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Model reduction by mean-field homogenization in viscoelastic composites. I. Primal theory

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Cite this article: Idiart MI, Lahellec N, Suquet P. 2020 Model reduction by mean-field homogenization in viscoelastic composites. I. Primal theory. *Proc. R. Soc. A* **476**: 20200407. http://dx.doi.org/10.1098/rspa.2020.0407

Received: 24 May 2020
Accepted: 28 September 2020

Subject Areas:
thermodynamics, mechanics, mathematical modelling

Keywords:
viscoelasticity, composites, homogenization, variational methods

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A homogenization scheme for viscoelastic composites proposed by Lahellec & Suquet (2007 *Int. J. Solids Struct.* **44**, 507–529 (doi:10.1016/j.ijsolstr.2006.04.038)) is revisited. The scheme relies upon an incremental variational formulation providing the inelastic strain field at a given time step in terms of the inelastic strain field from the previous time step, along with a judicious use of Legendre transforms to approximate the relevant functional by an alternative functional depending on the inelastic strain fields only through their first and second moments over each constituent phase. As a result, the approximation generates a reduced description of the microscopic state of the composite in terms of a finite set of internal variables that incorporates information on the intraphase fluctuations of the inelastic strain and that can be evaluated by mean-field homogenization techniques. In this work we provide an alternative derivation of the scheme, relying on the Cauchy–Schwarz inequality rather than the Legendre transform, and in so doing we expose the mathematical structure of the resulting approximation and generalize the exposition to fully anisotropic material systems.

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1. Introduction

The time-dependent mechanical behaviour of composite media is the result of intricate interactions between elastic and inelastic deformation processes operating within the different constitutive phases. A key consequence of these interactions is that microscopic constitutive descriptions based on finite sets of internal variables give rise to macroscopic constitutive descriptions with an infinity of internal variables (e.g. [1,2]). This fact has motivated several attempts to generate approximate macroscopic descriptions based on reduced sets of effective internal variables that provide a partial but hopefully accurate characterization of the evolving microscopic state of the composite.

When all constitutive phases exhibit linear viscoelastic behaviour, macroscopic descriptions are often fashioned in spectral form by means of the Laplace transform and the so-called correspondence principle (e.g. [3–6]). The multiplication of internal variables is thereby manifested by the multiplication of relaxation times: composites exhibiting a discrete spectrum with a finite number of relaxation times—i.e. short-memory effects—at the microscale display a continuous spectrum with an infinite number of relaxation times—i.e. long-memory effects—at the macroscale (e.g. [7,8]). The number of effective relaxation times is closely related to the microstructural arrangement and the location of poles of the effective complex moduli on the real negative axis [9]. The analytic representation of these moduli can be used to derive bounds on the effective creep/relaxation functions with incomplete information on the microstructure [10]. A class of microstructures for which macroscopic long-memory effects result in a finer but still discrete spectrum has been reported by Ricaud & Masson [11]. More generally, a so-called collocation method is often employed to generate approximate macroscopic descriptions with a finite number of relaxation times (e.g. [6,12,13]). Interestingly, these simplified descriptions can be recast in terms of effective internal variables [11,14], although the physical meaning of these internal variables and their relationship to the microscopic internal variable fields is, to the best of our knowledge, still unknown.

When constitutive phases exhibit non-linear viscoelastic behaviour, macroscopic descriptions based on effective internal variables are invariably employed. One of the earliest approximations consisted in assuming that the inelastic strain within each phase of the composite is uniform and therefore characterized by a single internal variable per phase [15]. While attractively simple, it is known that this approximate scheme, commonly referred to as 'transformation field analysis', can be severely inaccurate when the local fields exhibit strong spatial fluctuations (e.g. [16]). In view of this observation, Michel & Suquet [16,17] refined the approximation by assuming non-uniform inelastic strain fields that can be expressed as a linear combination of a finite number of predefined fields, so that the amplitudes constitute a finite set of internal variables for which evolution laws can be provided. This refinement significantly improves the accuracy of the approximation but introduces the need to carry out full-field numerical computations.

In parallel developments, Lahellec & Suquet [18,19] proposed a conceptually different approximation that enables the use of mean-field homogenization techniques for linearly elastic composites to avoid full-field numerical computations. This approximate scheme hinges upon an incremental variational formulation of the evolution law that provides the inelastic strain field at a given time step in terms of the inelastic strain field from the previous time step (e.g. [20–22]). Similarly to previous works on purely elastic/viscous composites (e.g. [23]), Legendre transforms are then used to approximate the relevant functional by an alternative functional that depends on the inelastic strain fields only through their first and second moments over each phase. As a result, the scheme generates a reduced description of the microscopic state of the composite that can be evaluated by mean-field homogenization techniques and, at the same time, incorporates information on the intraphase fluctuations of the inelastic strain field through a finite set of effective internal variables endued with clear physical meaning. This description can significantly improve on the original transformation field analysis without requiring full-field computations. The scheme has already been applied to linear viscoelastic composites [18,24] as well as non-linear viscoelastic composites [19,25,26] and elastoplastic composites [27–29] with relative success. A variant of the scheme for elastoplastic composites can also be found in Lucchetta *et al.* [30]. In this work we revisit the scheme in the context of linear viscoelastic composites. We provide an alternative derivation relying on the Cauchy–Schwarz inequality instead of the Legendre transform, and in so doing we expose the mathematical structure of the resulting approximation and generalize the exposition to fully anisotropic material systems. In contrast to some of the works alluded to above, the focus here is placed on the *structure* of the effective viscoelastic constitutive relations, and on accurate approximations of those relations granted the underlying microstructure is known with sufficient precision so that purely *elastic* properties can be accurately determined. The derivation is carried out within a primal variational formulation of the mechanical problem, in the sense that the mechanical fields entering the potentials are the various strains and their rates. Estimates for rigidly reinforced viscoelastic solids subject to complex deformation histories are reported in a companion paper to highlight the capabilities and limitations of the scheme and to identify possible improvements.

2. Problem setting

We consider a representative volume element of a composite material made up of N constituent phases, and denote by Ω and $\Omega^{(r)}$ $(r=1,\ldots,N)$ the domains occupied by the element and the phases within it, respectively, so that $\Omega=\cup_{r=1}^{N}\Omega^{(r)}$. Also, we denote by $\chi^{(r)}(\mathbf{x})$ the characteristic function of each subdomain $\Omega^{(r)}$. The local viscoelastic response is described within the framework of generalized standard materials by constitutive relations of the form [31]

$$\boldsymbol{\sigma}=\frac{\partial w}{\partial \boldsymbol{\varepsilon}}(\mathbf{x},\boldsymbol{\varepsilon},\boldsymbol{\alpha}) \quad \text{and} \quad \frac{\partial w}{\partial \boldsymbol{\alpha}}(\mathbf{x},\boldsymbol{\varepsilon},\boldsymbol{\alpha})+\frac{\partial \varphi}{\partial \dot{\boldsymbol{\alpha}}}(\mathbf{x},\dot{\boldsymbol{\alpha}})=\mathbf{0}, \tag{2.1}$$

where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\alpha}$ denote the infinitesimal and inelastic strains relative to a stress-free reference configuration, $\boldsymbol{\sigma}$ denotes the Cauchy stress, the dot over a variable denotes a time derivative, and the potential functions w and φ are, respectively, the Helmholtz free-energy density and the dissipation potential of the composite, which can be expressed in terms of the corresponding phase potentials as

$$w(\mathbf{x},\boldsymbol{\varepsilon},\boldsymbol{\alpha})=\sum_{r=1}^{N}\chi^{(r)}(\mathbf{x})w^{(r)}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) \quad \text{and} \quad \varphi(\mathbf{x},\dot{\boldsymbol{\alpha}})=\sum_{r=1}^{N}\chi^{(r)}(\mathbf{x})\varphi^{(r)}(\dot{\boldsymbol{\alpha}}). \tag{2.2}$$

These potentials are convex functions of the mechanical fields with suitable growth conditions. The dissipation potentials are, at the same time, positive functions vanishing at the origin.

The homogenized response relates the macroscopic stress $\overline{\boldsymbol{\sigma}}$ to the macroscopic strain $\overline{\boldsymbol{\varepsilon}}$, which are the averages of the local stress and strain fields over the representative volume element. This relation can be written in terms of the macroscopic free-energy density and dissipation potential as (e.g. [2])

$$\overline{\boldsymbol{\sigma}}=\frac{\partial \overline{w}}{\partial \overline{\boldsymbol{\varepsilon}}}(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha}) \quad \text{and} \quad \frac{\delta \overline{w}}{\delta \boldsymbol{\alpha}(\mathbf{x})}(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha})+\frac{\delta \overline{\varphi}}{\delta \dot{\boldsymbol{\alpha}}(\mathbf{x})}(\dot{\boldsymbol{\alpha}})=\mathbf{0}, \tag{2.3}$$

where

$$\overline{w}(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha})=\inf_{\boldsymbol{\varepsilon}\in\mathcal{K}(\overline{\boldsymbol{\varepsilon}})}\langle w(\mathbf{x},\boldsymbol{\varepsilon},\boldsymbol{\alpha})\rangle \quad \text{and} \quad \overline{\varphi}(\dot{\boldsymbol{\alpha}})=\langle \varphi(\mathbf{x},\dot{\boldsymbol{\alpha}})\rangle. \tag{2.4}$$

In these expressions, $\mathcal{K}(\overline{\boldsymbol{\varepsilon}})$ is the set of kinematically admissible strain fields with average $\overline{\boldsymbol{\varepsilon}}$, $\langle\cdot\rangle$ denotes volume averaging over the representative volume element and the δ operator denotes a functional derivative. It is observed that the macroscopic free-energy density and dissipation potential are the volume averages of their microscopic counterparts, and are therefore functionals of the microscopic inelastic strain field and its rate. These functionals inherit the convexity of the local potentials. Thus, homogenization preserves the generalized standard structure of the local reponse, with the microscopic inelastic strain field playing the role of a macroscopic internal variable albeit of infinite dimension. The purpose of the approximate scheme presented below is to reduce the dimensionality of the macroscopic internal variables to a finite number.

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To ease the exposition, we restrict our attention to viscoelastic phases characterized by generalized Kelvin–Voigt potentials—also referred to as Poynting–Thomson potentials—of the form

$$w^{(r)}(\boldsymbol{\varepsilon},\boldsymbol{\alpha}) = \frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})\cdot\mathbb{L}^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}) + \frac{1}{2}\boldsymbol{\alpha}\cdot\mathbb{H}^{(r)}\boldsymbol{\alpha} \quad \text{and} \quad \varphi^{(r)}(\dot{\boldsymbol{\alpha}}) = \frac{1}{2}\dot{\boldsymbol{\alpha}}\cdot\mathbb{M}^{(r)}\dot{\boldsymbol{\alpha}}, \tag{2.5}$$

where $\mathbb{L}^{(r)}$, $\mathbb{H}^{(r)}$ and $\mathbb{M}^{(r)}$ are positive-definite tensors of elastic, hardening and viscous moduli. In this case, the macroscopic potentials can be written as

$$\overline{w}(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha}) = \inf_{\boldsymbol{\varepsilon}\in\mathcal{K}(\overline{\boldsymbol{\varepsilon}})} \sum_{r=1}^{N} c^{(r)} \left\langle \frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})\cdot\mathbb{L}^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha}) + \frac{1}{2}\boldsymbol{\alpha}\cdot\mathbb{H}^{(r)}\boldsymbol{\alpha} \right\rangle^{(r)} \tag{2.6}$$

and

$$\overline{\varphi}(\dot{\boldsymbol{\alpha}}) = \sum_{r=1}^{N} c^{(r)} \frac{1}{2} \left\langle \dot{\boldsymbol{\alpha}}\cdot\mathbb{M}^{(r)}\dot{\boldsymbol{\alpha}} \right\rangle^{(r)}, \tag{2.7}$$

where $\langle\cdot\rangle^{(r)}$ and $c^{(r)}$ denote the volume average over phase r and its volume fraction. It is emphasized, however, that the main conclusions of this study remain relevant to more general linear models; for nonlinear models see §3c.

3. Model reduction

(a) Reduced macroscopic potentials

Following Lahellec & Suquet [18], the evolution law $(2.3)_2$ is discretized in time using an implicit Euler scheme of the form

$$\frac{\delta\overline{w}}{\delta\boldsymbol{\alpha}(\mathbf{x})}(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha}) + \frac{\delta\overline{\varphi}}{\delta\dot{\boldsymbol{\alpha}}(\mathbf{x})}\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right) = \mathbf{0}, \tag{3.1}$$

with time step Δt. This expression constitutes an equation for the inelastic strain field $\boldsymbol{\alpha}$ at the current time step given the inelastic strain field $\boldsymbol{\alpha}_n$ at the previous time step. Central to the approximate scheme of Lahellec & Suquet [18] is the fact that, in view of the convexity of the macroscopic potentials, the algebraic equation (3.1) is equivalent to the variational problem

$$\inf_{\boldsymbol{\alpha}} \left[\overline{w}(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha}) + \Delta t\, \overline{\varphi}\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right) \right]. \tag{3.2}$$

Variational formulations of this sort permit the derivation of rigorous bounds and facilitate the confection of accurate approximations. Indeed, making use of the Cauchy–Schwarz inequality, it is shown in appendix A that

$$\left\langle \left(\frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right)\cdot\mathbb{M}^{(r)}\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right) \right\rangle^{(r)} \lesseqgtr \frac{\langle\boldsymbol{\alpha}\rangle^{(r)}-\langle\boldsymbol{\alpha}_n\rangle^{(r)}}{\Delta t}\cdot\mathbb{M}^{(r)}\frac{\langle\boldsymbol{\alpha}\rangle^{(r)}-\langle\boldsymbol{\alpha}_n\rangle^{(r)}}{\Delta t} + m^{(r)}\left(\frac{\mathrm{C}_{\alpha}^{(r)1/2} \pm \mathrm{C}_{\alpha_n}^{(r)1/2}}{\Delta t}\right)^2, \tag{3.3}$$

where

$$\mathrm{C}_{\alpha}^{(r)} = \left\langle (\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)})\cdot\breve{\mathbb{M}}^{(r)}(\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)}) \right\rangle^{(r)}, \tag{3.4}$$

and $\mathrm{C}_{\alpha_n}^{(r)}$ are traces of the intraphase fluctuations of the inelastic strain field at each time step, $m^{(r)} = ||\mathbb{M}^{(r)}||$ is the Euclidean norm of the viscosity tensor $\mathbb{M}^{(r)}$, and $\breve{\mathbb{M}}^{(r)} = \mathbb{M}^{(r)}/m^{(r)}$. The sense of the bound depends on the sign adopted inside the last squared term. However, the purpose here is not to insist on the bounding character of the right-hand side of (3.3) but rather to exploit it as a sensible approximation of the left-hand side. The form of the right-hand side suggests the use of the bound corresponding to the negative sign—regardless of its upper or lower character—to be consistent with the time continuous limit $\Delta t \to 0$; indeed, selecting the positive sign leads

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to an unbounded right-hand side in that limit. The discretized dissipation *functional* is thus approximated by the discretized dissipation *function*

$$\overline{\varphi}\left(\frac{\boldsymbol{\alpha}-\boldsymbol{\alpha}_n}{\Delta t}\right)\approx\hat{\varphi}\left(\frac{\langle\boldsymbol{\alpha}\rangle^{(1)}-\langle\boldsymbol{\alpha}_n\rangle^{(1)}}{\Delta t},\ldots,\frac{C_{\alpha}^{(1)^{1/2}}-C_{\alpha_n}^{(1)^{1/2}}}{\Delta t},\ldots\right)$$
$$=\sum_{r=1}^{N}c^{(r)}\left[\frac{1}{2}\frac{\langle\boldsymbol{\alpha}\rangle^{(r)}-\langle\boldsymbol{\alpha}_n\rangle^{(r)}}{\Delta t}\cdot\mathbb{M}^{(r)}\frac{\langle\boldsymbol{\alpha}\rangle^{(r)}-\langle\boldsymbol{\alpha}_n\rangle^{(r)}}{\Delta t}+\frac{1}{2}m^{(r)}\left(\frac{C_{\alpha}^{(r)^{1/2}}-C_{\alpha_n}^{(r)^{1/2}}}{\Delta t}\right)^2\right], \quad (3.5)$$

which depends on the inelastic strain field only through the phase averages and intraphase fluctuations. Making use of this approximation in the discretized evolution law (3.2) we obtain

$$\inf_{\boldsymbol{\alpha}}\left[\overline{w}(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha})+\Delta t\,\hat{\varphi}\left(\frac{\langle\boldsymbol{\alpha}\rangle^{(1)}-\langle\boldsymbol{\alpha}_n\rangle^{(1)}}{\Delta t},\ldots,\frac{C_{\alpha}^{(1)^{1/2}}-C_{\alpha_n}^{(1)^{1/2}}}{\Delta t},\ldots\right)\right], \quad (3.6)$$

and therefore, partitioning the infimum problem,

$$\inf_{\substack{\overline{\boldsymbol{\alpha}}^{(r)}\\ \widetilde{\alpha}^{(r)}\geq 0}}\left[\hat{w}\left(\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\alpha}}^{(1)},\ldots,\widetilde{\alpha}^{(1)},\ldots,\right)+\Delta t\hat{\varphi}\left(\frac{\overline{\boldsymbol{\alpha}}^{(1)}-\overline{\boldsymbol{\alpha}}_n^{(1)}}{\Delta t},\ldots,\frac{\widetilde{\alpha}^{(1)}-\widetilde{\alpha}_n^{(1)}}{\Delta t},\ldots\right)\right], \quad (3.7)$$

where

$$\hat{w}\left(\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\alpha}}^{(1)},\ldots\widetilde{\alpha}^{(1)},\ldots\right)=\inf_{\boldsymbol{\alpha}\in\mathcal{J}(\overline{\boldsymbol{\alpha}}^{(1)},\ldots\widetilde{\alpha}^{(1)},\ldots)}\overline{w}\left(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha}\right) \quad (3.8)$$

and

$$\mathcal{J}(\overline{\boldsymbol{\alpha}}^{(1)},\ldots\widetilde{\alpha}^{(1)},\ldots)=\Big\{\boldsymbol{\alpha}\,:\,\langle\boldsymbol{\alpha}\rangle^{(r)}=\overline{\boldsymbol{\alpha}}^{(r)}\quad\text{and}$$
$$\left\langle(\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)})\cdot\breve{\mathbb{M}}^{(r)}(\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)})\right\rangle^{(r)^{1/2}}=\widetilde{\alpha}^{(r)}\quad\text{for}\quad r=1,\ldots,N\Big\}. \quad (3.9)$$

It is now evident that the infimum problem (3.7) constitutes an implicit Euler discretization of the continuous evolution laws

$$\frac{\partial\hat{w}}{\partial\overline{\boldsymbol{\alpha}}^{(r)}}\left(\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\alpha}}^{(1)},\ldots,\overline{\boldsymbol{\alpha}}^{(N)},\widetilde{\alpha}^{(1)},\ldots,\widetilde{\alpha}^{(N)}\right)+\frac{\partial\hat{\varphi}}{\partial\dot{\overline{\boldsymbol{\alpha}}}^{(r)}}\left(\dot{\overline{\boldsymbol{\alpha}}}^{(1)},\ldots,\dot{\overline{\boldsymbol{\alpha}}}^{(N)},\dot{\widetilde{\alpha}}^{(1)},\ldots,\dot{\widetilde{\alpha}}^{(N)}\right)=\mathbf{0}, \quad (3.10)$$

$$\frac{\partial\hat{w}}{\partial\widetilde{\alpha}^{(r)}}\left(\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\alpha}}^{(1)},\ldots,\overline{\boldsymbol{\alpha}}^{(N)},\widetilde{\alpha}^{(1)},\ldots,\widetilde{\alpha}^{(N)}\right)+\frac{\partial\hat{\varphi}}{\partial\dot{\widetilde{\alpha}}^{(r)}}\left(\dot{\overline{\boldsymbol{\alpha}}}^{(1)},\ldots,\dot{\overline{\boldsymbol{\alpha}}}^{(N)},\dot{\widetilde{\alpha}}^{(1)},\ldots,\dot{\widetilde{\alpha}}^{(N)}\right)=0, \quad (3.11)$$

for $r=1,\ldots,N$, where

$$\hat{w}\left(\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\alpha}}^{(1)},\ldots,\widetilde{\alpha}^{(1)},\ldots\right)=\inf_{\substack{\boldsymbol{\varepsilon}\in\mathcal{K}(\overline{\boldsymbol{\varepsilon}})\\ \boldsymbol{\alpha}\in\mathcal{J}(\overline{\boldsymbol{\alpha}}^{(1)},\ldots\widetilde{\alpha}^{(1)},\ldots)}}\sum_{r=1}^{N}c^{(r)}\left\langle\frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})\cdot\mathbb{L}^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})+\frac{1}{2}\boldsymbol{\alpha}\cdot\mathbb{H}^{(r)}\boldsymbol{\alpha}\right\rangle^{(r)} \quad (3.12)$$

and

$$\hat{\varphi}\left(\dot{\overline{\boldsymbol{\alpha}}}^{(1)},\ldots,\dot{\widetilde{\alpha}}^{(1)},\ldots\right)=\sum_{r=1}^{N}c^{(r)}\left[\frac{1}{2}\dot{\overline{\boldsymbol{\alpha}}}^{(r)}\cdot\mathbb{M}^{(r)}\dot{\overline{\boldsymbol{\alpha}}}^{(r)}+\frac{1}{2}m^{(r)}\,\dot{\widetilde{\alpha}}^{(r)^2}\right] \quad (3.13)$$

constitute reduced-order effective potentials which describe the internal state of the composite via a finite set of effective internal variables representing the first moments of the inelastic strain over each phase and the second moments of their intraphase fluctuations. Furthermore, in view of the definition (3.8), the partial derivatives of the macroscopic free-energy density and its reduced version with respect to the macroscopic strain are identical, and therefore

$$\overline{\boldsymbol{\sigma}}=\frac{\partial\hat{w}}{\partial\overline{\boldsymbol{\varepsilon}}}\left(\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{\alpha}}^{(1)},\ldots,\overline{\boldsymbol{\alpha}}^{(N)},\widetilde{\alpha}^{(1)},\ldots,\widetilde{\alpha}^{(N)}\right). \quad (3.14)$$

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(b) Mean-field homogenization

Instrumental to the above order reduction is the fact that the reduced free-energy density (3.12) can be evaluated via mean-field homogenization techniques. Indeed, carrying out the minimization with respect to the inelastic strain field as in appendix B, the reduced free-energy density takes the form

$$\hat{w}\left(\bar{\boldsymbol{\varepsilon}}, \bar{\boldsymbol{\alpha}}^{(1)}, \ldots, \tilde{\alpha}^{(1)}, \ldots\right) = \inf_{\boldsymbol{\varepsilon}\in\mathcal{K}(\bar{\boldsymbol{\varepsilon}})} \sum_{r=1}^{N} c^{(r)} f^{(r)}\left(\langle\boldsymbol{\varepsilon}\rangle^{(r)}, \langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)}, \bar{\boldsymbol{\alpha}}^{(r)}, \tilde{\alpha}^{(r)}\right), \tag{3.15}$$

where the functions $f^{(r)}$ are defined as

$$\begin{aligned} f^{(r)}\left(\boldsymbol{\varepsilon}^{(r)}, \mathbb{E}^{(r)}, \bar{\boldsymbol{\alpha}}^{(r)}, \tilde{\alpha}^{(r)}\right) = {} & \frac{1}{2}\left(\boldsymbol{\varepsilon}^{(r)} - \bar{\boldsymbol{\alpha}}^{(r)}\right)\cdot\mathbb{L}^{(r)}\left(\boldsymbol{\varepsilon}^{(r)} - \bar{\boldsymbol{\alpha}}^{(r)}\right) + \frac{1}{2}\bar{\boldsymbol{\alpha}}^{(r)}\cdot\mathbb{H}^{(r)}\bar{\boldsymbol{\alpha}}^{(r)} \\ & + \frac{1}{2}\left[\left(\mathbb{I} - \mathbb{G}^{(r)^T}\right)\mathbb{L}^{(r)}\left(\mathbb{I} - \mathbb{G}^{(r)}\right) + \mathbb{G}^{(r)^T}\mathbb{H}^{(r)}\mathbb{G}^{(r)}\right] \\ & \cdot\left(\mathbb{E}^{(r)} - \boldsymbol{\varepsilon}^{(r)}\otimes\boldsymbol{\varepsilon}^{(r)}\right) \end{aligned} \tag{3.16}$$

in terms of tensors $\mathbb{G}^{(r)}$ given by

$$\mathbb{G}^{(r)} = \left(\mathbb{L}^{(r)} + \mathbb{H}^{(r)} + 2\lambda^{(r)}\check{\mathbb{M}}^{(r)}\right)^{-1}\mathbb{L}^{(r)}, \tag{3.17}$$

with scalars $\lambda^{(r)}$ being the solution to the equations

$$\mathbb{G}^{(r)^T}\check{\mathbb{M}}^{(r)}\mathbb{G}^{(r)}\cdot\left(\mathbb{E}^{(r)} - \boldsymbol{\varepsilon}^{(r)}\otimes\boldsymbol{\varepsilon}^{(r)}\right) = \tilde{\alpha}^{(r)^2}, \tag{3.18}$$

and with fourth-order tensors $\mathbb{E}^{(r)}$ being identified with the second moments of the strain field and therefore being positive semi-definite.

Expression (3.15) now requires the solution of a *nonlinear* minimization problem with respect to the strain field. However, the associated Euler–Lagrange equations are those of a linear ‘thermoelastic’ comparison solid with the same microstructure as the viscoelastic composites but with piecewise uniform stiffness tensor and eigenstress field given by [32]

$$\tau_0^{(r)} = \frac{\partial f^{(r)}}{\partial\boldsymbol{\varepsilon}^{(r)}}\left(\langle\boldsymbol{\varepsilon}\rangle^{(r)}, \langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)}, \bar{\boldsymbol{\alpha}}^{(r)}, \tilde{\alpha}^{(r)}\right) \quad\text{and}\quad \mathbb{L}_0^{(r)} = 2\frac{\partial f^{(r)}}{\partial\mathbb{E}^{(r)}}\left(\langle\boldsymbol{\varepsilon}\rangle^{(r)}, \langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)}, \bar{\boldsymbol{\alpha}}^{(r)}, \tilde{\alpha}^{(r)}\right). \tag{3.19}$$

Therefore, the minimizing strain field in (3.15) coincides exactly with that of a *linear* comparison problem

$$\hat{w}_0(\bar{\boldsymbol{\varepsilon}}) = \underset{\boldsymbol{\varepsilon}\in\mathcal{K}(\bar{\boldsymbol{\varepsilon}})}{\text{stat}} \sum_{r=1}^{N} c^{(r)}\left\langle\frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbb{L}_0^{(r)}\boldsymbol{\varepsilon} + \tau_0^{(r)}\cdot\boldsymbol{\varepsilon}\right\rangle^{(r)} \tag{3.20}$$

with constitutive tensors given self-consistently by (3.19). The stationary rather than extremal character of the variational problem (3.20) is due to the fact that the tensors $\mathbb{L}_0^{(r)}$ are not in general positive definite. In any event, this comparison energy can be expressed as

$$\hat{w}_0(\bar{\boldsymbol{\varepsilon}}) = \frac{1}{2}\bar{\boldsymbol{\varepsilon}}\cdot\widetilde{\mathbb{L}}_0\bar{\boldsymbol{\varepsilon}} + \tilde{\tau}_0\cdot\bar{\boldsymbol{\varepsilon}} + \tilde{g}_0, \tag{3.21}$$

where $\widetilde{\mathbb{L}}_0$, $\tilde{\tau}_0$ and $\tilde{g}_0$ are effective properties that can be determined with any suitable mean-field homogenization technique for N-phase linear thermoelastic solids. The first and second moments of the strain field within each phase can then be determined from this comparison energy by evaluating the derivatives

$$\langle\boldsymbol{\varepsilon}\rangle^{(r)} = \frac{1}{c^{(r)}}\frac{\partial\hat{w}_0}{\partial\tau_0^{(r)}}(\bar{\boldsymbol{\varepsilon}}) \quad\text{and}\quad \langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)} = \frac{2}{c^{(r)}}\frac{\partial\hat{w}_0}{\partial\mathbb{L}_0^{(r)}}(\bar{\boldsymbol{\varepsilon}}) \quad \text{for } r = 1, \ldots, N, \tag{3.22}$$

which follow from well-known relations for field statistics in linear heterogeneous media (e.g. [32,33]) and, together with relations (3.19), constitute a set of algebraic non-linear equations for those moments. Whenever these equations exhibit multiple roots, the root giving the minimum value of (3.15) with positive-definite phase covariances of the strain field must be selected. On the other hand, the tensors $\mathbb{L}_0^{(r)}$ need not be positive definite in view of the purely stationary character of the comparison problem (3.20); whenever they are all positive (negative) definite, the stationary point in (3.20) corresponds to a minimum (maximum) point. We do not dwell on the analysis of necessary conditions for the existence of solutions and defer it to applications of the scheme in specific contexts. In any event, the reduced free-energy density (3.15) and its derivatives are completely determined by the mean-field homogenization scheme of choice.

The macroscopic constitutive relation can now be written more explicitly as

$$\overline{\boldsymbol{\sigma}} = \sum_{r=1}^{N} c^{(r)}\mathbb{L}^{(r)}\left(\langle\boldsymbol{\varepsilon}\rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right), \tag{3.23}$$

and the evolution laws as

$$\mathbb{M}^{(r)}\dot{\overline{\boldsymbol{\alpha}}}^{(r)} + (\mathbb{L}^{(r)} + \mathbb{H}^{(r)})\overline{\boldsymbol{\alpha}}^{(r)} = \mathbb{L}^{(r)}\langle\boldsymbol{\varepsilon}\rangle^{(r)} \quad \text{and} \quad m^{(r)}\dot{\widetilde{\alpha}}^{(r)} - 2\lambda^{(r)}\widetilde{\alpha}^{(r)} = 0, \tag{3.24}$$

where the $\lambda^{(r)}$'s are the solution to (3.18) and thus depend on the intraphase fluctuations of the strain and inelastic strain fields. Alternatively, the macroscopic constitutive relation can be obtained from the comparison energy as

$$\overline{\boldsymbol{\sigma}} = \frac{\partial\hat{w}_0}{\partial\overline{\boldsymbol{\varepsilon}}}(\overline{\boldsymbol{\varepsilon}}) = \widetilde{\mathbb{L}}_0\overline{\boldsymbol{\varepsilon}} + \widetilde{\tau}_0, \tag{3.25}$$

since the stress fields associated with both minimization problems agree exactly. Finally, relations (3.22) provide the first- and second-order intraphase statistics of the underlying strain field. And given the local relations (2.1) and (B 4), these strain statistics imply the corresponding stress statistics

$$\langle\boldsymbol{\sigma}\rangle^{(r)} = \mathbb{L}^{(r)}\left(\langle\boldsymbol{\varepsilon}\rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right) \tag{3.26}$$

and

$$\langle\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}\rangle^{(r)} - \langle\boldsymbol{\sigma}\rangle^{(r)}\otimes\langle\boldsymbol{\sigma}\rangle^{(r)} = \mathbb{L}^{(r)}\left(\mathbb{I} - \mathbb{G}^{(r)}\right)\left(\langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)} - \langle\boldsymbol{\varepsilon}\rangle^{(r)}\otimes\langle\boldsymbol{\varepsilon}\rangle^{(r)}\right)\left(\mathbb{I} - \mathbb{G}^{(r)^T}\right)\mathbb{L}^{(r)}. \tag{3.27}$$

(c) Discussion

Expressions (3.10)–(3.14) constitute the time-continuous form of the mean-field homogenization scheme initially proposed by Lahellec & Suquet [18]. A proof of the equivalence between the two formulations is provided in appendix C. Several comments are in order at this point. First, the availability of an upper and lower bound for the second moments (3.3) corresponds to the presence of multiple roots in the original formulation. But unlike the original formulation requiring the evaluation of all roots and an *a posteriori* identification of the most suitable root, the present formulation provides an *a priori* criterion for root selection. The selected bound is only used here as an approximation, regardless of its bounding sense. Second, the present formulation makes it plain that this approximate homogenization scheme entails in effect a model reduction that preserves the two-potential structure of the exact constitutive relations. From a practical standpoint, in fact, it constitutes a refinement of the 'transformation field analysis' of Dvorak [15] by incorporating second-order statistics of the inelastic strain field through the additional effective internal variables $\widetilde{\alpha}^{(r)}$. Unfortunately, the reduced dissipation potential is convex but the reduced free-energy density is not in view of the fact that the set of admissible internal variables $\mathcal{J}$, as given by (3.9), is non-convex. Thus, the present formulation also makes it plain that this model reduction does not preserve the generalized standard structure of the exact constitutive relations. However, it is shown in appendix D that the reduced free-energy density does attain its convexification within the range delimited by the inequalities

$$\left[\mathbb{I} + \mathbb{H}^{(r)}\mathbb{L}^{(r)^{-1}}\right]^{-1}\check{\mathbb{M}}^{(r)}\left[\mathbb{I} + \mathbb{L}^{(r)^{-1}}\mathbb{H}^{(r)}\right]^{-1}\cdot\left(\langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)} - \langle\boldsymbol{\varepsilon}\rangle^{(r)}\otimes\langle\boldsymbol{\varepsilon}\rangle^{(r)}\right) \geq \widetilde{\alpha}^{(r)^2}, \tag{3.28}$$

for $r = 1, \ldots, N$, at least for constitutive tensors $\mathbb{L}^{(r)}$, $\mathbb{H}^{(r)}$ and $\mathbb{M}^{(r)}$ having a common set of eigentensors within each phase. Thus, the reduced free-energy density is convex in the Cartesian product space of macroscopic strains and effective internal variables whenever all intraphase fluctuations of the inelastic strain field are smaller than those of the total strain field as measured by the above projections. This range contains the initial state of the composite with vanishing inelastic strains. Outside that range, the reduced free-energy density is, in general, non-convex. The consequences of this non-convexity on the constitutive relations are discussed in the context of specific examples in the companion paper. Nonetheless, the approximation does have the capability to reproduce the exact response in some situations, as will be seen in the examples.

Third, the reduced homogenization scheme furnishes not only the macroscopic constitutive response but also the first- and second-order intraphase statistics of the underlying strain and stress fields. These field statistics are uniquely given by expressions (3.22), (3.26) and (3.27). This is in contrast to field statistics furnished by the 'decoupled' schemes commonly employed in the context of linear and non-linear viscoelastic solids exhibiting Maxwellian rheologies. By assuming a kinematically compatible elastic strain field, these schemes effectively decouple the viscoelastic homogenization problem into a purely elastic problem and a purely viscous problem (e.g. [25,34]), but in so doing generate two different sets of field statistics associated with each problem and thus introduce some arbitrariness (see, for instance, [35]).

It is also noted that a more general approximation can be made by considering the spectral decomposition of the viscosity tensors and invoking the Cauchy–Schwarz inequality for the various inner products and associated norms defined by the eigentensors of each phase. It is reasonable to expect that such an approximation will produce more accurate descriptions at the expense of increasing the number of internal variables. At any rate, it is observed that the reduced version (3.12) of the macroscopic free-energy density incorporates a dependence on the local viscous anisotropy through the set of admissible internal variable fields.

Finally, it is easy to extend the reduced homogenization scheme to non-linearly viscous composites with local dissipation potentials of the form

$$\varphi^{(r)}(\dot{\boldsymbol{\alpha}}) = F^{(r)}(\dot{\boldsymbol{\alpha}} \cdot \mathbb{M}^{(r)}\dot{\boldsymbol{\alpha}}), \tag{3.29}$$

where the functions $F^{(r)}$ are concave on the space of positive reals. In this case, use can be made of Jensen's inequality to bound from above the dissipation potential of the composite by

$$\overline{\varphi}(\dot{\boldsymbol{\alpha}}) = \sum_{r=1}^{N} c^{(r)} \left\langle F^{(r)}(\dot{\boldsymbol{\alpha}} \cdot \mathbb{M}^{(r)}\dot{\boldsymbol{\alpha}}) \right\rangle^{(r)} \le \sum_{r=1}^{N} c^{(r)} F^{(r)} \left(\left\langle \dot{\boldsymbol{\alpha}} \cdot \mathbb{M}^{(r)}\dot{\boldsymbol{\alpha}} \right\rangle^{(r)} \right), \tag{3.30}$$

and subsequently use the Cauchy–Schwarz inequality (3.3) to approximate the arguments of the functions $F^{(r)}$ and generate the reduced dissipation potential

$$\hat{\varphi}\left(\dot{\overline{\boldsymbol{\alpha}}}^{(1)}, \ldots, \dot{\tilde{\alpha}}^{(1)}, \ldots\right) = \sum_{r=1}^{N} c^{(r)} F^{(r)} \left(\dot{\overline{\boldsymbol{\alpha}}}^{(r)} \cdot \mathbb{M}^{(r)} \dot{\overline{\boldsymbol{\alpha}}}^{(r)} + m^{(r)}\, \dot{\tilde{\alpha}}^{(r)^2} \right), \tag{3.31}$$

which inherits the convexity of the local dissipation potential. The approximation (3.30) amounts to a linearization of the non-linear dissipation potentials of the secant type (e.g. [32,36]), and the resulting estimates constitute the time-continuous form of the non-linear estimates initially proposed by Lahellec & Suquet [19] for isotropic solids with power-law dissipation potentials. Other approximations in (3.30) incorporating additional statistics of the inelastic strain rate field can be similarly exploited (e.g. [23,37]).

4. Isotropic phases

The above formulae simplify considerably when the constitutive responses of all phases are isotropic and characterized by constitutive tensors of the form

$$\mathbb{L}^{(r)} = 3\kappa^{(r)}\mathbb{J} + 2\mu^{(r)}\mathbb{K}, \quad \mathbb{H}^{(r)} = +\infty\ \mathbb{J} + 2h^{(r)}\mathbb{K} \quad \text{and} \quad \mathbb{M}^{(r)} = 2\eta^{(r)}\mathbb{K}, \tag{4.1}$$

where $\mathbb{J}$ and $\mathbb{K}$ are the standard fourth-order isotropic bulk and shear projection tensors, respectively, $\kappa^{(r)}$ and $\mu^{(r)}$ are the bulk and shear elastic moduli, respectively, $h^{(r)}$ is a hardening

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modulus and $\eta^{(r)}$ is the viscosity. The form of the tensor $\mathbb{H}^{(r)}$ implies that volumetric changes are purely elastic.

Given the above form of constitutive tensors, the tensors (3.17) simplify to

$$\mathbb{G}^{(r)} = \frac{\mu^{(r)}}{\mu^{(r)} + h^{(r)} + \lambda^{(r)}}\mathbb{K}, \tag{4.2}$$

and therefore the functions (3.16) reduce to

$$f^{(r)}\left(\boldsymbol{\varepsilon}^{(r)}, \mathbb{E}^{(r)}, \overline{\boldsymbol{\alpha}}^{(r)}, \widetilde{\alpha}^{(r)}\right) = \frac{9}{2}\kappa^{(r)}\varepsilon_m^{(r)2} + \mu^{(r)}\left(\boldsymbol{\varepsilon}_d^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right)^2 + h^{(r)}\overline{\boldsymbol{\alpha}}^{(r)2} + \frac{\mu^{(r)}h^{(r)} + (h^{(r)} + \lambda^{(r)})^2}{(\mu^{(r)} + h^{(r)} + \lambda^{(r)})^2}\mu^{(r)}\mathbb{K}\cdot\left(\mathbb{E}^{(r)} - \boldsymbol{\varepsilon}^{(r)}\otimes\boldsymbol{\varepsilon}^{(r)}\right), \tag{4.3}$$

where $\varepsilon_m^{(r)} = \mathrm{tr}\boldsymbol{\varepsilon}^{(r)}/3$ and $\boldsymbol{\varepsilon}_d^{(r)} = \boldsymbol{\varepsilon}^{(r)} - \varepsilon_m^{(r)}\mathbf{I}$ denote the mean and deviatoric parts of $\boldsymbol{\varepsilon}^{(r)}$, respectively, $\mathbf{I}$ denotes the second-order identity tensor, $\mathrm{tr}\overline{\boldsymbol{\alpha}}^{(r)} = 0$, the square of a second-order tensor $\mathbf{a}$ has been defined as $\mathbf{a}^2 = \mathbf{a}\cdot\mathbf{a}$, and the scalars $\lambda^{(r)}$ have been rescaled by an inconsequential factor $\|\mathbb{K}\|^{-1} = 1/\sqrt{5}$. In turn, equations (3.18) reduce to

$$\left(\frac{\mu^{(r)}}{\mu^{(r)} + h^{(r)} + \lambda^{(r)}}\right)^2\mathbb{K}\cdot\left(\mathbb{E}^{(r)} - \boldsymbol{\varepsilon}^{(r)}\otimes\boldsymbol{\varepsilon}^{(r)}\right) = \widetilde{\alpha}^{(r)2}, \tag{4.4}$$

where the effective internal variables $\widetilde{\alpha}^{(r)}$ have been rescaled by a factor $\|\mathbb{K}\|^{1/2} = 5^{1/4}$; these equations are solved by

$$\lambda^{(r)} = \frac{\pm\left[\mathbb{K}\cdot\left(\mathbb{E}^{(r)} - \boldsymbol{\varepsilon}^{(r)}\otimes\boldsymbol{\varepsilon}^{(r)}\right)\right]^{1/2} - \widetilde{\alpha}^{(r)}}{\widetilde{\alpha}^{(r)}}\mu^{(r)} - h^{(r)}. \tag{4.5}$$

As anticipated in the previous subsection, multiple roots are observed; the root delivering the minimum value of the functions $f^{(r)}$ should be selected. Thus,

$$f^{(r)}\left(\boldsymbol{\varepsilon}^{(r)}, \mathbb{E}^{(r)}, \overline{\boldsymbol{\alpha}}^{(r)}, \widetilde{\alpha}^{(r)}\right) = \frac{9}{2}\kappa^{(r)}\varepsilon_m^{(r)2} + \mu^{(r)}\left(\boldsymbol{\varepsilon}_d^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right)^2 + h^{(r)}\left(\overline{\boldsymbol{\alpha}}^{(r)2} + \widetilde{\alpha}^{(r)2}\right) + \mu^{(r)}\left(\left[\mathbb{K}\cdot\left(\mathbb{E}^{(r)} - \boldsymbol{\varepsilon}^{(r)}\otimes\boldsymbol{\varepsilon}^{(r)}\right)\right]^{1/2} - \widetilde{\alpha}^{(r)}\right)^2. \tag{4.6}$$

In view of this expression for the functions $f^{(r)}$, the reduced-order effective potentials become

$$\hat{w}\left(\overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\alpha}}^{(1)}, \ldots, \widetilde{\alpha}^{(1)}, \ldots\right) = \inf_{\boldsymbol{\varepsilon}\in\mathcal{K}(\overline{\boldsymbol{\varepsilon}})}\sum_{r=1}^{N} c^{(r)}\left[\frac{9}{2}\kappa^{(r)}\langle\varepsilon_m^2\rangle^{(r)} + \mu^{(r)}\left(\langle\boldsymbol{\varepsilon}_d\rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right)^2 + \mu^{(r)}\left(\left(\langle\boldsymbol{\varepsilon}_d^2\rangle^{(r)} - \langle\boldsymbol{\varepsilon}_d\rangle^{(r)2}\right)^{1/2} - \widetilde{\alpha}^{(r)}\right)^2 + h^{(r)}\left(\overline{\boldsymbol{\alpha}}^{(r)2} + \widetilde{\alpha}^{(r)2}\right)\right] \tag{4.7}$$

and

$$\hat{\varphi}\left(\dot{\overline{\boldsymbol{\alpha}}}^{(1)}, \ldots, \dot{\widetilde{\alpha}}^{(1)}, \ldots\right) = \sum_{r=1}^{N} c^{(r)}\eta^{(r)}\left[\dot{\overline{\boldsymbol{\alpha}}}^{(r)2} + \dot{\widetilde{\alpha}}^{(r)2}\right]. \tag{4.8}$$

The macroscopic constitutive relation is then given by

$$\overline{\boldsymbol{\sigma}} = \sum_{r=1}^{N} c^{(r)}\left[3\kappa^{(r)}\langle\varepsilon_m\rangle^{(r)}\mathbf{I} + 2\mu^{(r)}\left(\langle\boldsymbol{\varepsilon}_d\rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right)\right], \tag{4.9}$$

while the evolution laws are given by

$$\frac{\eta^{(r)}}{\mu^{(r)}}\dot{\overline{\boldsymbol{\alpha}}}^{(r)} + \left(1 + \frac{h^{(r)}}{\mu^{(r)}}\right)\overline{\boldsymbol{\alpha}}^{(r)} = \langle\boldsymbol{\varepsilon}_d\rangle^{(r)} \quad\text{and}\quad \frac{\eta^{(r)}}{\mu^{(r)}}\dot{\widetilde{\alpha}}^{(r)} + \left(1 + \frac{h^{(r)}}{\mu^{(r)}}\right)\widetilde{\alpha}^{(r)} = \widetilde{\varepsilon}_d^{(r)}, \tag{4.10}$$

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where $\tilde{\varepsilon}_d^{(r)}$ is an isotropic measure of the intraphase fluctuations of the deviatoric strain defined by

$$\tilde{\varepsilon}_d^{(r)} = \left(\langle \boldsymbol{\varepsilon}_d^2 \rangle^{(r)} - \langle \boldsymbol{\varepsilon}_d \rangle^{(r)^2} \right)^{1/2}. \tag{4.11}$$

In turn, the stress statistics (3.26) and (3.27) are given by

$$\langle \boldsymbol{\sigma} \rangle^{(r)} = \left[3\kappa^{(r)} \langle \varepsilon_m \rangle^{(r)} \mathbf{I} + 2\mu^{(r)} \left(\langle \boldsymbol{\varepsilon}_d \rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)} \right) \right] \tag{4.12}$$

and

$$\langle \boldsymbol{\sigma} \otimes \boldsymbol{\sigma} \rangle^{(r)} - \langle \boldsymbol{\sigma} \rangle^{(r)} \otimes \langle \boldsymbol{\sigma} \rangle^{(r)} = \left(3\kappa^{(r)}\right)^2 \left(\langle \varepsilon_m^2 \rangle^{(r)} - \langle \varepsilon_m \rangle^{(r)^2} \right) \mathbf{I} \otimes \mathbf{I} + \left(2\mu^{(r)}\right)^2 \left(\frac{\tilde{\varepsilon}_d^{(r)} - \tilde{\alpha}^{(r)}}{\tilde{\varepsilon}_d^{(r)}} \right)^2 \left(\langle \boldsymbol{\varepsilon}_d \otimes \boldsymbol{\varepsilon}_d \rangle^{(r)} - \langle \boldsymbol{\varepsilon}_d \rangle^{(r)} \otimes \langle \boldsymbol{\varepsilon}_d \rangle^{(r)} \right). \tag{4.13}$$

In view of these relations, the thermodynamic forces associated with the effective internal variables are connected to the microscopic stress field by

$$\overline{\mathbf{a}}^{(r)} = -\frac{\partial \hat{w}}{\partial \overline{\boldsymbol{\alpha}}^{(r)}} = c^{(r)} \left[\overline{\boldsymbol{\sigma}}_d^{(r)} - 2h^{(r)} \overline{\boldsymbol{\alpha}}^{(r)} \right] \tag{4.14}$$

and

$$\tilde{a}^{(r)} = -\frac{\partial \hat{w}}{\partial \tilde{\alpha}^{(r)}} = c^{(r)} \left[\tilde{\sigma}_d^{(r)} \operatorname{sgn}(\tilde{\varepsilon}_d^{(r)} - 2\tilde{\alpha}^{(r)}) - 2h^{(r)} \tilde{\alpha}^{(r)} \right], \tag{4.15}$$

where $\overline{\boldsymbol{\sigma}}_d^{(r)} = \langle \boldsymbol{\sigma}_d \rangle^{(r)}$ is the average of the deviatoric stress field over phase r as given by the deviatoric part of (4.12), and $\tilde{\sigma}_d^{(r)^2} = \langle \boldsymbol{\sigma}_d^2 \rangle^{(r)} - \langle \boldsymbol{\sigma}_d \rangle^{(r)^2}$ is a measure of the deviatoric stress fluctuations over that phase as given by the deviatoric trace of (4.13).

The above expressions require the first and second moments of the strain field within each phase. These moments can be obtained from the comparison problem (3.20), which simplifies to

$$\hat{w}_0 (\overline{\boldsymbol{\varepsilon}}) = \underset{\boldsymbol{\varepsilon} \in \mathcal{K}(\overline{\boldsymbol{\varepsilon}})}{\text{stat}} \sum_{r=1}^{N} c^{(r)} \left[\frac{9}{2} \kappa^{(r)} \langle \varepsilon_m^2 \rangle^{(r)} + \mu_0^{(r)} \langle \boldsymbol{\varepsilon}_d^2 \rangle^{(r)} + \tau_0^{(r)} \cdot \langle \boldsymbol{\varepsilon}_d \rangle^{(r)} \right], \tag{4.16}$$

with

$$\tau_0^{(r)} = 2\mu^{(r)} \left(\frac{\tilde{\alpha}^{(r)}}{\tilde{\varepsilon}_d^{(r)}} \langle \boldsymbol{\varepsilon}_d \rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)} \right) \quad \text{and} \quad \mu_0^{(r)} = \mu^{(r)} \left(1 - \frac{\tilde{\alpha}^{(r)}}{\tilde{\varepsilon}_d^{(r)}} \right). \tag{4.17}$$

The required moments then follow from the identities (3.22), which reduce to

$$\langle \boldsymbol{\varepsilon}_d \rangle^{(r)} = \frac{1}{c^{(r)}} \frac{\partial \hat{w}_0}{\partial \tau_0^{(r)}} (\overline{\boldsymbol{\varepsilon}}), \quad \langle \boldsymbol{\varepsilon}_d^2 \rangle^{(r)} = \frac{1}{c^{(r)}} \frac{\partial \hat{w}_0}{\partial \mu_0^{(r)}} (\overline{\boldsymbol{\varepsilon}}) \quad \text{and} \quad \langle \boldsymbol{\varepsilon}_m^2 \rangle^{(r)} = \frac{1}{c^{(r)}} \frac{2}{9} \frac{\partial \hat{w}_0}{\partial \kappa^{(r)}} (\overline{\boldsymbol{\varepsilon}}). \tag{4.18}$$

Once again, expressions (4.17) and (4.18) constitute a set of non-linear equations for the first and second moments of the strain field within each phase which may have multiple solutions, but only those entailing positive strain fluctuations are admissible, i.e. $\mu_0^{(r)} \leq \mu^{(r)}$. Among those admissible roots, the one providing the minimum reduced free-energy density (4.7) must be selected. The sign of the comparison moduli $\mu_0^{(r)}$'s, on the other hand, is unrestricted. However, we note that the convexity conditions (3.28) reduce to

$$\tilde{\varepsilon}_d^{(r)} \geq \left(1 + \frac{h^{(r)}}{\mu^{(r)}} \right) \tilde{\alpha}^{(r)} \quad \text{or equivalently } \dot{\tilde{\alpha}}^{(r)} \geq 0 \tag{4.19}$$

in view of the evolution laws $(4.10)_2$. Thus, the reduced free-energy density (4.7) attains its convexification whenever the intraphase fluctuations of the inelastic strain field increase in every phase.

5. Concluding remarks

We have provided an alternative—but equivalent—formulation of the mean-field homogenization scheme of Lahellec & Suquet [18,19] for viscoelastic composites that exposes its mathematical structure and generalizes it to fully anisotropic material behaviours. It was found that the scheme entails in effect a model reduction that preserves the two-potential structure of the macroscopic constitutive behaviour being approximated but without preserving its global convexity. Thus, the ensuing reduced model is not a generalized standard material model. The capabilities and limitations of this model are explored in light of these findings in the companion paper.

We conclude by recalling that alternative mean-field homogenization schemes for viscoelastic and elasto-viscoplastic composites are already available from the works of Lahellec & Suquet [38] and Idiart & Lahellec [25]. These schemes hinge upon an incremental formulation of the macroscopic evolution law based on strain rates rather than strains. As a result, the natural variable requiring an order reduction is the stress field rather than the inelastic strain field within the composite. The question arises then as to whether a certain duality between the various mean-field schemes can be established and eventually exploited to expose their mathematical structure along the lines of the present work. This question will be addressed in a separate contribution.

Data accessibility. This work does not contain any additional data.
Authors' contributions. All the authors of this work are equal contributors, having worked together throughout the entire process of research, writing and revising. All authors agree to be accountable for all aspects of the work.
Competing interests. We declare we have no competing interests.
Funding. The work was funded by a CONICET-CNRS international collaboration through grants PCB-II D5054 and PICS 06400. M.I.I. acknowledges additional support by the Air Force Office of Scientific Research (USA) under award no. FA9550-19-1-0377; N.L. and P.S. acknowledge additional support from Excellence Initiative of Aix-Marseille University - A*MIDEX, a French 'Investissements d'Avenir' programme in the framework of the Labex MEC.

Appendix A. Bounds on the dissipation potential via Cauchy–Schwarz inequality

Given a particular phase r with viscosity tensor $\mathbb{M}^{(r)}$, we introduce the inner product and associated norm

$$(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)^{(r)} = \left\langle \boldsymbol{\alpha}_1 \cdot \breve{\mathbb{M}}^{(r)} \boldsymbol{\alpha}_2 \right\rangle^{(r)} \quad \text{and} \quad \|\boldsymbol{\alpha}\|^{(r)} = \left\langle \boldsymbol{\alpha} \cdot \breve{\mathbb{M}}^{(r)} \boldsymbol{\alpha} \right\rangle^{(r)^{1/2}}, \tag{A 1}$$

where $\breve{\mathbb{M}}^{(r)} = \mathbb{M}^{(r)}/\|\mathbb{M}^{(r)}\|$ is a unitary viscosity tensor. The contribution from that phase to the discretized dissipation can then be written as

$$\begin{aligned}
&\left\langle \left(\frac{\boldsymbol{\alpha} - \boldsymbol{\alpha}_n}{\Delta t} \right) \cdot \breve{\mathbb{M}}^{(r)} \left(\frac{\boldsymbol{\alpha} - \boldsymbol{\alpha}_n}{\Delta t} \right) \right\rangle^{(r)} \\
&\quad = \left\| \frac{\boldsymbol{\alpha} - \boldsymbol{\alpha}_n}{\Delta t} \right\|^{(r)^2} \\
&\quad = \left\| \frac{\langle \boldsymbol{\alpha} \rangle^{(r)} - \langle \boldsymbol{\alpha}_n \rangle^{(r)}}{\Delta t} + \frac{(\boldsymbol{\alpha} - \langle \boldsymbol{\alpha} \rangle^{(r)}) - (\boldsymbol{\alpha}_n - \langle \boldsymbol{\alpha}_n \rangle^{(r)})}{\Delta t} \right\|^{(r)^2} \\
&\quad = \left\| \frac{\langle \boldsymbol{\alpha} \rangle^{(r)} - \langle \boldsymbol{\alpha}_n \rangle^{(r)}}{\Delta t} \right\|^{(r)^2} + \left\| \frac{(\boldsymbol{\alpha} - \langle \boldsymbol{\alpha} \rangle^{(r)}) - (\boldsymbol{\alpha}_n - \langle \boldsymbol{\alpha}_n \rangle^{(r)})}{\Delta t} \right\|^{(r)^2} \\
&\quad = \left\| \frac{\langle \boldsymbol{\alpha} \rangle^{(r)} - \langle \boldsymbol{\alpha}_n \rangle^{(r)}}{\Delta t} \right\|^{(r)^2} + \frac{\|\boldsymbol{\alpha} - \langle \boldsymbol{\alpha} \rangle^{(r)}\|^{(r)^2}}{} \\
&\qquad \frac{-2(\boldsymbol{\alpha} - \langle \boldsymbol{\alpha} \rangle^{(r)}, \boldsymbol{\alpha}_n - \langle \boldsymbol{\alpha}_n \rangle^{(r)})^{(r)} + \|\boldsymbol{\alpha}_n - \langle \boldsymbol{\alpha}_n \rangle^{(r)}\|^{(r)^2}}{\Delta t^2},
\end{aligned} \tag{A 2}$$

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where use has been made of the fact that

$$\left(\langle\boldsymbol{\alpha}\rangle^{(r)} - \langle\boldsymbol{\alpha}_n\rangle^{(r)}, \boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right)^{(r)} = \left(\langle\boldsymbol{\alpha}\rangle^{(r)} - \langle\boldsymbol{\alpha}_n\rangle^{(r)}, \boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}\right)^{(r)} = 0. \tag{A 3}$$

Now, in view of the Cauchy–Schwarz inequality

$$\left(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}, \boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}\right)^{(r)} \lesseqgtr \pm \left\|\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right\|^{(r)} \left\|\boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}\right\|^{(r)}, \tag{A 4}$$

the dissipation (A 2) can be bounded by

$$\begin{aligned}\left\langle\left(\frac{\boldsymbol{\alpha} - \boldsymbol{\alpha}_n}{\Delta t}\right) \cdot \check{\mathbb{M}}^{(r)} \left(\frac{\boldsymbol{\alpha} - \boldsymbol{\alpha}_n}{\Delta t}\right)\right\rangle^{(r)} &\lesseqgtr \left\|\frac{\langle\boldsymbol{\alpha}\rangle^{(r)} - \langle\boldsymbol{\alpha}_n\rangle^{(r)}}{\Delta t}\right\|^{(r)^2} + \frac{\|\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\|^{(r)^2}}{} \\ &\quad \frac{\pm 2\|\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\|^{(r)}\|\boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}\|^{(r)} + \|\boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}\|^{(r)^2}}{\Delta t^2} \\ &= \left\|\frac{\langle\boldsymbol{\alpha}\rangle^{(r)} - \langle\boldsymbol{\alpha}_n\rangle^{(r)}}{\Delta t}\right\|^{(r)^2} \\ &\quad + \left(\frac{\|\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\|^{(r)} \pm \|\boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}\|^{(r)}}{\Delta t}\right)^2,\end{aligned} \tag{A 5}$$

where the sense of the bound depends on the sign adopted inside the last squared term. The bounds (3.3) simply follow from multiplying this inequality by $||\mathbb{M}^{(r)}||$ and identifying the quantities $\widetilde{\alpha}^{(r)}$ and $\widetilde{\alpha}_n^{(r)}$ with the norms $||\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}||^{(r)}$ and $||\boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}||^{(r)}$.

Appendix B. Inelastic strain field in the reduced free-energy density

The reduced free-energy density (3.12) can be rewritten as

$$\begin{aligned}\hat{w}\left(\overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\alpha}}^{(1)}, \ldots, \widetilde{\alpha}^{(1)}, \ldots\right) = \inf_{\boldsymbol{\varepsilon}\in\mathcal{K}(\overline{\boldsymbol{\varepsilon}})} \inf_{\boldsymbol{\alpha}} \sup_{\boldsymbol{\Lambda}^{(r)}} \sup_{\lambda^{(r)}} \sum_{r=1}^{N} c^{(r)} \left\langle \frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}) \cdot \mathbb{L}^{(r)}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}) + \frac{1}{2}\boldsymbol{\alpha} \cdot \mathbb{H}^{(r)}\boldsymbol{\alpha}\right\rangle^{(r)} \\ + \boldsymbol{\Lambda}^{(r)} \cdot \left(\langle\boldsymbol{\alpha}\rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right) + \lambda^{(r)}\left(\left\langle\left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)}\right) \cdot \check{\mathbb{M}}^{(r)} \left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)}\right)\right\rangle^{(r)} - \widetilde{\alpha}^{(r)^2}\right).\end{aligned} \tag{B 1}$$

Within each phase, the optimal inelastic strain field is thus given by the optimality condition

$$-\mathbb{L}^{(r)}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}) + \mathbb{H}^{(r)}\boldsymbol{\alpha} + \boldsymbol{\Lambda}^{(r)} + 2\lambda^{(r)}\check{\mathbb{M}}^{(r)}\left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)}\right) = \mathbf{0}. \tag{B 2}$$

Phase averaging this expression and taking into account the optimality condition with respect to $\boldsymbol{\Lambda}^{(r)}$, namely $\langle\boldsymbol{\alpha}\rangle^{(r)} = \overline{\boldsymbol{\alpha}}^{(r)}$, we obtain

$$-\mathbb{L}^{(r)}\left(\langle\boldsymbol{\varepsilon}\rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right) + \mathbb{H}^{(r)}\overline{\boldsymbol{\alpha}}^{(r)} + \boldsymbol{\Lambda}^{(r)} = \mathbf{0}. \tag{B 3}$$

Eliminating $\boldsymbol{\Lambda}^{(r)}$ by substracting these two expressions we obtain

$$\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)} = \mathbb{G}^{(r)}\left(\boldsymbol{\varepsilon} - \langle\boldsymbol{\varepsilon}\rangle^{(r)}\right), \tag{B 4}$$

with the tensor $\mathbb{G}^{(r)}$ defined by (3.17). Thus,

$$\left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)}\right) \cdot \check{\mathbb{M}}^{(r)} \left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)}\right) = \left(\boldsymbol{\varepsilon} - \langle\boldsymbol{\varepsilon}\rangle^{(r)}\right) \cdot \mathbb{G}^{(r)^T}\check{\mathbb{M}}^{(r)}\mathbb{G}^{(r)}\left(\boldsymbol{\varepsilon} - \langle\boldsymbol{\varepsilon}\rangle^{(r)}\right). \tag{B 5}$$

Finally, phase averaging this last expression and taking into account the optimality condition with respect to $\lambda^{(r)}$ as given by

$$\left\langle\left(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right) \cdot \check{\mathbb{M}}^{(r)} \left(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right)\right\rangle^{(r)} = \widetilde{\alpha}^{(r)^2}, \tag{B 6}$$

we obtain the equation

$$\mathbb{G}^{(r)^T}\breve{\mathbb{M}}^{(r)}\mathbb{G}^{(r)}\cdot\left(\langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)}-\langle\boldsymbol{\varepsilon}\rangle^{(r)}\otimes\langle\boldsymbol{\varepsilon}\rangle^{(r)}\right)=\widetilde{\alpha}^{(r)^2},\tag{B 7}$$

for the optimal Lagrange multiplier $\lambda^{(r)}$. Expression (B 4) therefore gives the optimal inelastic strain within each phase in terms of $\overline{\boldsymbol{\alpha}}^{(r)}$, $\widetilde{\alpha}^{(r)}$, $\langle\boldsymbol{\varepsilon}\rangle^{(r)}$ and $\langle\boldsymbol{\varepsilon}\otimes\boldsymbol{\varepsilon}\rangle^{(r)}$.

Appendix C. Equivalence between original and present formulations

The original formulation proposed by Lahellec & Suquet [18] expresses the incremental problem (3.7) as

$$\operatorname*{stat}_{\boldsymbol{\alpha}_0^{(r)},\lambda_0^{(r)}}\left[\widetilde{w}_0\left(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha}_0^{(r)},\lambda_0^{(r)}\right)+\sum_{r=1}^{N}c^{(r)}v^{(r)}\left(\boldsymbol{\alpha}_0^{(r)},\lambda_0^{(r)};\boldsymbol{\alpha}_n\right)\right],\tag{C 1}$$

where

$$\begin{aligned}\widetilde{w}_0\left(\overline{\boldsymbol{\varepsilon}},\boldsymbol{\alpha}_0^{(r)},\lambda_0^{(r)}\right)=\inf_{\boldsymbol{\varepsilon}\in\mathcal{K}(\overline{\boldsymbol{\varepsilon}})}\inf_{\boldsymbol{\alpha}}\sum_{r=1}^{N}c^{(r)}\Big\langle&\frac{1}{2}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})\cdot\mathbb{L}^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})+\frac{1}{2}\boldsymbol{\alpha}\cdot\mathbb{H}^{(r)}\boldsymbol{\alpha}\\&+\frac{\lambda_0^{(r)}}{2\Delta t}\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0^{(r)}\right)\cdot\mathbb{M}^{(r)}\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0^{(r)}\right)\Big\rangle^{(r)}\end{aligned}\tag{C 2}$$

and

$$\begin{aligned}v\left(\boldsymbol{\alpha}_0^{(r)},\lambda_0^{(r)};\boldsymbol{\alpha}_n\right)=\operatorname*{stat}_{\hat{\boldsymbol{\alpha}}}\sum_{r=1}^{N}c^{(r)}\Big\langle&\frac{1}{2\Delta t}(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_n)\cdot\mathbb{M}^{(r)}(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_n)\\&-\frac{\lambda_0^{(r)}}{2\Delta t}(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0^{(r)})\cdot\mathbb{M}^{(r)}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0^{(r)}\right)\Big\rangle^{(r)},\end{aligned}\tag{C 3}$$

and identifies the total strain, inelastic strain and stress fields in $\widetilde{w}_0$ with those fields at the current time step. To show the equivalence of this formulation with that of §3 we begin by spelling out the various optimality conditions,

$$\nabla\cdot\boldsymbol{\sigma}=\mathbf{0}\quad\text{in }\Omega,\tag{C 4}$$

$$\boldsymbol{\sigma}=\mathbb{L}^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})\quad\text{in }\Omega^{(r)},\tag{C 5}$$

$$-\mathbb{L}^{(r)}(\boldsymbol{\varepsilon}-\boldsymbol{\alpha})+\mathbb{H}^{(r)}\boldsymbol{\alpha}+\frac{\lambda_0^{(r)}}{\Delta t}\mathbb{M}^{(r)}\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0^{(r)}\right)=\mathbf{0}\quad\text{in }\Omega^{(r)},\tag{C 6}$$

$$(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_n)-\lambda_0^{(r)}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0^{(r)}\right)=\mathbf{0}\quad\text{in }\Omega^{(r)},\tag{C 7}$$

$$\langle\boldsymbol{\alpha}\rangle^{(r)}=\langle\hat{\boldsymbol{\alpha}}\rangle^{(r)},\tag{C 8}$$

$$\left\langle\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0^{(r)}\right)\cdot\mathbb{M}^{(r)}\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0^{(r)}\right)\right\rangle^{(r)}=\left\langle\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0^{(r)}\right)\cdot\mathbb{M}^{(r)}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0^{(r)}\right)\right\rangle^{(r)}.\tag{C 9}$$

Thus, for a given field $\boldsymbol{\alpha}$, the field equations (C 4) and (C 5) generate the same displacement field as (3.12). Now, to determine the field $\boldsymbol{\alpha}$ we substract its phase average to (C 6) to obtain

$$\left(\mathbb{L}^{(r)}+\mathbb{H}^{(r)}+\frac{\lambda_0^{(r)}}{\Delta t}\mathbb{M}^{(r)}\right)\left(\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)}\right)=\mathbb{L}^{(r)}\left(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}\rangle^{(r)}\right);\tag{C 10}$$

therefore

$$\left(\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)}\right)=\mathbb{G}^{(r)}\left(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}\rangle^{(r)}\right)\tag{C 11}$$

and

$$\left\langle\left(\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)}\right)\cdot\breve{\mathbb{M}}^{(r)}\left(\boldsymbol{\alpha}-\langle\boldsymbol{\alpha}\rangle^{(r)}\right)\right\rangle^{(r)}=\left\langle\left(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}\rangle^{(r)}\right)\cdot\mathbb{G}^{T}\breve{\mathbb{M}}^{(r)}\mathbb{G}\left(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}\rangle^{(r)}\right)\right\rangle^{(r)},\tag{C 12}$$

where

$$\check{\mathbb{M}}^{(r)} = \|\mathbb{M}^{(r)}\|^{-1}\mathbb{M}^{(r)}, \quad \mathbb{G}^{(r)} = \left(\mathbb{L}^{(r)} + \mathbb{H}^{(r)} + 2\lambda^{(r)}\check{\mathbb{M}}^{(r)}\right)^{-1}\mathbb{L}^{(r)}, \quad \lambda^{(r)} = \frac{\lambda_0^{(r)}}{2\Delta t}\|\mathbb{M}^{(r)}\|. \tag{C 13}$$

Thus, for a given set of moments $\langle\boldsymbol{\alpha}\rangle^{(r)}$ and $\langle(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}) \cdot \check{\mathbb{M}}^{(r)}(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)})\rangle^{(r)}$, equations (C 10)–(C 13) generate the same inelastic strain field as (3.12); see appendix B. To determine the first moments, we phase average (C 6) and (C 7), and take into account (C 8), to obtain

$$\mathbb{M}^{(r)}\frac{\langle\boldsymbol{\alpha}\rangle^{(r)} - \langle\boldsymbol{\alpha}_n\rangle^{(r)}}{\Delta t} + \left(\mathbb{L}^{(r)} + \mathbb{H}^{(r)}\right)\langle\boldsymbol{\alpha}\rangle^{(r)} = \langle\boldsymbol{\varepsilon}\rangle^{(r)}. \tag{C 14}$$

To determine the corresponding equations for the fluctuations, we substract its phase average from (C 7) to obtain

$$\left(1 - \lambda_0^{(r)}\right)\left(\hat{\boldsymbol{\alpha}} - \langle\hat{\boldsymbol{\alpha}}\rangle^{(r)}\right) = \left(\boldsymbol{\alpha}_n - \langle\boldsymbol{\alpha}_n\rangle^{(r)}\right). \tag{C 15}$$

At the same time, (C 8) and (C 9) imply that

$$\left\langle\left(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right) \cdot \check{\mathbb{M}}^{(r)}\left(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right)\right\rangle^{(r)} = \left\langle\left(\hat{\boldsymbol{\alpha}} - \langle\hat{\boldsymbol{\alpha}}\rangle^{(r)}\right) \cdot \check{\mathbb{M}}^{(r)}\left(\hat{\boldsymbol{\alpha}} - \langle\hat{\boldsymbol{\alpha}}\rangle^{(r)}\right)\right\rangle^{(r)}. \tag{C 16}$$

Introducing the former into the latter and taking the square root we obtain

$$(1 - \lambda_0^{(r)})\, C_{\alpha}^{(r)^{1/2}} = \pm C_{\alpha_n}^{(r)^{1/2}}. \tag{C 17}$$

Upon choosing the positive root in this expression, as in Lahellec & Suquet [18], and using (C 13)$_3$, we finally obtain

$$\|\mathbb{M}^{(r)}\|\frac{C_{\alpha}^{(r)^{1/2}} - C_{\alpha_n}^{(r)^{1/2}}}{\Delta t} - 2\lambda^{(r)}C_{\alpha}^{(r)^{1/2}} = 0. \tag{C 18}$$

Equations (C 14) and (C 18) are the implicit Euler discretizations of the evolutions laws (3.24). Thus, the total and inelastic strain fields, and therefore the corresponding stress field, are the same as the corresponding fields of the reduced-order model of §3.

Appendix D. Convex range of the reduced free-energy density

Consider a function $\hat{w}_c$ defined by the same expression (3.12) for $\hat{w}$ but with the set of admissible inelastic strains $\mathcal{J}$ replaced by the set

$$\mathcal{J}_c(\overline{\boldsymbol{\alpha}}^{(1)}, \ldots \widetilde{\alpha}^{(1)}, \ldots) = \Big\{\boldsymbol{\alpha} : \langle\boldsymbol{\alpha}\rangle^{(r)} = \overline{\boldsymbol{\alpha}}^{(r)} \quad \text{and}$$
$$\left\langle\left(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right) \cdot \check{\mathbb{M}}^{(r)}\left(\boldsymbol{\alpha} - \langle\boldsymbol{\alpha}\rangle^{(r)}\right)\right\rangle^{(r)^{1/2}} \le \widetilde{\alpha}^{(r)} \quad \text{for} \quad r = 1, \ldots, N\Big\}. \tag{D 1}$$

The set $\mathcal{J}_c$ is convex and, therefore, the function $\hat{w}_c$ is convex in the Cartesian product space of macroscopic strains and effective internal variables. It is evident that $\mathcal{J} \subseteq \mathcal{J}_c$ and therefore that

$$\hat{w}\left(\overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\alpha}}^{(1)}, \ldots, \widetilde{\alpha}^{(1)}, \ldots\right) \ge \hat{w}_c\left(\overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\alpha}}^{(1)}, \ldots, \widetilde{\alpha}^{(1)}, \ldots\right). \tag{D 2}$$

Now, the function $\hat{w}_c$ can be rewritten as

$$\hat{w}_c\left(\overline{\boldsymbol{\varepsilon}}, \overline{\boldsymbol{\alpha}}^{(1)}, \ldots, \widetilde{\alpha}^{(1)}, \ldots\right)$$
$$= \inf_{\boldsymbol{\varepsilon}\in\mathcal{K}(\overline{\boldsymbol{\varepsilon}})} \inf_{\boldsymbol{\alpha}} \sup_{\boldsymbol{\Lambda}^{(r)}} \sup_{\lambda^{(r)}\ge 0} \sum_{r=1}^{N} c^{(r)}\left\langle\frac{1}{2}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}) \cdot \mathbb{L}^{(r)}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}) + \frac{1}{2}\boldsymbol{\alpha} \cdot \mathbb{H}^{(r)}\boldsymbol{\alpha}\right\rangle^{(r)}$$
$$+ \boldsymbol{\Lambda}^{(r)} \cdot \left(\langle\boldsymbol{\alpha}\rangle^{(r)} - \overline{\boldsymbol{\alpha}}^{(r)}\right) + \lambda^{(r)}\left(\left\langle\left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)}\right) \cdot \check{\mathbb{M}}^{(r)}\left(\boldsymbol{\alpha} - \overline{\boldsymbol{\alpha}}^{(r)}\right)\right\rangle^{(r)} - \widetilde{\alpha}^{(r)^2}\right). \tag{D 3}$$

Within each phase, the optimal inelastic strain field is thus given by the same set of optimality conditions (B 2)–(B 7) as those of $\hat{w}$, provided those equations are satisfied by positive Lagrange multipliers $\lambda^{(r)} \geq 0$. Assuming the constitutive tensors $\mathbb{L}^{(r)}$, $\mathbb{H}^{(r)}$ and $\mathbb{M}^{(r)}$ have a common set of eigentensors within each phase, and given that those constitutive tensors, as well as the strain fluctuations tensor, are all positive definite, the left-hand side of (B 7) is a decreasing function of $\lambda^{(r)}$, and so that optimality condition holds whenever

$$\left[\mathbb{I} + \mathbb{H}^{(r)}\mathbb{L}^{(r)^{-1}}\right]^{-1} \breve{\mathbb{M}}^{(r)} \left[\mathbb{I} + \mathbb{L}^{(r)^{-1}}\mathbb{H}^{(r)}\right]^{-1} \cdot \left(\langle \boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon} \rangle^{(r)} - \langle \boldsymbol{\varepsilon} \rangle^{(r)} \otimes \langle \boldsymbol{\varepsilon} \rangle^{(r)}\right) \geq \widetilde{\alpha}^{(r)^2}. \qquad \text{(D 4)}$$

Thus, equality in (D 2) holds whenever the inequality (D 4) holds for all $r = 1, \ldots, N$. Within that range, the function $\hat{w}$ attains its convexification, as given by the function $\hat{w}_c$. Outside that range, the function $\hat{w}_c$ need not constitute the convexification of the function $\hat{w}$.

References

1. Rice JR. 1971 Inelastic constitutive relations for solids: an internal-variable theory and its application to metal plasticity. *J. Mech. Phys. Solids* **19**, 433–455. (doi:10.1016/0022-5096(71)90010-X)
2. Suquet P. 1985 Local and global aspects in the mathematical theory of plasticity. In *Plasticity today: modelling, methods and applications* (eds A Sawczuk, G Bianchi), pp. 279–310. London, UK: Elsevier.
3. Beurthey S, Zaoui A. 2000 Structural morphology and relaxation spectra of viscoelastic heterogeneous materials. *Eur. J. Mech./A Solids* **19**, 1–16. (doi:10.1016/S0997-7538(00)00157-1)
4. Hashin Z. 1970 Complex moduli of viscoelastic composites–I. General theory and application to particulate composites. *Int. J. Solids Struct.* **6**, 539–552. (doi:10.1016/0020-7683(70)90029-6)
5. Rougier Y, Stolz C, Zaoui A. 1993 Représentation spectrale en viscoélasticité linéaire des matériaux hétérogènes. *C. R. Acad. Sci. Paris* **316**, 1517–1522.
6. Schapery RA. 1967 Stress analysis of viscoelastic composite materials. *J. Composite Mater.* **1**, 228–267. (doi:10.1177/002199836700100302)
7. Francfort G, Suquet P. 1986 Homogenization and mechanical dissipation in thermoviscoelasticity. *Arch. Rat. Mech. Anal.* **96**, 265–293. (doi:10.1007/BF00251909)
8. Sanchez-Hubert J, Sanchez-Palencia E. 1978 Sur certains problèmes physiques d'homogénéisation donnant lieu à des phénomènes de relaxation. *C. R. Acad. Sci. Paris A* **286**, 903–906.
9. Milton GW. 2002 *The theory of composites*. Cambridge, UK: Cambridge University Press.
10. Mattei O, Milton GW. 2016 Bounds for the response of viscoelastic composites under antiplane loadings. In *Extending the theory of composites to other areas of science* (ed. GW Milton), ch. 6. Pennsauken, NJ: Milton and Patton Publishing (produced by BookBaby.com).
11. Ricaud J-M, Masson R. 2009 Effective properties of linear viscoelastic heterogeneous media: internal variables formulation and extension to ageing behaviours. *Int. J. Solids Struct.* **46**, 1599–1606. (doi:10.1016/j.ijsolstr.2008.12.007)
12. Laws N, McLaughlin R. 1978 Self-consistent estimates for the viscoelastic creep compliance of composite materials. *Proc. R. Soc. Lond. A* **359**, 251–273. (doi:10.1098/rspa.1978.0041)
13. Rekik A, Brenner R. 2011 Optimization of the collocation inversion method for the linear viscoelastic homogenization. *Mech. Res. Commun.* **38**, 305–308. (doi:10.1016/j.mechrescom.2011.04.003)
14. Vu QH, Brenner R, Castelnau O, Moulinec H, Suquet P. 2012 A self-consistent estimate for linear viscoelastic polycrystals with internal variables inferred from the collocation method. *Modell. Simul. Mater. Sci. Eng.* **20**, 0240034. (doi:10.1088/0965-0393/20/2/024003)
15. Dvorak G. 1992 Transformation field analysis of inelastic composite materials. *Proc. R. Soc. Lond. A* **437**, 311–327. (doi:10.1098/rspa.1992.0063)
16. Michel JC, Suquet P. 2003 Nonuniform transformation field analysis. *Int. J. Solids Struct.* **40**, 6937–6955. (doi:10.1016/S0020-7683(03)00346-9)
17. Michel JC, Suquet P. 2016 A model-reduction approach in micromechanics of materials preserving the variational structure of constitutive relations. *J. Mech. Phys. Solids* **90**, 254–285. (doi:10.1016/j.jmps.2016.02.005)
18. Lahellec N, Suquet P. 2007 Effective behavior of linear viscoelastic composites: a time-integration approach. *Int. J. Solids Struct.* **44**, 507–529. (doi:10.1016/j.ijsolstr.2006.04.038)

royalsocietypublishing.org/journal/rspa *Proc. R. Soc. A* **476**: 20200407

19. Lahellec N, Suquet P. 2007 On the effective behavior of nonlinear inelastic composites: I. Incremental variational principles. *J. Mech. Phys. Solids* **55**, 1932–1963. (doi:10.1016/j.jmps.2007.02.003)
20. Mialon P. 1986 Eléments d'analyse et de résolution numérique des relations de l'élasto-plasticité. *E.D.F. Bulletin de la Direction des Etudes et Recherches. Série C, Mathématiques, Informatique* **3**, 57–89.
21. Miehe C. 2002 Strain-driven homogenization of inelastic microstructures and composites based on an incremental variational formulation. *Int. J. Numer. Meth. Eng.* **55**, 1285–1322. (doi:10.1002/nme.515)
22. Ortiz M, Stainier L. 1999 The variational formulation of viscoplastic constitutive updates. *Comput. Methods Appl. Mech. Eng.* **171**, 419–444. (doi:10.1016/S0045-7825(98)00219-9)
23. Ponte Castañeda P, Willis JR. 1999 Variational second-order estimates for nonlinear composites. *Proc. R. Soc. Lond. A* **455**, 1799–1811. (doi:10.1098/rspa.1999.0380)
24. Tressou B, Vaziri R, Nadot-Martin C. 2018 Application of the incremental variational approach (EIV model) to the linear viscoelastic homogenization of different types of microstructures: long fiber-, particle-reinforced and strand-based composites. *Eur. J. Mech./A Solids.* **68**, 104–116. (doi:10.1016/j.euromechsol.2017.10.006)
25. Idiart MI, Lahellec N. 2016 Estimates for the overall linear properties of pointwise heterogeneous solids and application to elasto-viscoplasticity. *J. Mech. Phys. Solids* **97**, 317–332. (doi:10.1016/j.jmps.2015.12.017)
26. Lahellec N, Suquet P. 2007 On the effective behavior of nonlinear inelastic composites: II. A second-order procedure. *J. Mech. Phys. Solids* **55**, 1964–1992. (doi:10.1016/j.jmps.2007.02.004)
27. Boudet J, Auslender F, Bornert M, Lapusta Y. 2016 An incremental variational formulation for the prediction of the effective work-hardening behaviour and field statistics of elasto-(visco)plastic composites. *Int. J. Solids Struct.* **83**, 90–113. (doi:10.1016/j.ijsolstr.2016.01.003)
28. Huang Y, Abou-Chakra Guéry A, Shao J. 2015 Incremental variational approach for time dependent deformation in clayey rock. *Int. J. Plasticity.* **64**, 88–103. (doi:10.1016/j.ijplas.2014.07.003)
29. Zhao L, Shao J, Zhu Q, Liu Z, Yurtdas I. 2019 Homogenization of rock-like materials with plastic matrix based on an incremental variational principle. *Int. J. Plasticity* **123**, 145–164. (doi:10.1016/j.ijplas.2019.07.015)
30. Lucchetta A, Auslender F, Bornert M, Kondo D. 2019 A double incremental variational procedure for elastoplastic composites with combined isotropic and linear kinematic hardening. *Int. J. Solids Struct.* **158**, 243–267. (doi:10.1016/j.ijsolstr.2018.09.012)
31. Germain P, Nguyen QS, Suquet P. 1983 Continuum thermodynamics. *J. Appl. Mech.* **50**, 1010. (doi:10.1115/1.3167184)
32. Ponte Castañeda P, Suquet P. 1998 Nonlinear composites. *Adv. Appl. Mech.* **34**, 171–302.
33. Idiart MI, Ponte Castañeda P. 2007 Field statistics in nonlinear composites. I. Theory. *Proc. R. Soc. Lond. A* **463**, 183–202.
34. Suquet P. 2012 Four exact relations for the effective relaxation function of linear viscoelastic composites. *C. R. Mecanique* **340**, 387–399. (doi:10.1016/j.crme.2012.02.022)
35. Zecevic M, Lebensohn RA. 2020 New robust self-consistent homogenization schemes of elasto-viscoplastic polycrystals. *Int. J. Solids Struct.* **202**, 434–453. (doi:10.1016/j.ijsolstr.2020.05.032)
36. Idiart MI, Ponte Castañeda P. 2007 Variational linear comparison bounds for nonlinear composites with anisotropic phases. I. General results. *Proc. R. Soc. A* **463**, 907–924. (doi:10.1098/rspa.2006.1797)
37. Michel JC, Suquet P. 2017 Effective potentials in nonlinear polycrystals and quadrature formulae. *Proc. R. Soc. A* **473**, 20170213. (doi:10.1098/rspa.2017.0213)
38. Lahellec N, Suquet P. 2013 Effective response and field statistics in elasto-plastic and elasto-viscoplastic composites under radial and non-radial loadings. *Int. J. Plasticity* **42**, 1–30. (doi:10.1016/j.ijplas.2012.09.005)