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On k-tree Containment Graphs of Paths in a Tree

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Abstract

A k-tree is either a complete graph on k vertices or a graph that contains a vertex whose neighborhood induces a complete graph on k vertices and whose removal results in a k-tree. If the comparability graph of a poset **P** is a k-tree, we say that **P** is a k-tree poset. In the present work, we study and characterize by forbidden subposets the k-tree posets that admit a containment model mapping vertices into paths of a tree (CPT k-tree posets). Furthermore, we characterize the dually-CPT and strong-CPT k-tree posets and their comparability graphs. The characterizations lead to efficient recognition algorithms for the respective classes.

Keywords Containment models · Comparability graphs · k-trees · CPT posets

1 Introduction and Definitions

Different classes of posets have been defined by imposing geometric restrictions to the sets used in their containment models [2, 4, 5, 8, 16]. Posets admitting a containment model using intervals of a line, which are called CI posets and are known to be the posets with dimension at most 2, are exactly the posets whose comparability graphs belong to the well understood class of permutation graphs [3]. In [1, 2], we have initiated the study of those posets that admit a containment model mapping vertices into paths of a tree, which are called CPT *posets* and clearly constitute a superclass of CI posets. We have found remarkable differences between CI and CPT posets. First, the dimension of CI posets is bounded above by 2, but the dimension of CPT posets is unbounded, this means that for every positive integer d there exists some CPT poset with dimension greater than d. Second, any

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CI poset admits a CI model using non trivial paths, however, there exist CPT posets that require the use of trivial paths in any CPT model. And third, the fact of being a CI poset is a comparability invariant, but the fact of being CPT is not. Figure 1 illustrates the previous observations, the poset $\mathbf{N}^d$ is CPT. In any CPT representation of $\mathbf{N}^d$ the vertex labelled b has to be represented by a trivial path. The dual poset $\mathbf{N}$ is non CPT [2]. Determining classes of posets in which being CPT is an invariant of comparability and understanding the structure behind it is a challenging problem. In Section 4, we solve this problem within a subclass of CPT posets.

As opposed to the previous differences, both classes are hereditary, meaning that any subposet of a CI (resp. CPT) poset is also CI (resp. CPT). Consequently, they admit a characterization by a family of minimal forbidden subposets. It is well known that the forbidden structures for being CI are the 3-irreducible posets, i.e. the minimal posets with dimension 3 [9, 14]. Do the 3-irreducible posets admit a CPT representation? In other words, can the obstacle that does not allow a 3-irreducible poset to have a containment model using paths of line be overcome by relaxing the host structure of the model? In Section 2, answering this question, we determine which of the 3-irreducible posets are CPT. In addition, using them, we describe an infinite family of posets which are minimal forbidden subposets for being CPT. The complete family of minimal forbidden subposets needed to characterize the class of CPT posets is unknown.

Finally, focusing in the algorithmic aspects of the recognition problems, CI poset can be recognized in linear time [15]. Determining the time complexity of recognizing CPT posets is an open problem.

Accordingly, it is natural to ask whether there are interesting classes of posets where these problems can be solved efficiently. In [2], we consider successfully two classes: the posets whose comparability graphs are split (the vertices can be partitioned into a clique and a stable set), and the posets whose comparability graphs are trees. Continuing with that work, in Section 3 of the present paper, we investigate (and solve) the characterization and the recognition problem in the class of posets whose comparability graphs are k-trees. In opposition to the small diameter, unbounded treewidth case of split graphs, we turn here to k-trees that are also chordal graphs, but with unbounded diameter and bounded treewidth [3]. In Section 5, we pose several open problems.

1.1 Definitions

A *partially ordered set* or *poset* is a pair $\mathbf{P} = (X, P)$ where X is a finite non-empty set and P is a reflexive, antisymmetric, and transitive binary relation on X. As usual, we write

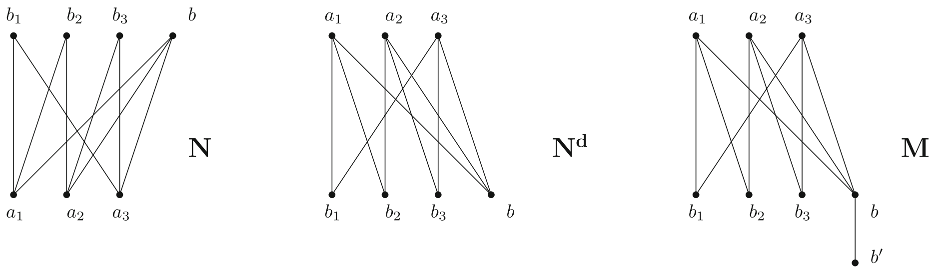

Fig. 1 Poset $\mathbf{N}$ is non CPT. Its dual $\mathbf{N}^d$ is CPT. In any CPT representation of $\mathbf{N}^d$, the vertex labelled b has to be represented by a trivial path. Thus, the poset $\mathbf{M}$ obtained from $\mathbf{N}^d$ by adding b' is non CPT

$x \le y$ in $\mathbf{P}$ for $(x, y) \in P$, and $x < y$ in $\mathbf{P}$ when $(x, y) \in P$ and $x \neq y$. If $x < y$ or $y < x$, we say that x and y are *comparable* in $\mathbf{P}$ and write $x \perp y$. An element x is *covered* by y in $\mathbf{P}$, denoted by $x <: y$ in $\mathbf{P}$, when $x < y$ and there is no element $z \in X$ for which $x < z$ and $z < y$. The *down-set* $\{x \in X : x < z\}$ and the *up-set* $\{x \in X : z < x\}$ of an element z are denoted by $D(z)$ and $U(z)$, respectively. We let $D[z] = D(z) \cup \{z\}$ and $U[z] = U(z) \cup \{z\}$. When $D(z) = \emptyset$, we say that z is a *minimal element* of $\mathbf{P}$, and that z is *maximal* when $U(z) = \emptyset$.

A *chain* in $\mathbf{P}$ is a subposet whose vertices are pairwise comparable. The *height* of $\mathbf{P}$ is the number of vertices in its maximum chain. The *restriction* of the relation P to a subset Y of X is denoted by $P(Y)$. We use $\mathbf{P}(Y)$ to refer to the subposet $(Y, P(Y))$ of $\mathbf{P}$. The *dual* of a poset $\mathbf{P} = (X, P)$ is the poset $\mathbf{P}^d = (X, P^d)$ where $x < y$ in $\mathbf{P}^d$ if and only if $y < x$ in $\mathbf{P}$.

A *containment model* $M_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ maps each element x of X into a set M_x in such a way that $x < y$ in $\mathbf{P}$ if and only if M_x is a proper subset of M_y. We identify the containment model $M_{\mathbf{P}}$ with the set family $(M_x)_{x \in X}$.

A poset $\mathbf{P} = (X, P)$ is a *containment order of paths in a tree*, or CPT poset for brevity, if it admits a containment model where every W_x is a path of a tree T, which is called the host tree of the model. When T is a path, $\mathbf{P}$ is said to be a *containment order of intervals* or CI poset for short.

The comparability graph $G_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ is the simple graph with vertex set $V(G_{\mathbf{P}}) = X$ and edge set $E(G_{\mathbf{P}}) = \{xy : x \perp y\}$. A graph G is a *comparability graph* if there exists some poset $\mathbf{P}$ such that $G = G_{\mathbf{P}}$. An undirected graph $G = (V, E)$ admits a *transitive orientation* $\overrightarrow{E}$ if $\overrightarrow{xy} \in \overrightarrow{E}$ and $\overrightarrow{yz} \in \overrightarrow{E}$, then $\overrightarrow{xz} \in \overrightarrow{E}$. The graphs whose edges can be transitively oriented are exactly the comparability graphs [7].

A set $M \subseteq V$ is a *module of a graph* $G = (V, E)$, (*homogeneous set* [6]) if and only if $N(x) - M = N(y) - M$ for all $x, y \in M$. The whole set V and the singleton sets $\{x\}$, for any $x \in V$, are modules of G. These modules are called *trivial modules*. A graph G is *prime* if all its modules are trivial. Otherwise G is *decomposable* or *degenerate*.

A set $M \subseteq X$ is a *module of a poset* $\mathbf{P} = (X, P)$ if M is a module of $G_{\mathbf{P}}$. So, for all x, $y \in M$ and $v \in X - M$, it is true that $x \perp v$ in $\mathbf{P}$ if and only if $y \perp v$ in $\mathbf{P}$. The trivial modules of $G_{\mathbf{P}}$, X and $\{x\}$ with $x \in X$, are also the trivial modules of $\mathbf{P}$. If all modules of a poset are trivial, we say that it is a *prime poset*. Otherwise, we say that it is *decomposable*.

If two posets are isomorphic, then their comparability graphs are also isomorphic. In general, the converse does not hold. We say that two posets are *associated* if their comparability graphs are isomorphic. In this paper, we do not distinguish between isomorphic posets (or graphs).

Theorem 1 ([6]) *Let* $\mathbf{P}$ *and* $\mathbf{P}'$ *be associated posets. Then the following statements hold.*

1. *If* $\mathbf{P}$ *is prime, then* $\mathbf{P}' = \mathbf{P}$ *or* $\mathbf{P}' = \mathbf{P}^d$.
2. *If* $\mathbf{S}$ *is a subposet of* $\mathbf{P}$ *and* $\mathbf{P}$ *is prime, then* $\mathbf{S}$ *or* $\mathbf{S}^d$ *is a subposet of* $\mathbf{P}'$.
3. *If* $\mathbf{S}$ *is a subposet of* $\mathbf{P}$ *and* $\mathbf{S}$ *is prime, then* $\mathbf{S}$ *or* $\mathbf{S}^d$ *is a subposet of* $\mathbf{P}'$.

2 Forbidden Structures for *CPT* Posets

The following necessary condition for being a CPT poset was stated in [2].

Lemma 2 *If* z *is a vertex of a* CPT *poset* $\mathbf{P}$*, then the subposet* $\mathbf{P}(D(z))$ *induced by the down-set of* z *is* CI.

The posets in Figs. 2 and 3 and their dual posets are said to be 3-irreducible. They form the family of forbidden induced subposets for being CI. This family was independently determined by Kelly [9], and Trotter and Moore [14]. A consequence of the previous lemma is that every 3-irreducible poset plus a least upper bound is a non CPT poset. This motivates the following definition.

Definition 3 Given a poset $\mathbf{P}$, we let $\widehat{\mathbf{P}}$ be the poset obtained by adding a maximum element to $\mathbf{P}$. We name $Top_{3-irred}$ the set $\{\widehat{\mathbf{P}} \mid$ with $\mathbf{P}$ any 3-irreducible poset $\}$.

Therefore, the $Top_{3-irred}$ posets are forbidden subposets for being CPT. A question that naturally arises is whether the $Top_{3-irred}$ posets are minimal forbidden subposets. Furthermore, are the 3-irreducible posets CPT? In a first inspection, we found without much difficulty that almost all 3-irreducible posets are CPT.

Lemma 4 *Every 3-irreducible poset except* $\mathbf{I_n}$ *with* $\boldsymbol{n} \geq 0$ *is* CPT.

Proof Let $\mathbf{P} = (X, P)$ be a 3-irreducible poset, see Figs. 2 and 3. First we consider the case when $\mathbf{P}$ has a maximal element x covering a unique element x'. Observe that in such case $\mathbf{P}$ is one of the following posets: $\mathbf{B}^d$, $\mathbf{C}$, $\mathbf{C}^d$, $\mathbf{CX_1}$, $\mathbf{CX_1^d}$, $\mathbf{CX_2}$, $\mathbf{EX_1}$, $\mathbf{EX_1^d}$, $\mathbf{EX_2} = \mathbf{EX_2^d}$, $\mathbf{FX_1^d}$, $\mathbf{E_n}$, $\mathbf{E_n^d}$, $\mathbf{F_n} = \mathbf{F_n^d}$ or $\mathbf{H_n} = \mathbf{H_n^d}$, with $n \geq 0$.

Since $\mathbf{P}$ is a minimal non CI poset, $\mathbf{P} - x$ admits a CI model M on a host path T. Let $q \in V(T)$ be an end vertex of $W_{x'}$. Let T' be the tree obtained from T by adding a pendant vertex q' adjacent to q. Also, let W_x be the path of T' obtained by adding the vertex q' to the path $W_{x'}$. Clearly, M plus the path W_x is a CPT model of $\mathbf{P}$ on the host tree T'.

When $\mathbf{P}$ is one of the posets $\mathbf{B}$ or $\mathbf{CX_2^d}$, a similar argument can be applied using a minimal vertex x covered by a unique vertex x'. Otherwise, if $\mathbf{P}$ is none of the posets cited above nor

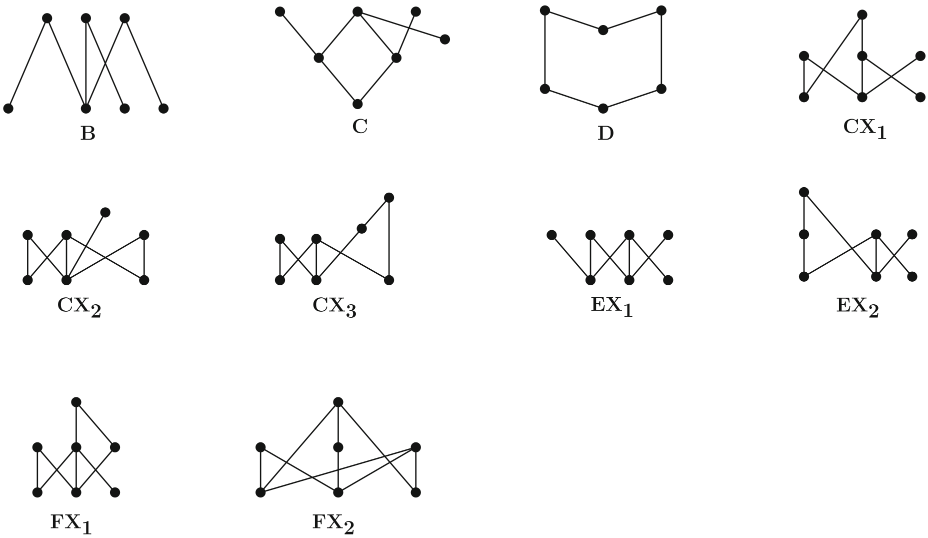

Fig. 2 These posets, the ones in Fig. 3, and their dual posets constitute the 3-irreducible family

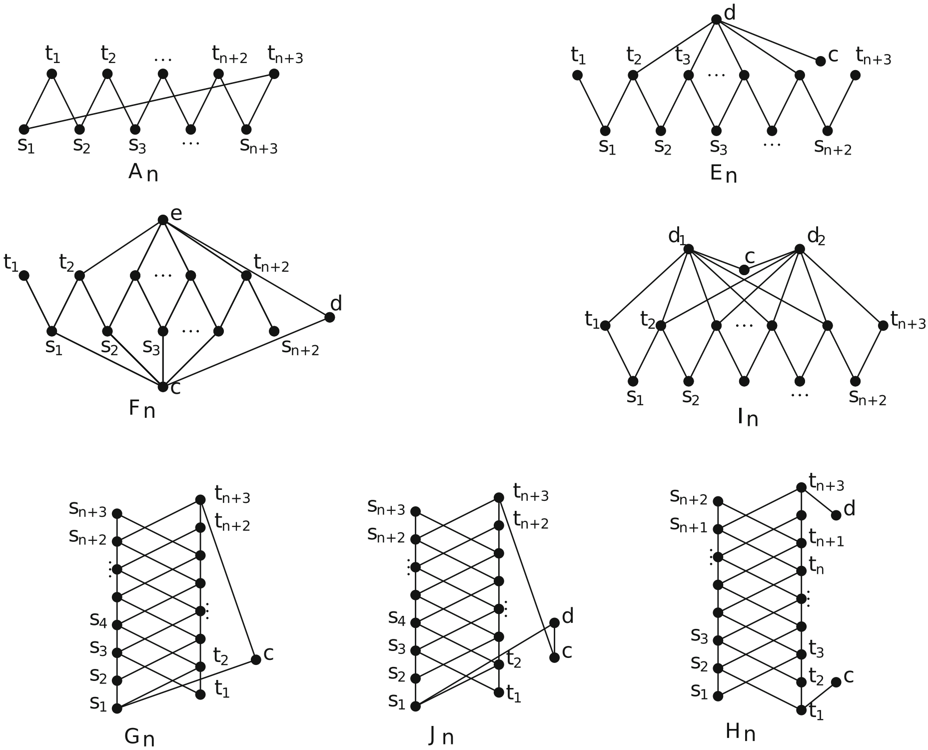

Fig. 3 These posets, the ones in Fig. 2, and their dual posets constitute the 3-irreducible family

$\mathbf{I}_n$ for $n \geq 0$, then $\mathbf{P}$ is one of the following posets: $\mathbf{D}$, $\mathbf{D}^d$, $\mathbf{CX_3}$, $\mathbf{CX_3^d}$, $\mathbf{FX_1}$, $\mathbf{FX_2} = \mathbf{FX_2^d}$, $\mathbf{A_n} = \mathbf{A_n^d}$, $\mathbf{I_n^d}$, $\mathbf{G_n} = \mathbf{G_n^d}$ or $\mathbf{J_n} = \mathbf{J_n^d}$, with $n \geq 0$. A CPT model of each one of these posets is shown in Figs. 4 and 5. □

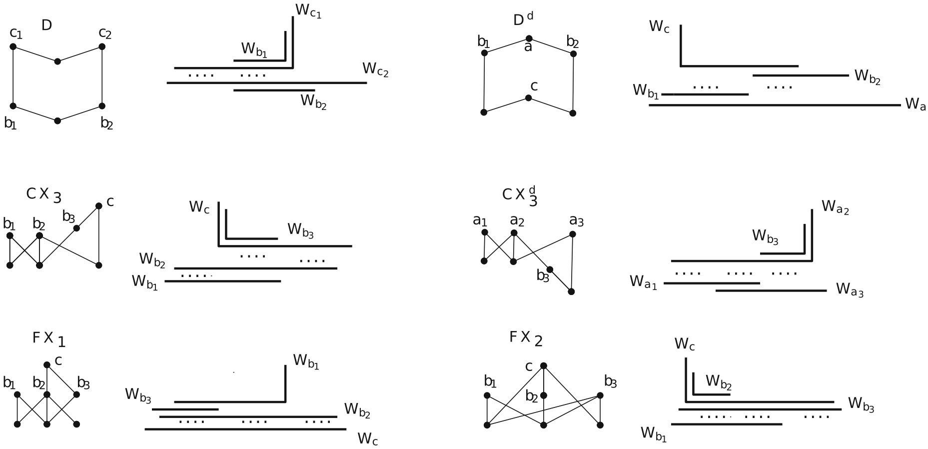

Fig. 4 CPT models of 3-irreducible posets $\mathbf{D}$, $\mathbf{D}^d$, $\mathbf{CX_3}$, $\mathbf{CX_3^d}$, $\mathbf{FX_1}$, $\mathbf{FX_2} = \mathbf{FX_2^d}$

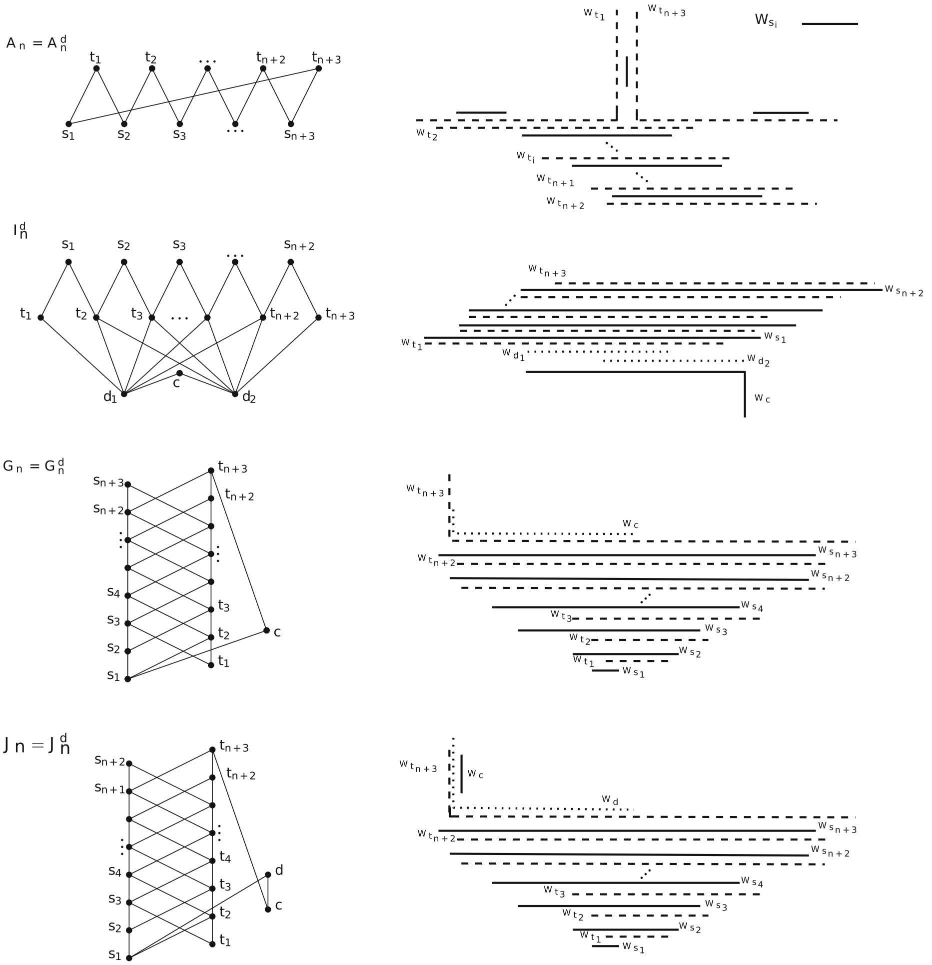

Fig. 5 CPT models of 3-irreducible posets $\mathbf{A_n} = \mathbf{A_n^d}$, $\mathbf{I_n^d}$, $\mathbf{G_n} = \mathbf{G_n^d}$ and $\mathbf{J_n} = \mathbf{J_n^d}$, with $n \geq 0$

Remark 5 Let $\mathbf{P}$ be a poset whose comparability graph is the chordless path $[v_1, v_2, \ldots, v_n]$, $n \geq 4$. Consider any CI model $(W_{v_i})_{1 \leq i \leq n}$ of $\mathbf{P}$, and let W_{v_i} be the interval $[l_i, r_i]$ for $1 \leq i \leq n$. Then

- $l_1 < l_3 < l_5 < \ldots,\ r_1 < r_3 < r_5 \ldots,\ l_2 < l_4 < l_6 < \ldots,$ and $r_2 < r_4 < r_6 < \ldots$; or

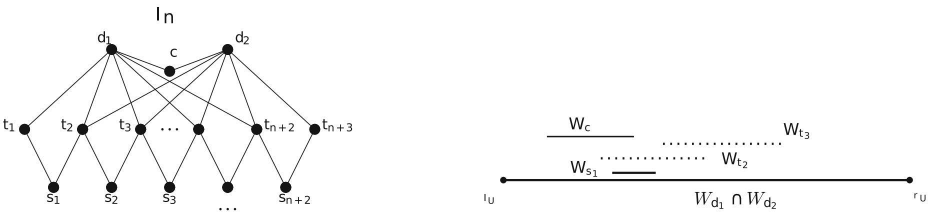

Fig. 6 Construction used in Lemma 6 to prove that the posets $\mathbf{I_n}$ are non CPT

- $l_1 > l_3 > l_5 \ldots$, $r_1 > r_3 > r_5 > \ldots$, $l_2 > l_4 > l_6 > \ldots$, and $r_2 > r_4 > r_6 > \ldots$.

In addition, when n is odd and $v_1 < v_2$, in the first case (second case,resp.), if I is any interval contained in $[l_2, r_{n-1}]$ ($[l_{n-1}, r_2]$, resp.), then I properly contains or is properly contained in some of the intervals W_{v_i} of the model.

Lemma 6 *For every $n \geq 0$, the 3-irreducible poset $\mathbf{I_n}$ in Fig. 3 is not CPT.*

Proof In order to derive a contradiction, suppose there exists a CPT model $(W_v)_{v \in V(\mathbf{I_n})}$ of $\mathbf{I_n}$. Label the vertices of $\mathbf{I_n}$ as in Fig. 6, and let A be the vertex set $D(d_1) \cap D(d_2)$. The subposet $\mathbf{I_n}(A)$ has a CI model contained in the path $U = W_{d_1} \cap W_{d_2}$. Let r_v and l_v be the right and left extremes, respectively, of the path W_v with $v \in A$. Without loss of generality, we assume that l_{t_2} is nearer to l_U than $l_{t_{n+2}}$ (see Fig. 6).

By Remark 5 and the fact that none of the paths W_v with $v \in A - \{c\}$ can either contain or be contained in W_c, we have that l_c is between l_U and l_{t_2}, or r_c is between $r_{t_{n+2}}$ and r_U. By symmetry, we can assume without loss of generality that l_c is between l_U and l_{t_2}. This implies that r_c is between l_c and r_{s_1}.

Moreover, the path W_{t_1} must be contained in W_{d_1}, contain W_{s_1} and not be contained in U, so l_U is an interior point of the path W_{t_1}. Thus, W_c turns out to be contained in W_{t_1}, contradicting the fact that c and t_1 are incomparable. □

Lemma 7 *The $Top_{3-irred}$ posets are forbidden subposets for being CPT. All of them except $\widehat{\mathbf{I_n}}$ with $n \geq 0$ are minimal.*

Proof The first statement is a direct consequence of Lemma 2. Let $\mathbf{P} = (X, P)$ be a 3-irreducible poset other than $\widehat{\mathbf{I_n}}$ and let u be the maximum of $\widehat{\mathbf{P}}$. By Lemma 4, $\widehat{\mathbf{P}} - u = \mathbf{P}$ is CPT. Now, let $x \in X$. Since $\mathbf{P}$ is 3-irreducible, then $\mathbf{P} - x$ is CI. Thus, $\mathbf{P} - x$ plus any maximum element is CI, which implies that $\widehat{\mathbf{P}} - x$ is CPT. □

In addition to the examples in the previous lemma, we have proved that the posets $\mathbf{I}_n$ for $n \geq 0$ and the posets $\mathbf{N}^d$ and $\mathbf{M}$ in Fig. 1 are forbidden subposets for the class CPT. While describing the complete list of forbidden subposets remains as an open problem, the next theorem shows that the family $Top_{3-irred}$ provides a complete characterization by forbidden subposets of CPT posets within any class in which the given necessary condition to be CPT is also sufficient. Later, we will use it in the particular case of k-tree posets.

Theorem 8 *Let $\mathcal{A}$ be any class of posets in which the necessary condition to be CPT given by Lemma 2 is also sufficient. A poset $\mathbf{P} \in \mathcal{A}$ is CPT if and only if $\mathbf{P}$ is $Top_{3-irred}$-free.*

Proof By Lemma 7, any CPT poset is $Top_{3-irred}$-free. Conversely, if $\mathbf{P}$ is $Top_{3-irred}$-free, then the down-set of each vertex is CI, i.e. $\mathbf{P}$ satisfies the necessary condition to be CPT. Since the necessary condition is also sufficient in $\mathcal{A}$, we have that $\mathbf{P}$ is CPT. □

Notice that if $\mathcal{A}$ is a hereditary class, then in Theorem 8, $Top_{3-irred}$ can be replaced by $Top_{3-irred} \cap \mathcal{A}$.

2.1 *CPT* Representations with no Trivial Paths

As we noted in the introduction, there exist posets that do not admit a CPT model unless some vertex is represented by a trivial path. Actually, the only minimal example of such a poset that we have found is the poset **N** in Fig. 1. Sometimes, in the process of modifying a CPT representation to justify the existence of some other particular representation, the possible existence of such vertices demands a special treatment. The following technical lemma will be used in the proof of Theorem 17.

Lemma 9 *Let* **P** *be a* CPT *poset. If* **P** *admits a* CPT *model in which no path is reduced to a vertex, then* **P** *admits a* CPT *model in which no path is reduced to a vertex and no two paths of the model have a common end vertex.*

Proof Let $M = (W_x)_{x \in X}$ be a CPT model of **P**, and assume that each W_x contains at least two vertices of the host tree T. We will proceed by induction on the number of coincident ends. If there are no common ends, there is nothing to be proved.

Let W_v be a shortest path between those sharing at least one of its end vertices with other path of the model. Let $q \in V(T)$ be one shared end vertex of W_v. Let $q_1, q_2, \ldots, q_k$ denote the neighbors of q in T. Since no path is reduced to a single vertex, we have that W_v must contain some other vertex q_i, say q_1. To obtain a new CPT model of **P**, we will consider two cases.

First assume no other path of the model is equal to W_x. Then, we proceed by subdividing the edge qq_1 of T with a vertex q_1' and doing the same subdivision in all the paths of M containing that edge. After that, replacing the path W_v with the path W_v' which is obtained by removing from W_v the extreme vertex q (in this case the new end of W_v' turns out to be q_1'; W_v' is not reduced into a vertex). The other paths of M are not modified.

Since in the new model, the only path that contains the vertex q_1' is W_v', it is clear that the number of paths with coincident ends has decreased. We claim that the *proper* containment relation between the paths has not been modified. Indeed, the only changes that could occur are that either W_v' is properly contained in a path in which W_v was not, which clearly is not possible; or that W_v' does not contain properly a path W_x that W_v did contain. This can not happen because in that case q has to be end vertex of W_x and W_x has to be shorter than W_v, contradicting its choice.

Now suppose that some paths of the model are exactly like W_v. By identifying all them, we can assume that W_u is the only such path. Let $p \in V(T)$ be the other end vertex of W_v. By subdividing edges if necessary we can assume that W_v contains at least two edges qq_1 and p_1p. Then, we proceed by subdividing the edges qq_1 and p_1p of T with new vertex q_1' and a nex vertex p_1', respectively, and doing the same subdivision in all the paths of M containing that edges. Thereafter, replacing the path W_v with the path W_v' which is obtained by removing from W_v the extreme vertex q (in this case the new end of W_v' turns out to be q_1' and W_v' it is not reduced into a vertex), and also replacing the path W_u with the path W_u' which is obtained by removing from W_u the extreme vertex p (in this case the new end of W_u' turns out to be p_1' and W_u' it is not reduced into a vertex). The other paths of M are not modified.

Clearly, the number of common end vertices between the paths of this new model is less than in the model M. Let us show that the *proper* containment relation did not change. This is clear between W_v and W_u. Then, by symmetry and because the paths W_v and W_u are the only ones that were modified, it is enough to see what happen between W_v' and W_x' for $x \neq v, u$. Since W_v and W_u were shortened to W_v' and W_u' respectively, we have that any path of the model that properly contains W_v (resp., W_u) also contains W'_v (resp., W'_u). On the other hand, if W_v properly contains a path W_x, by election of v, W_x and W_v do not share an end vertex, so W'_v contains W'_x, and the proof is complete. □

3 The Class of *k*-tree Posets

A graph G is a *k-tree* if it can be built recursively from a complete graph on k vertices by adding in each step a new vertex adjacent to exactly k neighbors which in turn induce a complete subgraph [12]. The 1-tree graphs are the trees, i.e. the connected graphs without cycles. The class of k-tree graphs has been widely studied [3]. In this paper, we introduce the naturally related notion of k-tree poset.

Definition 10 A poset $\mathbf{P}$ is a k-tree if its comparability graph $G_{\mathbf{P}}$ is a k-tree graph.

Just like k-tree graphs, k-tree posets can be defined recursively.

Construction process 11 *Given a poset $\mathbf{P}' = (X', P')$, a new poset $\mathbf{P}$ is obtained by adding a vertex z to X' through one of the following procedures.*

- *(u) Chose $a \in X'$ such that $D[a]$ induces a chain of size k and do $a <: z$; or*
- *(d) chose $a \in X'$ such that $U[a]$ induces a chain of size k and do $z <: a$; or*
- *(m) chose $a, c \in X'$ such that $D[c] \cup U[a]$ induces a chain of size k and do $c <: z <: a$.*

Clearly the given poset $\mathbf{P}'$ is a subposet of the new poset $\mathbf{P}$.

Lemma 12 *A poset $\mathbf{P}$ is a k-tree if and only if $\mathbf{P}$ can be obtained from a chain of k elements by repeatedly applying the construction process 11.*

Proof Clearly any poset built from a k-chain using the described procedure is a k-tree. Thus, let $\mathbf{P} = (X, P)$ be any k-tree poset and let us show by induction on $|X|$ that $\mathbf{P}$ can be obtained using the construction process 11.

If $|X| = k$, then $\mathbf{P}$ itself is a chain on k vertices; and if $|X| = k + 1$, then $\mathbf{P}$ is obtain using procedure (i). So let $|X| > k + 1$.

Since $G_{\mathbf{P}}$ is a k-tree, there is a vertex $z \in X$ such that $G' = G_{\mathbf{P}} - z$ is a k-tree, and $N(z) = \{z_1, z_2, \ldots, z_k\}$ is a clique. Therefore the k-tree poset $\mathbf{P}' = (X - \{z\}, P(X - \{z\}))$, can be obtained using the construction process 11; in what follows we will show that $\mathbf{P}$ can be obtained from $\mathbf{P}'$ in one more step. Since every maximal clique of a k-tree on at least $k + 1$ vertices has size $k + 1$, then there exists $u \in X - N[z]$ such that $N(z) \cup \{u\} = \{z_1, z_2, \ldots, z_k, u\}$ induces a maximal chain in $\mathbf{P}'$. We will consider three cases.

Vertex u is maximal in $\mathbf{P}'$: in such a case we can assume $z_1 < z_2 < \ldots < z_k < u$. Since $z \parallel u$ and $z_k \perp z$, we have $z_k < z$; which implies that $D[z_k]$ is exactly the k-chain $z_1 < z_2 < \ldots < z_k$. We conclude that z can be added by procedure (u) doing $z_k <: z$.

Vertex u is minimal in $\mathbf{P}'$: in such a case we can assume $u < z_1 < z_2 < \ldots < z_k$. Since $z \parallel u$ and $z \perp z_1$, we have that $z < z_1$; which implies that $U[z_1]$ is exactly the k-chain $z_1 < z_2 < \ldots < z_k$. Hence, z can be added by procedure (d) doing $z <: z_1$.

Otherwise, we can assume $z_1 < z_2 < \ldots < z_\ell < u < z_{\ell+1} < \ldots < z_k$ for some $1 \le \ell \le k$. Since $z \parallel u$, $z \perp z_\ell$ and $z \perp z_{\ell+1}$, we have that $z_\ell < z < z_{\ell+1}$, which implies that $D[z_\ell] \cup U[z_{\ell+1}]$ is exactly the k-chain $z_1 < z_2 < \ldots < z_\ell < z_{\ell+1} < \ldots < z_k$. Therefore, z can be added by procedure (m) doing $z_\ell <: z <: z_{\ell+1}$. □

Definition 13 The elements of a k-tree poset **P** can be totally ordered as $z_1, z_2, \ldots, z_n$ in accordance with the order in which they appear in a given construction process. The first k elements are the ones in the initial chain, so $z_1 < z_2 < \ldots < z_k$ in **P**. Such a total order of the vertices is called a **construction sequence** of **P**. The construction sequence is called **good** when $z_j < z_r$ in **P** for all j with $r < j \leq n$, where z_r is the last element in the sequence that is a maximal element of **P**.

The following remark is a direct consequence of the construction process itself and the fact that every maximal clique of a k-tree graph on at least $k+1$ vertices has size $k+1$.

Remark 14 Let $z_1, z_2, \ldots, z_k, \ldots, z_n$ be a construction sequence of **P**. Then,

(1) Since the first vertex z_{k+1} can always be added using procedure (u), we will assume $z_k < z_{k+1}$.

(2) An element z_j is maximal in **P** if and only if it was added by procedure (u).

(3) An element $z_j \neq z_1$ is minimal in **P** if and only if it was added by procedure (d).

(4) If z_j was added by procedure (d), then there is a unique $i < j$ such that z_i is maximal and $z_j < z_i$ in **P**.

(5) If z_j was added by procedure (m), then there is a unique $i < j$ such that z_i is maximal and $z_j < z_i$ in **P**.

Next, we will show that every k-tree poset admits a good construction sequence. An example is shown in Fig. 7.

Lemma 15 *Let* **P** *be a k-tree poset. There exists a construction sequence* $z_1, z_2, \ldots, z_n$ *of* **P** *such that if* $r = max\ \{i : z_i \text{ is maximal in } \mathbf{P}\}$*, then* $z_i \in D[z_r]$ *for all* $i \geq r$*.*

Proof We will proceed by induction on n, the number of elements of **P**. Let $z_1, z_2, \ldots, z_n$ be any construction sequence of **P**. If z_n is maximal, then the result is trivial. Hence, we can assume that $n \geq k+3$ and that z_n was added using procedure (m) or (d). In any case, z_n is comparable with exactly k elements which together with an element $u \| z_n$ induce a maximal $k+1$-chain C of **P**. Let w be the maximum of C, we have $u < w$ and $z_n < w$. Let $z'_1, z'_2, \ldots, z'_{n-1}$ be a construction sequence of $\mathbf{P}' = \mathbf{P} - \{z_n\}$ such that $z'_i \in D(z'_t)$ for all $i > t$ where z'_t is the last maximal element in this construction sequence. Keep in mind that the down and the up-set of every vertex is totally independent of the construction sequence. Clearly, if $z_n < z'_t$, then z_n can be added at the end of the construction process of $\mathbf{P}'$ and the proof follows.

Thus assume $z_n \notin D(z'_t)$ and let $I' \subseteq \{1, 2, \ldots, n-1\}$ such that the vertices z'_i with $i \in I'$ are the ones in the maximal chain C. We claim that $i < t$ for every $i \in I'$, what allows to add in the process of $\mathbf{P}'$ the vertex z_n immediately before z'_t, finishing the proof. Indeed, let $s \in I'$ such that $w = z'_s$. Since $z_n < w$ and $z_n \not< z'_t$, then $w \neq z'_t$ and so $s < t$. In addition, any vertex z'_i with $i \in I' - \{s\}$ was added using procedure (m) or (d). Thus, if $i > t$, then $s < t < i$ and the two maximal elements ($w = z'_s$ and z'_t) are greater than z'_i in contradiction with either (4) or (5) of Remark 14. We concluded that $i < t$ and the proof follows. □

The following lemma supplies the key to the proof of the main Theorem 17.

Lemma 16 *Let* z_r *be the last maximal element in a good construction sequence* $z_1, z_2, \ldots, z_n$ *of* **P**; $n \geq k+1$*. Then,*

(i) $\{z_1, z_2, \ldots, z_{r-1}\} \cap D(z_r)$ *is a chain* C *of size* k*; and*
(ii) *if* $s < r \leq j$ *and* $z_s \perp z_j$*, then* $z_s < z_j$ *and* $z_s \in C$.

Proof Since z_r is maximal, then it has label (u). Therefore, (i) and (ii) for $j = r$ hold trivially.

To prove (ii) for $r < j$, first recall that in a good order, $r < j$ implies $z_j < z_r$. Let $t \leq s$ be such that z_t is maximal and $z_s \leq z_t$. If $z_j < z_s$, then $z_j < z_t$, which implies that z_t and z_r are maximal elements greater than z_j in contradiction with either (4) or (5) of Remark 14. Thus $z_s < z_j$, and so $z_s \in D(z_r)$. By item (i), $z_s \in C$. □

Theorem 17 *A* k*-tree poset* $\mathbf{P}$ *is* CPT *if and only if* $\mathbf{P}(D[z])$ *is* CI *for each maximal element* z *of* $\mathbf{P}$*. Moreover, in such a case,* $\mathbf{P}$ *admits a* CPT *model with no path reduced to a single vertex and no two paths with a common end vertex.*

Proof The necessary condition follows from Lemma 2. So let $\mathbf{P} = (X, P)$ be a k-tree such that $D[z]$ induces a CI poset for any maximal element z. We will prove by induction on $|X|$ that $\mathbf{P}$ admits a CPT model such that no path is reduced to a vertex and no two paths have a common end vertex.

If $|X| = k$ or $k + 1$, then $\mathbf{P}$ is a chain and the proof is trivial. Accordingly, we assume $|X| \geq k + 2$.

By Lemma 15, there is a good construction sequence $z_1, z_2, \ldots, z_n$ of $\mathbf{P}$. Let z_r be the last maximal element in this order and let X' be the vertex set $\{z_1, z_2, \ldots, z_{r-1}\}$. The k-tree poset $\mathbf{P}' = \mathbf{P}(X')$ admits a CPT model $M' = (W'_x)_{x \in X'}$ such that no path is reduced to a single vertex and no two paths have a common end vertex. On the other hand, by hypothesis, $\mathbf{P}'' = \mathbf{P}(D[z_r])$ is CI. Thus $\mathbf{P}''$ admits a CI model $M'' = (W''_x)_{x \in D[z_r]}$ such that no path is reduced to a vertex and no two paths have coincident ends [4].

Next, we will obtain a CPT model of $\mathbf{P}$ using M' and M''. By Lemma 16 (i), $X' \cap D[z_r]$ is a chain C of size k. Therefore, the vertices of C are represented in M' and in M''. Notice that in both models the paths do not have coincident ends and no path is reduced to a vertex. Consequently, by subdividing edges where necessary, we can make sure that for each vertex x of the chain C, the paths W'_x and W''_x in M' and M'', respectively, are equal.

Let T' be the host tree of the model M'. Let T denote the tree obtained from T' by adding two new branches, one in each end vertex of W'_w, where w is the top element of the chain C. Let M be the model on the host tree T obtained from M' by adding (along the path defined by the two new branches) the model M'' in such a way that, for each vertex x in the chain C, the paths W'_x and W''_x are coincident and become a single path which will be denoted by W_x. See an example in Fig. 7.

Thus we have the path family $M = (W_x)_{x \in X}$ on the host tree T with

$$W_x = \begin{cases} W'_x, & \text{if } x \in X' - C; \\ W''_x, & \text{if } x \in D[z_r] - C; \\ W'_x = W''_x, & \text{if } x \in C. \end{cases}$$

Now we will prove that M is a CPT model of $\mathbf{P}$. Let x and y be any two vertices of $\mathbf{P}$. Clearly if one of the vertices belong to C, or both vertices belong to $X' - C$, or both vertices belong to $D[z_r] - C$, then $x < y$ in $\mathbf{P}$ if and only if $W_x \subset W_y$. So we just need to see what happens when one vertex is in $X' - C$ and the other is in $D[z_r] - C$, say $x = X' - C$ and $y \in D[z_r] - C$. Therefore, $x = z_s$ for some $s < r$ and $y = z_t$ for some $t \geq r$. By Lemma 16, $z_s \| z_r$ in $\mathbf{P}$.

Suppose $W_{z_s} \subset W_{z_t}$, then $W'_{z_s} \subset W''_{z_t}$, which implies $W'_{z_s} \subset W'_w$ (see Fig. 7); and so $s < w$ in $\mathbf{P}$, in contradiction with the fact that $z_s \notin C$.

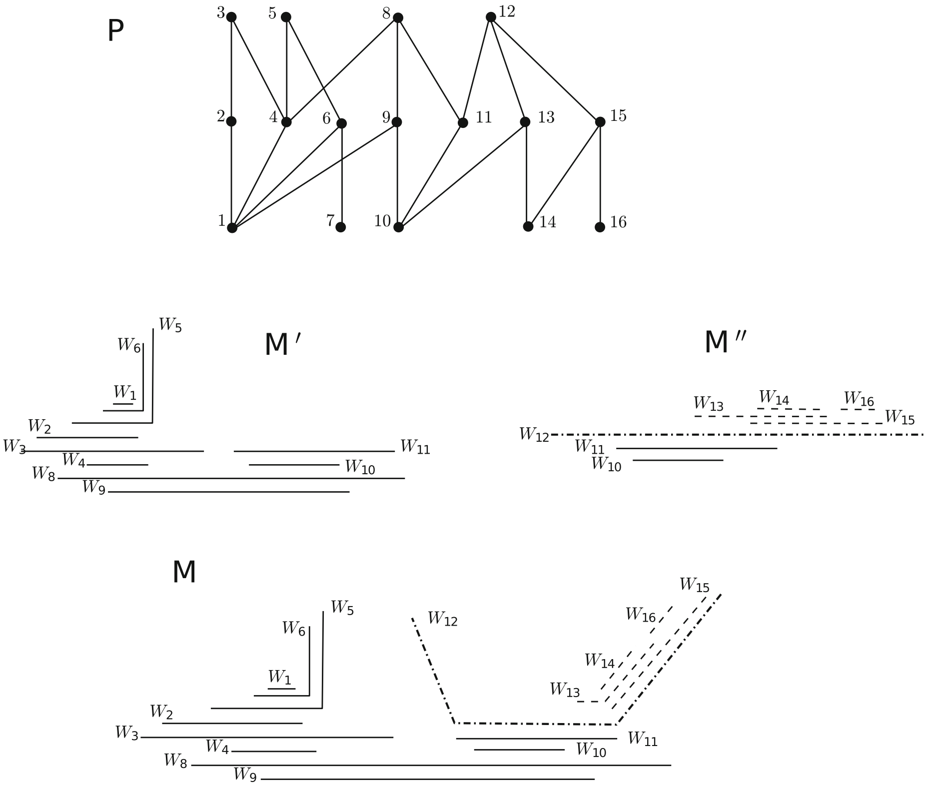

Fig. 7 A poset **P** with the vertices labelled in a good construction sequence (Definition 13). The CPT model M' for $\mathbf{P}[1, 2, \ldots, 11]$ and the CI model M'' for $\mathbf{P}[D[12]]$ are put together into the CPT model M for **P**, illustrating the proof of Theorem17

Now we will show that $W_{z_t} \not\subset W_{z_s}$. If $t = r$, then the proof is clear since W_{z_r} is contained in no path of the model M.

If $t > r$ and $W_{z_t} \subset W_{z_s}$, then $W''_{z_t} \subset W'_{z_s}$, which implies $W''_{z_t} \subset W''_w$ and so $z_t < w$ in **P** in contradiction with Lemma 16.

Clearly, no path of the model M is reduced to a single vertex. Therefore by the Lemma 9, **P** admits a CPT model such that no path is reduced to a vertex and no two paths have a common end vertex; which concludes the inductive proof. □

The following corollary, stated for emphasis, follows inmediately from Theorems 8 and 17.

Corollary 18 *A k-tree poset is CPT if and only it is $Top_{3-irred}$-free.*

Another significant consequence of the previous theorem is the existence of an efficient algorithm for recognizing CPT k-tree posets.

Corollary 19 *Determining whether a given poset is CPT and k-tree can be done in polynomial time.*

Proof First, determining whether the comparability graph of a given poset **P** is a k-tree can be solved in polynomial time [13]. In case of a positive answer, to determine if **P** is CPT, by Theorem 17, it is enough to see whether the down-set of each maximal element of **P** induces a dimension 2 poset. This can be done in linear time [15]. □

We finish this section with a theorem which improves Corollary 18 in the sense that it gives a characterization of CPT k-tree posets using a *minimal* family of forbidden subposets.

Theorem 20 *Let* **B**, **C**, $\mathbf{E_n}$ *and* $\mathbf{F_n}$ *be the posets in* Figs. 2 and 3.

i) *A 2-tree poset* **P** *is* CPT *if and only if* **P** *does not have* $\widehat{\mathbf{B}}$ *or* $\widehat{\mathbf{B}^d}$ *as subposet.*
ii) *A 3-tree poset* **P** *is* CPT *if and only if* **P** *does not have* $\widehat{\mathbf{B}}$, $\widehat{\mathbf{B}^d}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{C}^d}$, $\widehat{\mathbf{E_n}}$ *or* $\widehat{\mathbf{E_n^d}}$ *with* $\mathbf{n} \geq 0$ *as subposet.*
iii) *A k-tree poset* **P** *with* $k \geq 4$ *is* CPT *if and only if* **P** *does not have* $\widehat{\mathbf{B}}$, $\widehat{\mathbf{B}^d}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{C}^d}$, $\widehat{\mathbf{E_n}}$, $\widehat{\mathbf{E_n^d}}$ *or* $\widehat{\mathbf{F_n}}$ *with* $\mathbf{n} \geq 0$ *as subposet.*

Proof Let $\widehat{\mathbf{H}}$ be a $Top_{3-irred}$ subposet of a k-tree poset **P**. Since k-tree graphs are chordal, we have that k-tree posets are acyclic. Therefore, **H** is none of the following posets nor their respective dual posets: **D**, $\mathbf{CX_1}$, $\mathbf{CX_2}$, $\mathbf{CX_3}$, $\mathbf{EX_1}$, $\mathbf{EX_2}$, $\mathbf{FX_1}$, $\mathbf{FX_2}$, $\mathbf{A_n}$, $\mathbf{I_n}$, $\mathbf{G_n}$, $\mathbf{J_n}$, $\mathbf{H_n}$, with $\mathbf{n} \geq 0$.

If **P** is 2-tree, then $h(\mathbf{P}) = 3$, thus $h(\mathbf{H}) < 3$. It implies that **H** is either **B** or $\mathbf{B}^d$.

Analogously, if **P** is 3-tree, then $h(\mathbf{H}) < 4$ and the proof follows. □

4 Dually and Strong-*CPT* Subclasses

A poset **P** is *dually-CPT* if it is CPT and $\mathbf{P}^d$ is also CPT. If **P** is CPT and every poset associated with **P** is also CPT, then **P** is *strong-CPT*. The *dually-CPT graphs* and *strong-CPT graphs* are the comparability graphs of dually-CPT posets and strong-CPT posets, respectively. It is clear that strong-CPT class is contained in dually-CPT class. However, it is not known whether the two classes are distinct [2].

Lemma 21 *Let* **P** *and* $\mathbf{P}'$ *be associated posets. If* **H** *is a prime poset and* $\widehat{\mathbf{H}}$ *is subposet of* **P**, *then* $\widehat{\mathbf{H}}$, $\widehat{\mathbf{H}}^d$, $\widehat{\mathbf{H}^d}$ *or* $\widehat{\mathbf{H}^d}^d$ *is subposet of* $\mathbf{P}'$.

Proof By item 3 of Theorem 1, either **H** or $\mathbf{H}^d$ is subposet of $\mathbf{P}'$. In the former case, since $\widehat{\mathbf{H}}$ is subposet of **P**, we have that in $\mathbf{P}'$ there is a vertex comparable with every vertex of **H**, say u. Since H is prime, then either u is greater than every vertex of **H** or u is smaller that every vertex of **H**, which means that $\widehat{\mathbf{H}}$ or $\widehat{\mathbf{H}^d}^d$ is subposet of $\mathbf{P}'$. In the latter case, in an analogous way, we have that $\widehat{\mathbf{H}^d}$ or $\widehat{\mathbf{H}}^d$ is subposet of $\mathbf{P}'$. □

Theorem 22 *Let* $\mathbf{P} = (X, P)$ *be a k-tree poset with* $k \geq 2$. *The following statements are equivalent.*

i) **P** *is strong-CPT.*
ii) **P** *is dually-CPT.*
iii) **P** *does not contain* $\widehat{\mathbf{B}}$, $\widehat{\mathbf{B}^d}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{C}^d}$, $\widehat{\mathbf{E_n}}$, $\widehat{\mathbf{E_n^d}}$, $\widehat{\mathbf{F_n}}$, *with* $\mathbf{n} \geq 0$, *or their respective dual posets as subposet.*

For $k = 2, 3$ the given family of forbidden subposets is not minimal. For $k = 2$, the list of posets in $iii)$ can be reduced to only $\widehat{\mathbf{B}}$ and $\widehat{\mathbf{B}^d}$. For $k = 3$, $\widehat{\mathbf{F_n}}$, with $\mathbf{n} \geq 0$, can be removed from the list.

Proof Clearly, any strong-CPT poset is dually-CPT. If $\mathbf{P}$ is dually-CPT, then, by definition of dually-CPT and Theorem 20, none of the posets $\widehat{\mathbf{B}}$, $\widehat{\mathbf{B}^d}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{C}^d}$, $\widehat{\mathbf{E}_n}$, $\widehat{\mathbf{E}_n^d}$, $\widehat{\mathbf{F}_n}$, with $\mathbf{n} \geq 0$, nor their respective dual posets can be subposets of $\mathbf{P}$. So we have proved $ii)$ implies $iii)$.

Let $\mathbf{P}$ be a k-tree poset satisfying $iii)$. By Theorem 20, $\mathbf{P}$ is CPT. Assume, to derive a contradiction, that $\mathbf{P}$ is non strong-CPT, thus there exist a poset $\mathbf{P}'$ associated with $\mathbf{P}$ which is non CPT. Notice that $\mathbf{P}'$ is a k-tree, then, by Theorem 20, $\mathbf{P}'$ contains $\widehat{\mathbf{B}}$, $\widehat{\mathbf{B}^d}$, $\widehat{\mathbf{C}}$, $\widehat{\mathbf{C}^d}$, $\widehat{\mathbf{E_n}}$, $\widehat{\mathbf{E_n^d}}$ or $\widehat{\mathbf{F_n}}$, with $\mathbf{n} \geq 0$, as subposet. Since every 3-irreducible poset is prime [9], the proof follows by Lemma 21. □

Theorem 23 *Let G be a comparability k-tree graph. The following conditions are equivalents.*

i) *G is strong-CPT.*
ii) *G is dually-CPT.*
iii) *G does not contain neither $G_{\widehat{\mathbf{B}}}$, $G_{\widehat{\mathbf{C}}}$, $G_{\widehat{\mathbf{E}_n}}$ nor $G_{\widehat{\mathbf{F_n}}}$ with $\mathbf{n} \geq 0$ as induced subgraph. For $k = 2, 3$ the given family of forbidden subgraphs is not minimal. For $k = 2$, the list of graphs in $iii)$ can be reduced to $G_{\widehat{\mathbf{B}}}$ and $G_{\widehat{\mathbf{B}^d}}$. For $k = 3$, $G_{\widehat{\mathbf{F_n}}}$, with $\mathbf{n} \geq 0$, can be removed from the list.*

Proof By Theorem 22, $i)$ and $ii)$ are equivalent, and $iii)$ implies $i)$.

Let $G = G_{\mathbf{P}}$ with $\mathbf{P}$ strong-CPT. Assume, to derive a contradiction, that $G_{\widehat{\mathbf{H}}}$ is an induced subgraph of G, where $\mathbf{H}$ is any one of the posets $\mathbf{B}$, $\mathbf{C}$, $\mathbf{E_n}$ or $\mathbf{F_n}$. Then $\mathbf{P}$ contain a subposet associated with $\widehat{\mathbf{H}}$. Since $\mathbf{H}$ is prime the only posets associated with $\widehat{\mathbf{H}}$ are $\widehat{\mathbf{H}}$, $\widehat{\mathbf{H}}^d$, $\widehat{\mathbf{H}^d}$ and $\widehat{\mathbf{H}^d}^d$. In any case, since $\mathbf{P}$ is strong-CPT, we have a contradiction with Theorem 22. We have proved that $i)$ implies $iii)$. □

5 Open Problems

Several problems related to CPT representations of posets remain open. The time complexity of the recognition problem of CPT posets is unknown.

Although a few minimal non CPT posets have been depicted, the whole family that would allow a total characterization of CPT posets by forbidden subposets has not been fully described. We have solved these issues restricted to the class of split posets and to the class of k-tree posets. However, we have not obtained a characterization by forbidden subgraphs of CPT split graphs or CPT k-tree graphs. An extra complexity in characterizing CPT graphs by forbidden structures is the fact that being CPT is not a comparability invariant. This means that a same graph may be the comparability graph of a poset that is CPT and of another poset which is non CPT.

In the process of reviewing the present paper, one of the anonymous referees provided a simple proof of the following lemma.

Lemma Let x and y be distinct maximal elements in a poset $\mathbf{P}$ with $D_{\mathbf{P}}(x) \subsetneq D_{\mathbf{P}}(y)$. Suppose $D_{\mathbf{P}}(x) \nsubseteq D_{\mathbf{P}}(z)$ for every element $z \neq x, y$. Let $\mathbf{P}'$ be the poset obtained from $\mathbf{P}$ by removing y and adding a new maximal element y' so that $D_{\mathbf{P}'}(y') = \{x\} \cup D_{\mathbf{P}}(y)$. If $\mathbf{P}$ is a CPT poset, then $\mathbf{P}'$ is also a CPT poset.

Let $\mathbf{P}$ be the poset obtained from the 3-irreducible poset $\mathbf{A_n}$ by adding a new maximal element y whose down-set contains all the minimal elements of $\mathbf{A_n}$ and any arbitrary subset of the maximal elements of $\mathbf{A_n}$. By applying the previous lemma as many times as necessary, the non CPT poset $\widehat{\mathbf{A_n}}$ is obtained. Therefore, $\mathbf{P}$ is non CPT. Even more, $\mathbf{P}$ is minimal non CPT. Notice that this implies that there are exponentially many different minimal forbidden subposets for the class CPT that have the same structure as the poset $\mathbf{N}$.

In addition, we have observed that new forbidden subposets can be obtained by applying the previous reasoning to some other 3-irreducible posets. However, for instance, the minimal non CPT poset $\mathbf{M}$ in Fig. 1 cannot be obtained from a 3-irreducible poset by applying the previous lemma. We conjecture there are other forbidden subposet that are obtained by adding an element below some other element which need to be represented by a trivial path. For short, call *singular* the CPT posets that require the use of some trivial path in any of its CPT representation. A challenging open problem is determining the minimal singular CPT posets.

It is not known whether the classes of strong-CPT posets and dually-CPT posets are distinct. We have shown they are equal when intersected with the split class or with the k-tree class. Also, we have been able to demonstrate that a non singular CPT poset is strong if and only if is dually (manuscript in preparation).

In [11] it is proved that any poset whose comparability graph is chordal has dimension at most 4. The fact that the inequality is tight was proved by Kierstead and Trotter in [10]. It is not known whether there are 4-dimensional k-tree posets.

Finally, in [2], we proved that for every positive integer d, there exists a CPT poset with dimension d. As opposed, we also showed that the dimension of a CPT poset is bounded above by the number of leaves of the host tree used in any of its CPT representations. It is not known whether dimension is bounded for either the class of dually-CPT posets or the class of strong-CPT posets.

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References

1. Alcón, L., Gudiño, N., Gutierrez, M.: On containment graphs of paths in a tree. Electron. Notes Discret. Math. **50**, 175–180 (2015)
2. Alcón, L., Gudiño, N., Gutierrez, M.: Recent results on containment graphs of paths in a tree. Discret. Appl. Math. **245**, 139–147 (2018)
3. Brandstädt, A., Le, V., Spinrad, J.: Graph Classes: a Survey. Society for Industrial and Applied Mathematics, Philadelphia (1999)
4. Dushnik, B., Miller, E.: Partially ordered sets. Am. J. Math. **63**, 600–610 (1941)
5. Fishburn, P.C., Trotter, W.T.: Geometric containment orders: a survey. Order **15**, 167–182 (1999)
6. Gallai, T.: Transitiv orientierbare graphen. Acta Mathematica Hungarica **18**, 25–66 (1967)
7. Ghouila-Houri, A.: Caractérization des graphes non orientés dont on peut orienter les arrtes de maniére á obtenir le graphe d'une relation d'ordre. C.R. Acad. Sci. Paris **254**, 1370–1371 (1962)

8. Golumbic, M.C., Scheinerman, E., Graphs, C.ontainment.: Posets, and related classes of graphs. Ann. N. Y. Acad. Sci **555**, 192–204 (1989)
9. Kelly, D.: The 3-irreducible partially ordered sets. Can. J. Math. **29**, 367–383 (1977)
10. Kierstead, H.A., Trotter, W.T.: The dimension of cycle-free orders. Order **9**, 103–110 (1992)
11. Ma, T., Spinrad, J.P.: Cycle-free partial orders and chordal comparability graphs. Order **8**, 49–61 (1991)
12. Rose, D.J.: On simple characterizations of k-trees. Discret. Math. **7**, 317–322 (1974)
13. Rose, D.J., Tarjan, R.E., Lueker, G.: Algorithmic aspects of vertex elimination on graphs. SIAM J. Comput. **5**, 266–283 (1976)
14. Trotter, W.T., Moore, J.I.: Characterization problems for graphs, partially ordered sets, lattices and families of sets. Discret. Math. **16**, 361–381 (1976)
15. Spinrad, J.P.: On comparability and permutation graphs. SIAM J. Comput. **14**, 658–670 (1985)
16. Urrutia, J.: Partial orders and euclidean geometry. In: Rival, I. (ed.) Algorithms and Order, pp. 327–346. Kluwer Academic, Dordrecht (1989)