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Principal congruences in weak Heyting algebras

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Abstract. Let A be a weak Heyting algebra and let $a, b \in A$. We give a description for the congruence generated by the pair (a, b), and we use it in order to give a necessary and sufficient condition for a function $f\colon A^k \to A$ to be compatible with every congruence of A. We also find conditions on a not necessarily polynomial function $g(a, b)$ in A that imply that the function $a \mapsto \min\{b \in A : g(a, b) \leq b\}$ is compatible when defined.

1. Introduction

A weak Heyting algebra, or *WH*-algebra for short [3, 11], is an algebra $\langle A, \wedge, \vee, \to, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$, where the reduct algebra $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\to$ is a binary map such that for all $a, b, c \in A$, it satisfies the following conditions:

(a) $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$,
(b) $(a \to c) \wedge (b \to c) = (a \vee b) \to c$,
(c) $(a \to b) \wedge (b \to c) \leq a \to c$,
(d) $a \to a = 1$.

We write WH to indicate the variety of *WH*-algebras. This variety was introduced in [11] as the algebraic counterpart of the least subintuitionistic logic wK considered in [10]. A *WH*-algebra is a bounded distributive lattice with a binary operation $\to$ with the properties of the strict implication in the modal logic K. Each one of the varieties of *WH*-algebras studied in [11] corresponds to two propositional logics wK_σ and sK_σ defined in [10]. The logics wK_σ and sK_σ are the strict implication fragments of the local and global consequence relations defined by means of Kripke models, respectively.

Examples of *WH*-algebras that appear in the literature are the basic algebras introduced by M. Ardeshir and W. Ruitenburg in [1], and the subresiduated lattices of G. Epstein and A. Horn given in [13]; these last structures were introduced as a generalization of Heyting algebras [2].

A basic algebra is a *WH*-algebra that in addition satisfies the inequality

(I) $a \leq 1 \to a$.

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A subresiduated lattice is a *WH*-algebra that in addition satisfies the following inequalities:

(T) $a \to b \leq c \to (a \to b)$,

(R) $a \wedge (a \to b) \leq b$.

Besides basic algebras and subresiduated lattices, other varieties of *WH*-algebras can be considered, defined by arbitrary combinations to the inequalities (R), (T) and (I) above. These varieties are the varieties of *WH*-algebras that correspond to certain subintuitionistic logics. In every *WH*-algebra, the inequality (I) implies (T). There are at most five subvarieties obtainable in that way, and in fact there are exactly five. They are the variety of subresiduated lattices (denoted by SRL), the variety of basic algebras (denoted by B), the variety of the *WH*-algebras that satisfy (R) (whose elements will be called *RWH*-algebras), the variety of the *WH*-algebras that satisfy (T) (whose elements will be called *TWH*-algebras), and finally the variety of Heyting algebras (which are the *WH*-algebras that satisfy the three inequalities (R), (T) and (I)). The variety of *RWH*-algebras will be denoted by RWH and the variety of *TWH*-algebras by TWH. We shall denote by H the variety of Heyting algebras.

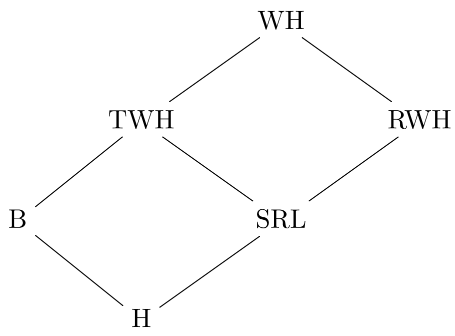

FIGURE 1. The order of the varieties

The five varieties mentioned above inherit some of the properties of their subvariety of Heyting algebras [11]. Let $A \in$ WH. It follows from [11, Proposition 4.22] that $A \in$ RWH if and only if for every $a, b, c \in A$, if $a \leq b \to c$ then $a \wedge b \leq c$. Also, it follows from [11, Proposition 4.20] that $A \in$ B if and only if for every $a, b, c \in A$, if $a \wedge b \leq c$, then $a \leq b \to c$.

Remark 1.1. Every bounded distributive lattice can be seen as an algebra in B if we define $a \to b = 1$ for every a, b.

Let A be an algebra and $a, b \in A$. By $\theta(a, b)$ we denote the principal congruence of A generated by (a, b), i.e., the smallest congruence of A that contains (a, b). A variety V has *equationally definable principal congruences* (EDPC) if there exists a finite family of quaternary terms $\{u_i, v_i\}_{i=1}^{n}$ such that for any principal congruence $\theta(a, b)$, $(c, d) \in \theta(a, b)$ if and only if we have

$u_i(a,b,c,d) = v_i(a,b,c,d)$ for every $i = 1,\dots,n$ [4]. This property is also of logical interest because a logic has some kind of deduction theorem if and only if the corresponding variety (obtained by the process of algebraization) has EDPC.

We show in this paper that in some cases, a good characterization of principal congruences is still possible, taking the following form: there exists a family of quaternary terms $\{u_{(i,m)}, v_{(i,m)}\}$ (with $i = 1,\dots,n$ and $m \geq 0$) such that for any principal congruence $\theta(a,b)$, $(c,d) \in \theta(a,b)$ if and only if there exists $k \geq 0$ such that $u_{(i,k)}(a,b,c,d) = v_{(i,k)}(a,b,c,d)$ for every $i = 1,\dots,n$. In [11], it was proved that TWH has EDPC, but WH and RWH do not. However, an explicit description of the principal congruences was not given.

Let $A \in$ WH, $a, b \in A$ and $n \in \mathbb{N}$ (the set of natural numbers). Abbreviate $1 \to a$ by $\Box(a)$; then the iterated operator $\Box^n$ is defined in the usual way. If $n = 0$, we define $\Box^0(a) = a$. As usual, define $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

Remark 1.2. Let A be an algebra and $a, b, c, d \in A$.

(a) If A is a distributive lattice, $(c,d) \in \theta(a,b)$ if and only if $a \wedge b \wedge c = a \wedge b \wedge d$ and $a \vee b \vee c = a \vee b \vee d$ [5, 17].

(b) Taking into account [18], we have that for $A \in$ RWH, $(c,d) \in \theta(a,b)$ if and only if there exists $n \in \mathbb{N}$ such that $\Box^n(a \leftrightarrow b) \leq c \leftrightarrow d$. Moreover, if $A \in$ B, then $(c,d) \in \theta(a,b)$ if and only if $a \leftrightarrow b \leq c \leftrightarrow d$.

It is interesting to note that if $A \in$ RWH, we obtain the following:

(i) There exists an order isomorphism between the lattice of congruences of A and the lattice of open filters of A, where an open filter is a filter closed by $\Box$ [11, Theorem 6.12].

(ii) If θ is a congruence of A, then $(a,b) \in \theta$ if and only if $(a \leftrightarrow b, 1) \in \theta$ [18, Lemma 2].

The characterization of the principal congruences of the algebras in RWH given in (b) of Remark 1.2 was proved in [18], taking into account (i) and (ii). However, if $A \in$ WH and $A \notin$ RWH, the assertions (i) and (ii) are not true in general.

In this paper, we provide a description of the principal congruences of any weak Heyting algebra. This description is motivated basically by the assertions in (a) and (b) of Remark 1.2. Taking into account that compatible functions are closely related with principal congruences, we give a necessary and sufficient condition on a function to be compatible. Finally, we find conditions on a not necessarily polynomial function $g(a,b)$ that imply that the map $a \mapsto \min\{b : g(a,b) \leq b\}$ is compatible when defined.

2. Principal congruences

In this section, we give an explicit description for the principal congruences of weak Heyting algebras.

Let $n \in \mathbb{N}$. We introduce the binary term

$$t_n(a,b) = (a \leftrightarrow b) \wedge \Box(a \leftrightarrow b) \wedge \cdots \wedge \Box^n(a \leftrightarrow b).$$

Let $A \in \mathrm{WH}$ and $a, b \in A$. Let $R(a,b)$ be the binary relation in A defined as follows: $(x,y) \in R(a,b)$ if and only if there exists $n \in \mathbb{N}$ satisfying

(C1) $x \wedge a \wedge b \wedge t_n(a,b) = y \wedge a \wedge b \wedge t_n(a,b)$,
(C2) $(x \vee a \vee b) \wedge t_n(a,b) = (y \vee a \vee b) \wedge t_n(a,b)$,
(C3) $t_n(a,b) \leq x \leftrightarrow y$.

We say that $n \in \mathbb{N}$ is *associated* with a pair (x,y) if (C1)–(C3) hold.

The following remark is part of the folklore of distributive lattices.

Remark 2.1. Let A be a distributive lattice, θ a congruence and $(x,y) \in A \times A$. If there exists $c \in A$ such that $(x \wedge c, y \wedge c) \in \theta$ and $(x \vee c, y \vee c) \in \theta$, then $(x,y) \in \theta$.

In what follows, we give one of the main results of this work.

Theorem 2.2. *Let $A \in \mathrm{WH}$ and $a, b \in A$. Then $\theta(a,b) = R(a,b)$.*

Proof. First we will prove that $R(a,b)$ is an equivalence relation. The reflexivity and symmetry are immediate. In order to prove that $R(a,b)$ is a transitive relation, let $(x,y), (y,z) \in R(a,b)$. Let n and m be two natural numbers associated with the pairs (x,y) and (y,z), and let $k = \max\{n,m\}$. It is clear that $x \wedge a \wedge b \wedge t_k(a,b) = z \wedge a \wedge b \wedge t_k(a,b)$ and $(x \vee a \vee b) \wedge t_k(a,b) = (z \vee a \vee b) \wedge t_k(a,b)$. Also, taking into account that

$$t_k(a,b) \leq (x \to y) \wedge (y \to x) \wedge (y \to z) \wedge (z \to y) \leq (x \to z) \wedge (z \to x),$$

we obtain that $t_k(a,b) \leq x \leftrightarrow z$. Thus, $R(a,b)$ is a transitive relation, and in consequence it is an equivalence relation.

Now we will see that $R(a,b)$ is a congruence. Let $(x,y), (z,w) \in R(a,b)$. Let n be a natural number associated with the pair (x,y), let m be a natural number associated with the pair (z,w), and let $k = \max\{n,m\}$.

(i) First we will show that $(x \wedge z, y \wedge w) \in R(a,b)$. It is clear that

$$x \wedge z \wedge a \wedge b \wedge t_k(a,b) = y \wedge w \wedge a \wedge b \wedge t_k(a,b).$$

Using the distributivity of the lattice,we have:

$$\begin{aligned} &((x \wedge z) \vee a \vee b) \wedge t_k(a,b) = (x \vee a \vee b) \wedge (z \vee a \vee b) \wedge t_k(a,b) \\ &= (x \vee a \vee b) \wedge (w \vee a \vee b) \wedge t_k(a,b) = t_k(a,b) \wedge (x \vee a \vee b) \wedge (w \vee a \vee b) \\ &= t_k(a,b) \wedge (y \vee a \vee b) \wedge (w \vee a \vee b) = t_k(a,b) \wedge ((y \wedge w) \vee a \vee b). \end{aligned}$$

Finally, we have

$$\begin{aligned} (x \wedge z) \leftrightarrow (y \wedge w) &\geq (x \to y) \wedge (z \to w) \wedge (y \to x) \wedge (w \to z) \\ &= (x \leftrightarrow y) \wedge (z \leftrightarrow w) \geq t_k(a,b). \end{aligned}$$

Hence, we have $(x \wedge z, y \wedge w) \in R(a,b)$.

(ii) Analogously, it can be proved that $(x \vee z, y \vee w) \in R(a,b)$.

(iii) In what follows, we will see that $(x \to z, y \to w) \in R(a, b)$. Note that

$$(x \to z) \wedge a \wedge b \wedge t_{k+1}(a, b) = a \wedge b \wedge t_{k+1}(a, b) \wedge (x \to z) \wedge (y \leftrightarrow x) \wedge (z \leftrightarrow w);$$

$$a \wedge b \wedge t_{k+1}(a, b) \wedge (x \to z) \wedge (y \leftrightarrow x) \wedge (z \leftrightarrow w) \leq a \wedge b \wedge t_{k+1}(a, b) \wedge (y \to w).$$

Hence, we have that $(x \to z) \wedge a \wedge b \wedge t_{k+1}(a, b) \leq a \wedge b \wedge t_{k+1}(a, b) \wedge (y \to w)$.

The other inequality can be proved in the same way, so we obtain that

$$(x \to z) \wedge a \wedge b \wedge t_{k+1}(a, b) = (y \to w) \wedge a \wedge b \wedge t_{k+1}(a, b).$$

Similarly it can be proved that $t_{k+1}(a, b) \wedge (x \to z) \leq (y \to w)$ and $t_{k+1}(a, b) \wedge (y \to w) \leq (x \to z)$, so this and the distributivity of the lattice yield $((x \to z) \vee a \vee b) \wedge t_{k+1}(a, b) = ((y \to w) \vee a \vee b) \wedge t_{k+1}(a, b)$.

In order to prove that $t_{k+1}(a, b) \leq (x \to z) \to (y \to w)$, note that the inequality $t_k(a, b) \wedge (x \to z) \leq (y \to w)$ implies that

$$(x \to z) \to t_k(a, b) \leq (x \to z) \to (y \to w). \tag{2.1}$$

On the other hand, using that $x \to z \leq 1$, we have that for every $l \geq 1$, it holds that $\Box^l(a \to b) \leq (x \to z) \to \Box^{l-1}(a \to b)$. Thus, we obtain the inequality $\Box^l(a \leftrightarrow b) \leq (x \to z) \to \Box^{l-1}(a \leftrightarrow b)$. Hence,

$$t_{k+1}(a, b) \leq (x \to z) \to t_k(a, b). \tag{2.2}$$

By equations (2.1) and (2.2), we conclude that

$$t_{k+1}(a, b) \leq (x \to z) \to t_k(a, b) \leq (x \to z) \to (y \to w).$$

Thus, $(x \to z, y \to w) \in R(a, b)$.

Thus, we have proved that $R(a, b)$ is a congruence which contains the pair (a, b). Let θ be a congruence with $(a, b) \in \theta$. We will show that $R(a, b) \subseteq \theta$. In order to see this, let $(x, y) \in R(a, b)$ and let n be a natural number associated with the pair (x, y). Taking into account that $(a, b) \in \theta$, we obtain that $(t_n(a, b), 1) \in \theta$. Put $c = a \wedge b \wedge t_n(a, b)$. We will show that $(x \vee c, y \vee c) \in \theta$ and $(x \wedge c, y \wedge c) \in \theta$. Easy computations show $(x \vee c, (x \vee a \vee b) \wedge t_n(a, b)) \in \theta$, $(y \vee c, (y \vee a \vee b) \wedge t_n(a, b)) \in \theta$, and $(x \vee a \vee b) \wedge t_n(a, b) = (y \vee a \vee b) \wedge t_n(a, b)$, so we obtain that $(x \vee c, y \vee c) \in \theta$. Taking into account that $x \wedge c = y \wedge c$, we have that $(x \wedge c, y \wedge c) \in \theta$. Thus, it follows from Remark 2.1 that $(x, y) \in \theta$. Hence, $R(a, b) \subseteq \theta$. Therefore, $\theta(a, b) = R(a, b)$. $\Box$

Remark 2.3. (1) Let $A \in$ RWH and $a, b \in A$. Taking into account that $\Box a = 1 \wedge (1 \to a) \leq a$, we have that $\Box^n(a) \leq a$ for every $n \in \mathbb{N}$. In particular, $\Box^n(a \leftrightarrow b) \leq a \leftrightarrow b$ for every $n \in \mathbb{N}$.

(2) Let $A \in$ RWH, $a, b, x, y \in A$ and $n \in \mathbb{N}$ such that $\Box^n(a \leftrightarrow b) \leq x \leftrightarrow y$. In particular, we have that $x \wedge \Box^n(a \leftrightarrow b) \leq y$ and $y \wedge \Box^n(a \leftrightarrow b) \leq x$. Thus, $x \wedge a \wedge b \wedge \Box^n(a \leftrightarrow b) \leq y \wedge a \wedge b \wedge \Box^n(a \leftrightarrow b)$. In a similar way, we can prove the other inequality, so $x \wedge a \wedge b \wedge \Box^n(a \leftrightarrow b) = y \wedge a \wedge b \wedge \Box^n(a \leftrightarrow b)$. Analogously (and taking into account the distributivity of the lattice), it is possible to show that $(x \vee a \vee b) \wedge \Box^n(a \leftrightarrow b) = (y \vee a \vee b) \wedge \Box^n(a \leftrightarrow b)$.

(3) Let $A \in \mathrm{TWH}$ and $a, b \in A$. Then $a \to b \leq 1 \to (a \to b) = \Box(a \to b)$, so $a \to b \leq \Box^n(a \to b)$ for every $n \in \mathbb{N}$. Moreover, $a \leftrightarrow b \leq \Box^n(a \leftrightarrow b)$ for every $n \in \mathbb{N}$. Thus, $t_n(a, b) = a \leftrightarrow b$.

(4) Let $A \in \mathrm{SRL}$ and $a, b \in A$. Then $\Box^n(a \leftrightarrow b) = a \leftrightarrow b$ for every $n \in \mathbb{N}$.

Proposition 2.4. *Let $A \in \mathrm{RWH}$, and $a, b \in A$. Then $(x, y) \in \theta(a, b)$ if and only if there exists $n \in \mathbb{N}$ such that $\Box^n(a \leftrightarrow b) \leq x \leftrightarrow y$. Moreover, if $A \in \mathrm{SRL}$, then $(x, y) \in \theta(a, b)$ if and only if $a \leftrightarrow b \leq x \leftrightarrow y$.*

The previous proposition is also a consequence from [18, Lemma 2].

Proposition 2.5. *Let $A \in \mathrm{TWH}$ and $a, b \in A$. Then $(x, y) \in \theta(a, b)$ if and only if the following conditions hold:*

(TC1) $x \wedge a \wedge b \wedge (a \leftrightarrow b) = y \wedge a \wedge b \wedge (a \leftrightarrow b)$,

(TC2) $(x \vee a \vee b) \wedge (a \leftrightarrow b) = (y \vee a \vee b) \wedge (a \leftrightarrow b)$,

(TC3) $a \leftrightarrow b \leq x \leftrightarrow y$.

In every basic algebra, we have the inequality $a \leq b \to a$. Then we obtain the following result.

Proposition 2.6. *Let $A \in \mathrm{B}$ and $a, b \in A$. Then $(x, y) \in \theta(a, b)$ if and only if the following conditions hold:*

(BC1) $x \wedge a \wedge b = y \wedge a \wedge b$,

(BC2) $(x \vee a \vee b) \wedge (a \leftrightarrow b) = (y \vee a \vee b) \wedge (a \leftrightarrow b)$,

(BC3) $a \leftrightarrow b \leq x \leftrightarrow y$.

Note that the well-known description of principal congruences for distributive lattices follows from Remark 1.1 and Proposition 2.6.

3. Compatible functions

Compatibility of functions is a classical topic in Universal Algebra. In [7], compatible functions were studied in Heyting algebras as the algebraic counterpart of intuitionistic connectives in the propositional intuitionistic calculus (see also [6]). In [18], these ideas were generalized in order to study compatible functions in the variety RWH. In this section, we study compatible functions in weak Heyting algebras using basically Theorem 2.2 and its consequences.

We start with the following.

Definition 3.1. Let A be an algebra and let $f\colon A^k \to A$ be a function.

(1) We say that f is *compatible with a congruence θ of A* if $(a_i, b_i) \in \theta$ for $i = 1, \ldots, k$ implies $(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)) \in \theta$.

(2) We say that f is a *compatible function* of A provided it is compatible with all the congruences of A.

Let A be an algebra. If $n \in \mathbb{N}$ and $f\colon A^n \to A$ is a function, then f is compatible if and only if the algebras A and $\langle A, f\rangle$ have the same congruences. For $n = 1$, f is compatible if and only if $(f(a), f(b)) \in \theta(a, b)$ for every

$a, b \in A$. The simplest examples of compatible functions on an algebra are the polynomial functions; note that in particular, all term functions (and constant functions) are compatible.

Corollary 3.2. *Let $A \in \mathrm{WH}$ and $f\colon A \to A$ a function. Then f is a compatible function if and only if for every $a, b \in A$, there exists $n \in \mathbb{N}$ that satisfies the following conditions:*

(C1) $f(a) \wedge a \wedge b \wedge t_n(a, b) = f(b) \wedge a \wedge b \wedge t_n(a, b)$,
(C2) $(f(a) \vee a \vee b) \wedge t_n(a, b) = (f(b) \vee a \vee b) \wedge t_n(a, b)$,
(C3) $t_n(a, b) \leq f(a) \leftrightarrow f(b)$.

Proof. This follows from Theorem 2.2. □

If L is a bounded distributive lattice, and if we define the operation $\to$ on its domain by

$$a \to b = \begin{cases} 1, & \text{if } a \leq b, \\ 0, & \text{if } a \not\leq b, \end{cases}$$

then the algebra $\langle L, \to \rangle$ is a *WH*-algebra. Moreover, we have that

$$a \leftrightarrow b = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{if } a \neq b. \end{cases}$$

Then by Corollary 3.2, every unary function on $\langle L, \to \rangle$ is compatible.

Corollary 3.3. *Let $A \in \mathrm{RWH}$ and $f\colon A \to A$ a function. Then f is a compatible function if and only if for every $a, b \in A$, there exists $n \in \mathbb{N}$ such that $\Box^n(a \leftrightarrow b) \leq f(a) \leftrightarrow f(b)$. Moreover, if $A \in \mathrm{SRL}$, then f is a compatible function if and only if $a \leftrightarrow b \leq f(a) \leftrightarrow f(b)$.*

Proof. This follows from Proposition 2.4. □

Corollary 3.3 is also a consequence of [18, Proposition 3] and of [18, Corollary 3].

Corollary 3.4. *Let $A \in \mathrm{TWH}$ and $f\colon A \to A$ a function. Then f is a compatible function if and only if for every $a, b \in A$, the following conditions hold:*

(TC1) $f(a) \wedge a \wedge b \wedge (a \leftrightarrow b) = f(b) \wedge a \wedge b \wedge (a \leftrightarrow b)$,
(TC2) $(f(a) \vee a \vee b) \wedge (a \leftrightarrow b) = (f(b) \vee a \vee b) \wedge (a \leftrightarrow b)$,
(TC3) $a \leftrightarrow b \leq f(a) \leftrightarrow f(b)$.

Proof. This follows from Proposition 2.5. □

Corollary 3.5. *Let $A \in \mathrm{B}$ and $f\colon A \to A$ a function. Then f is a compatible function if and only if for every $a, b \in A$, the following conditions hold:*

(BC1) $f(a) \wedge a \wedge b = f(b) \wedge a \wedge b$,
(BC2) $(f(a) \vee a \vee b) \wedge (a \leftrightarrow b) = (f(b) \vee a \vee b) \wedge (a \leftrightarrow b)$,
(BC3) $a \leftrightarrow b \leq f(a) \leftrightarrow f(b)$.

Proof. This follows from Proposition 2.6. □

Let A be an algebra, $f\colon A^k \to A$ a function, and $a = (a_1, \ldots, a_k) \in A^k$. For $i = 1, \ldots, k$, we define unary functions $f_i^a\colon A \to A$ by

$$f_i^a(x) := f(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_k).$$

Then we have the following characterization for the compatibility of a k-ary function f.

Lemma 3.6. *Let A be an algebra and $f\colon A^k \to A$ a function. The following conditions are equivalent:*

(a) *f is compatible.*
(b) *For every $a \in A^k$ and every $i = 1, \ldots, k$, the functions $f_i^a\colon A \to A$ are compatible.*

Let $a = (a_1, \ldots, a_k) \in A^k$. For every $m = 1, \ldots, k$, we define $a_l^m := (a_1, \ldots, a_m)$ and $a_r^m := (a_m, \ldots, a_k)$. Corollary 3.2 together with Lemma 3.6 allow us to characterize the compatible k-ary functions on a *WH*-algebra.

Corollary 3.7. *Let $A \in \mathrm{WH}$ and $f\colon A^k \to A$ a function $(k \geq 2)$. The following conditions are equivalent:*

(1) *f is compatible.*
(2) *For every $a, b \in A^k$, there exists $n \in \mathbb{N}$ satisfying the following conditions for every $i = 2, \ldots, k$:*
   (G1) $f(a) \wedge a_1 \wedge b_1 \wedge t_n(a_1, b_1) = f(b_1, a_r^2) \wedge a_1 \wedge b_1 \wedge t_n(a_1, b_1)$,
   $f(b_l^{i-1}, a_r^i) \wedge a_i \wedge b_i \wedge t_n(a_i, b_i) = f(b_l^i, a_r^{i+1}) \wedge a_i \wedge b_i \wedge t_n(a_i, b_i)$,
   (G2) $(f(a) \vee a_1 \vee b_1) \wedge t_n(a_1, b_1) = (f(b_1, a_r^2) \vee a_1 \vee b_1) \wedge t_n(a_1, b_1)$,
   $(f(b_l^{i-1}, a_r^i) \vee a_i \vee b_i) \wedge t_n(a_i, b_i) = (f(b_l^i, a_r^{i+1}) \vee a_i \vee b_i) \wedge t_n(a_i, b_i)$,
   (G3) $\bigwedge_{j=1}^k t_n(a_j, b_j) \leq f(a) \leftrightarrow f(b)$.

Proof. Suppose that f is compatible, and let $a, b \in A^k$. Let n_i be a natural number associated with the pair (a_i, b_i). Consider $n = \max\{n_1, \ldots, n_k\}$. Straightforward computations prove the conditions (G1) and (G2). Also, we have that

$$\begin{gathered} t_n(a_1, b_1) \leq f(a_1, a_2, \ldots, a_k) \leftrightarrow f(b_1, a_2, \ldots, a_k) \\ t_n(a_2, b_2) \leq f(b_1, a_2, \ldots, a_k) \leftrightarrow f(b_1, b_2, a_3, \ldots, a_k) \\ \vdots \\ t_n(a_k, b_k) \leq f(b_1, b_2, \ldots, b_{k-1}, a_k) \leftrightarrow f(b_1, \ldots, b_k) \end{gathered}$$

Since in *WH*-algebras we have that $(x \to y) \wedge (y \to z) \leq x \to z$, we obtain that $\bigwedge_{j=1}^k t_n(a_j, b_j) \leq f(a_1, \ldots, a_k) \leftrightarrow f(b_1, \ldots, b_k)$.

Conversely, assume condition (2) holds and let $a \in A^k$. Let $x, y \in A$. For the k-tuples $(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_k)$ and $(a_1, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_k)$, there exists a natural number n which satisfies the conditions (G1), (G2), and (G3).

Then Corollary 3.2 together with Lemma 3.6 allow us to claim that f is compatible. □

An algebra A is *affine complete* if any compatible function of A is given by a polynomial of A. It is *locally affine complete* provided that any compatible function is given by a polynomial on each finite subset of A. The variety of boolean algebras is affine complete [16]. The variety of Heyting algebras is not affine complete but it is locally affine complete [7]. Moreover, the variety of residuated lattices is locally affine complete [9].

It follows from [18, Corollary 7] that the variety RWH is locally affine complete. However, the variety B is not locally affine complete (this follows from Remark 1.1 and [17, Theorem 5.3.6]).

4. Some applications

In what follows, we shall use similar ideas to those in [8, 14, 19] in order to study compatible functions in the variety WH in terms of the minimum operator.

Definition 4.1. Let A be a poset and let $g\colon A \times A \to A$ be a function. We say that g satisfies condition (**M**) if the following condition holds:

$$\text{for all } a, b, c \in A,\ \ c \geq b \text{ implies } g(a, c) \leq g(a, b). \tag{\textbf{M}}$$

If A is a $\vee$-semilattice and g is a function which satisfies condition (**M**), then $g(a, g(a, b) \vee b) \leq g(a, b) \vee b$ for every $a, b \in A$.

Let A be a poset and g a binary function. When an expression of the type $\min\{b \in A : g(a, b) \leq b\}$ is used, it means that the minimum of the set $\{b \in A : g(a, b) \leq b\}$ exists for every $a \in A$.

Lemma 4.2. *Let A be a $\vee$-semilattice, and let $g\colon A \times A \to A$ be a function which satisfies the condition* (**M**)*. The following conditions are equivalent.*

(a) *There is a map $f\colon A \to A$ given by $f(a) = \min\{b \in A : g(a, b) \leq b\}$.*
(b) *There exists a map $h\colon A \to A$ which satisfies the following conditions for every $a, b \in A$:*
  (i) $g(a, h(a)) \leq h(a)$,
  (ii) $h(a) \leq g(a, b) \vee b$.
  Moreover, in this case we have that $f = h$.

Proof. This follows from [8, Lemma 15]. □

Then we have the following characterization for unary compatible functions in WH.

Proposition 4.3. *Let $A \in$ WH and let $f\colon A \to A$ be a function. The following conditions are equivalent.*

(1) *f is compatible.*

(2) *There exists a function* $g\colon A\times A\to A$ *which satisfies* (**M**), *compatible in the first variable and such that* $f(a)=\min\{b\in A : g(a,b)\le b\}$.
(3) *There exists a function* $\hat{g}\colon A\times A\to A$ *which satisfies* (**M**), *compatible in the first variable and such that it satisfies the following conditions for every* $a,b\in A$*:*
   (i) $\hat{g}(a,f(a))\le f(a)$,
   (ii) $f(a)\le \hat{g}(a,b)\vee b$.

Moreover, in this case we have that $g=\hat{g}$.

Proof. Suppose that f is compatible; define $g\colon A\times A\to A$ by $g(a,b)=f(a)$. Hence, the condition (2) is obtained. The equivalence between (2) and (3) follows from Lemma 4.2.

Finally, we shall prove that condition (3) implies condition (1). Let $a,b\in A$. Taking into account that g is compatible in the first variable and Corollary 3.2, we have that there exists $n\in\mathbb{N}$ such that

$$\begin{aligned} g(a,f(b))\wedge a\wedge b\wedge t_n(a,b) &= g(b,f(b))\wedge a\wedge b\wedge t_n(a,b),\\ g(a,f(b))\vee a\vee b\wedge t_n(a,b) &= g(b,f(b))\vee a\vee b\vee t_n(a,b),\\ t_n(a,b) &\le g(a,f(b))\leftrightarrow g(b,f(b)). \end{aligned}$$

We define $T_{(n,f)}(a,b):=f(b)\wedge a\wedge b\wedge t_n(a,b)$. Then we obtain that

$$\begin{aligned} f(a)\wedge a\wedge b\wedge t_n(a,b) &\le (f(b)\vee g(a,f(b)))\wedge a\wedge b\wedge t_n(a,b)\\ &= T_{(n,f)}(a,b)\vee (g(a,f(b))\wedge a\wedge b\wedge t_n(a,b))\\ &= T_{(n,f)}(a,b)\vee (g(b,f(b))\wedge a\wedge b\wedge t_n(a,b))\\ &\le T_{(n,f)}(a,b)\vee T_{(n,f)}(a,b)=T_{(n,f)}(a,b). \end{aligned}$$

The other inequality is proved in a similar way. Then (C1) holds and the proof of condition (C2) is similar. We also have that

$$t_n(a,b)\le g(a,f(b))\leftrightarrow g(b,f(b))\le g(a,f(b))\to g(b,f(b)). \tag{4.1}$$

Also, taking into account that $g(b,f(b))\le f(b)$, we obtain that

$$g(a,f(b))\to g(b,f(b))\le g(a,f(b))\to f(b). \tag{4.2}$$

Taking into account that $f(a)\le g(a,f(b))\vee f(b)$, and the inequalities (4.1) and (4.2), we have that

$$\begin{aligned} f(a)\to f(b) &\ge (g(a,f(b))\vee f(b))\to f(b)=g(a,f(b))\to f(b)\\ &\ge g(a,f(b))\to g(b,f(b))\ge t_n(a,b). \end{aligned}$$

Hence, $t_n(a,b)\le f(a)\to f(b)$. Taking into account that $t_n(a,b)=t_n(b,a)$, we conclude condition (C3). Therefore, by Corollary 3.2, we have that f is compatible. □

In what follows, we introduce a definition we shall use in order to give some examples of compatible functions.

Definition 4.4. Let V be a variety of algebras of type F and let $\epsilon(C)$ be a set of identities of type $F \cup C$, where C is a family of new function symbols. We say that $\epsilon(C)$ defines C *implicitly* if in each algebra $A \in V$, there is at most one family $\{f_A \colon A^n \to A\}_{f\in C}$ such that $(A, f_A)_{f\in C}$ satisfies the universal closure of the equations in $\epsilon(C)$. In this case, we say that each f is *implicitly defined* in V.

Let $A \in \mathrm{WH}$. We define the unary compatible function $\hat{S}$ by

$$\hat{S}a = \min\{b \in A : b \to a \leq b\}.$$

Equivalently, $\hat{S}$ can be implicitly defined by the inequalities

(S1) $\hat{S}a \to a \leq \hat{S}a$,
(S2) $\hat{S}a \leq b \vee (b \to a)$.

The map $\hat{S}$ generalizes the successor function considered in [7] on Heyting algebras. In [12], another generalization of the successor function was introduced for weak Heyting algebras; the operator was denoted by S and it was given by the inequalities $a \leq Sa$, $Sa \leq b \vee (b \to a)$, and $Sa \to a \leq a$. It follows from [12, Lemma 3.3] that if there exists S in a weak Heyting algebra, then there exists $\hat{S}$ (moreover, $S = \hat{S}$). The converse of this property is not true in general (see [12, Example 3.7]). In Heyting algebras, S exists if and only if $\hat{S}$ exists.

Let $A \in \mathrm{WH}$. We define the unary compatible function $\hat{\gamma}$ by

$$\hat{\gamma}a = \min\{b \in A : a \vee \neg b \leq b\},$$

where $\neg b = b \to 0$. This map can be also implicitly defined through the inequalities

(γ1) $a \vee \neg\hat{\gamma}a \leq \hat{\gamma}a$,
(γ2) $\hat{\gamma}a \leq a \vee \neg b \vee b$.

Equivalently, $\hat{\gamma}$ can be implicitly defined through the inequalities

(Γ1) $\neg\hat{\gamma}0 \leq \hat{\gamma}0$,
(Γ2) $\hat{\gamma}0 \leq b \vee \neg b$,
(Γ3) $\hat{\gamma}a = a \vee \hat{\gamma}0$.

In particular, if $\hat{\gamma}$ exists in A, then $\hat{\gamma}$ is a polynomial function on A. The map $\hat{\gamma}$ is a possible generalization of the gamma function studied in [7] for the case of Heyting algebras. In [12], there was given another possible generalization of the gamma function on weak Heyting algebras; this operator was called γ and it was defined by $\neg\gamma 0 = 0$, $\gamma a \leq b \vee (b \to a)$ and $\gamma a = a \vee \gamma 0$. It follows from [12] that given a weak Heyting algebra, if γ exists, then $\hat{\gamma}$ exists, with $\gamma = \hat{\gamma}$. However, taking into account [12, Example 3.7], we have that the converse of the previous property is not true in general. In Heyting algebras, γ exists if and only if $\hat{\gamma}$ exists.

Let $A \in \mathrm{WH}$. We define the unary compatible function G by

$$Ga = \min\{b \in A : (b \to a) \wedge \neg\neg a \leq b\}.$$

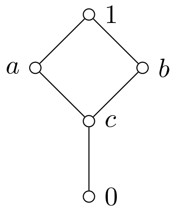

Figure 2

In an equivalent way, G can be implicitly defined through the inequalities

(G1) $(Ga \to a) \wedge \neg\neg a \leq Ga$,

(G2) $Ga \leq b \vee ((b \to a) \wedge \neg\neg a)$.

The map G is a possible generalization of Gabbay's function studied in [7] for the case of Heyting algebras (see also [15, 19]).

Consider the bounded distributive lattice given in Figure 2, which will be called A. In what follows, we define binary operations $\to$ on A in order to obtain weak Heyting algebras $\langle A, \to \rangle$, and we use these algebras in order to give examples of compatible functions.

Example 4.5. We define the following binary operation:

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	b	1	1	1	1
a	b	1	1	1	1
b	c	a	a	1	1
1	c	a	a	1	1

Then $\langle A, \to \rangle \in$ B and it is not a Heyting algebra because, for example, we have $a \to c = 1 \neq b$. We also have that

$\to$	$\hat{S}x$	$\hat{\gamma}x$	G
0	b	b	0
c	1	b	c
a	1	1	c
b	1	b	b
1	1	1	1

Example 4.6. We define the following binary operation:

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	1	1	1	1	1
a	c	c	1	c	1
b	c	c	c	1	1
1	c	c	c	c	1

Then $\langle A, \rightarrow \rangle \in$ TWH and it is not a basic algebra because, for example, we have $a \nleq c = 1 \rightarrow a$. Straightforward computations show that it is not possible to define $\hat{S}$ and $\hat{\gamma}$ (consider the element 0). However, $G0 = 0$, $Gc = c$, $Ga = b$, $Gb = a$, and $G1 = 1$.

Example 4.7. We define the following binary operation:

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	1	1	1	1	1
a	1	1	1	1	1
b	b	b	b	1	1
1	b	b	b	1	1

Then $\langle A, \rightarrow \rangle \in$ TWH and it is not a basic algebra because, for instance, we have $a \nleq b = 1 \rightarrow a$. We also have that

$\rightarrow$	$\hat{S}x$	$\hat{\gamma}x$	G
0	b	b	0
c	b	b	b
a	b	1	b
b	1	b	1
1	1	1	1

Finally, we define the following binary operation:

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	a	1	1	1	1
a	a	1	1	1	1
b	c	b	b	1	1
1	c	b	b	1	1

Then $\langle A, \rightarrow \rangle \in$ WH. We have that $\langle A, \rightarrow \rangle \notin$ RWH because $a \rightarrow 0 = a$ and $a \wedge (a \rightarrow 0) \nleq 0$. But, because $b \rightarrow a = b$ and $a \rightarrow 0 \nleq b \rightarrow (a \rightarrow 0)$, we have $\langle A, \rightarrow \rangle \notin$ TWH. It is possible to prove that $\hat{S}$, $\hat{\gamma}$, and G do not exist (for $\hat{S}$ and $\hat{\gamma}$, consider the element 0, and for G, consider the element c).

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