^2 + \xi x^2/\Upsilon)}.$$

Introduce the abbreviation: $\mu = (\xi + \Upsilon)/(\xi\Upsilon)$ and write down G as

$$\begin{aligned} G &\equiv \frac{1}{\pi\sqrt{\xi\Upsilon}}\, e^{-\mu(x^2 - 2yx/\xi\mu + y^2/\xi\mu)} \\ &= \frac{1}{\pi\sqrt{\xi\Upsilon}}\, e^{-\mu(x-A)^2 - \mu B}, \end{aligned} \tag{A.2}$$

"completing the square". We have

$$x^2 + A^2 - 2Ax + B = x^2 + y^2/\xi\mu - 2yx/\xi\mu, \quad (A.3)$$

so that $A = y/\xi\mu$, $B = A^2 - y^2/\xi\mu = (y^2/\xi\mu)(1 - 1/\xi\mu)$, and then

$$G = \frac{1}{\pi\sqrt{\xi\Upsilon}}\, e^{-\mu(x-A)^2-(y^2/\xi)(1-1/\xi\mu)}. \quad (A.4)$$

Thus,

$$\tilde{f}(y) = \int dx\, G(x-y) = \frac{1}{\sqrt{\pi(\xi+\Upsilon)}}\, e^{-y^2/(\xi+\Upsilon)}, \quad (A.5)$$

and we see, comparing (A.1) with (A.5), that the smearing means that the "width" of the Gaussian grew from Υ to $\Upsilon + \xi$.

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