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Operator Domains and SUSY Breaking in a Model of SUSYQM with a Singular Potential

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Abstract. The self-adjoint extension of the symmetric supercharges and Hamiltonian of a model of Supersymmetric Quantum Mechanics on the half-line, for the case of a singular superpotential, is analyzed. The compatibility of the domains of definition of the different operators and the possibility of effectively implement the graded superalgebra in a dense subspace of the Hilbert space is considered. As a consequence, conditions for SUSY breaking in this model are established.

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1. Introduction

Supersymmetry (SUSY) [1, 2, 3, 4, 5, 6, 8, 9, 10, 11] gives desirable features to quantum field theories, like an improved ultraviolet behavior, but also predicts superpartner states with degenerate mass which are not observed experimentally. Therefore, this symmetry is expected to be *spontaneously broken*.

Let us mention that a symmetry of the Hamiltonian is said to be spontaneously broken if the ground state does not exhibit this symmetry. For a continuous symmetry, this occurs when the ground state is not annihilated by the generators of these transformations.

Several schemes have been developed to try to solve the SUSY breaking problem, including the idea of non-perturbative breaking by instantons. In this context, the simplest model is the *Supersymmetric Quantum Mechanics* (SUSYQM), introduced by Witten [8] and Cooper and Freedman [10]. This is a toy model of

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a one-dimensional quantum-mechanical system whose *formal* Hamiltonian can be written as

$$H = \{\mathcal{Q}, \widetilde{\mathcal{Q}}\}_+ = \mathcal{Q}\widetilde{\mathcal{Q}} + \widetilde{\mathcal{Q}}\mathcal{Q}\,, \tag{1.1}$$

where the supercharges

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\mathcal{Q}} = \begin{pmatrix} 0 & \widetilde{A} \\ 0 & 0 \end{pmatrix} \tag{1.2}$$

are nilpotent operators,

$$\mathcal{Q}^2 = \widetilde{\mathcal{Q}}^2 = 0\,, \tag{1.3}$$

which commute with the Hamiltonian,

$$[H, \mathcal{Q}] = 0 = \left[H, \widetilde{\mathcal{Q}}\right]\,. \tag{1.4}$$

These conserved supercharges are the generators of the SUSY transformations.

Here, A and $\widetilde{A}$ are differential operators defined on a suitable dense subspace of functions where the necessary operator compositions in these equations are well defined,

$$A = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + W(x)\right), \quad \widetilde{A} = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + W(x)\right), \tag{1.5}$$

where $W(x)$ is called the superpotential.

Notice that, for A closed and $\widetilde{A} = A^\dagger$ (the adjoint of A), the Hamiltonian is non-negative,

$$\langle\phi|H|\phi\rangle = \left\|\mathcal{Q}\,|\phi\rangle\right\|^2 + \left\|\widetilde{\mathcal{Q}}\,|\phi\rangle\right\|^2 \geq 0\,, \tag{1.6}$$

and then the ground state $|\phi_0\rangle$ is invariant under SUSY transformations if and only if it has a vanishing energy eigenvalue

$$\mathcal{Q}\,|\phi_0\rangle = 0 = \widetilde{\mathcal{Q}}\,|\phi_0\rangle \quad \Leftrightarrow \quad \langle\phi_0|H|\phi_0\rangle = 0\,. \tag{1.7}$$

When considering these models, several authors have suggested that singular potentials could break SUSY through nonstandard mechanisms, leading to non-degenerate energy levels and even to negative energy eigenstates [12, 13, 14, 15, 16, 17].

In particular, Jevicki and Rodrigues [12] have considered the singular superpotential $W(x) = g/x - x$, with real g. Based on the square integrable solutions of the differential operator related to the Hamiltonian of this system, previously obtained by Lathouwers [18], they concluded that, for a certain range of the parameter g, SUSY is broken with a negative energy ground state. However, they have not established that all the functions they considered correspond to eigenvectors of the same self-adjoint Hamiltonian.

Later, Das and Pernice [19] have reconsidered this problem in the framework of a SUSY preserving regularization of the singular superpotential, finding that SUSY is recovered exactly at the end, when the regularization is removed. They conclude that SUSY is robust at short distances (high energies), and the singularities that occur in quantum mechanical models are unlike to break SUSY.

We addressed this subject [20] by studying the *self-adjoint extensions* (SAE) [21] of the Hamiltonian defined by the superpotential $W(x) = g/x - x$ in the half-line $\mathbf{R}^+$. In so doing, we have considered the SAE of the symmetric supercharges and the possibility of effectively implement the algebra of SUSY in a dense subspace of the Hilbert space.

We have shown that there is a range of values of g for which the supercharges admit a one-parameter family of SAE, corresponding to a one-parameter family of SAE of the Hamiltonian.

We found that only for two particular SAE, whose domains are scale invariant, the algebra of $N = 2$ SUSY can be realized, one with exact SUSY and the other with spontaneously broken SUSY. For other values of this continuous parameter, only the $N = 1$ SUSY algebra is obtained, with spontaneously broken SUSY and non degenerate energy spectrum.

We should mention that SAE of supercharges and Hamiltonian for the SUSYQM of the free particle with a point singularity in the line and the circle have been considered in [22, 23, 24, 25], where $N = 1, 2$ realization of SUSY are described. They have also been considered in the framework of the Landau Hamiltonian for two-dimensional particles in nontrivial topologies in [26] (see also [27]).

Moreover, a *hidden* supersymmetric structure similar to that described in the following also appears when considering the quantum-mechanical behavior of particles in a plane in the presence of a singular Aharonov-Bohm magnetic flux [28, 29].

2. The model and its supercharges

We consider a quantum mechanical system living in the half-line $\mathbf{R}^+$ and subject to a superpotential given by

$$W(x) = \frac{g}{x} - x \tag{2.1}$$

for $x > 0$ and real g. Then, the two differential operators in the expression of the supercharges take the form

$$\begin{aligned} A &= \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + \frac{g}{x} - x\right), \\ \widetilde{A} &= \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + \frac{g}{x} - x\right). \end{aligned} \tag{2.2}$$

Let us now introduce an operator $Q_+ = \widetilde{\mathcal{Q}} + \mathcal{Q}$, defined on the dense subspace $\mathcal{D}(Q_+) = \mathcal{C}_0^\infty(\mathbf{R}^+ \backslash \{\mathbf{0}\})$, over which its action is given by

$$Q_+\Psi = \begin{pmatrix} 0 & \widetilde{A} \\ A & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{2.3}$$

Its square, which is well defined within this domain, satisfies

$$Q_+{}^2 = \{\mathcal{Q}, \widetilde{\mathcal{Q}}\}_+ = H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{2.4}$$

where H is the Hamiltonian of the system, and $H_+ = \widetilde{A}A$ and $H_- = A\widetilde{A}$ are the partner Hamiltonians. It can be easily verified that Q_+ is a symmetric operator, but it is neither self-adjoint nor even closed.

Remark. Given a SAE of Q_+, its *square* gives a SAE of the Hamiltonian H, by virtue of a theorem due to von Neumann [30].

The first step in getting the SAE of Q_+ consists in the construction of its adjoint, $Q_+^\dagger$, and the determination of the *deficiency subspaces*

$$\mathcal{K}_\pm := \mathrm{Ker}(Q_+^\dagger \mp i).$$

Notice that, within the same domain $\mathcal{C}_0^\infty(\mathbf{R}^+\backslash\{\mathbf{0}\})$, a linearly independent combination of supercharges leads to the operator

$$Q_- = i(\widetilde{\mathcal{Q}} - \mathcal{Q}) = \imath Q_+ \sigma_3\,, \tag{2.5}$$

which is also symmetric, anticommutes with Q_+, $\{Q_+, Q_-\}_+ = 0$, and satisfies $Q_-{}^2 = H$. Here, the Pauli matrix $\sigma_3 = \mathrm{diag}\,(1, -1)$ is the *grading operator*, which distinguishes *fermionic* from *bosonic* states.

Since Q_- can also be obtained from Q_+ through a unitary transformation given by

$$Q_- = e^{i\sigma_3\pi/4} Q_+ e^{-i\sigma_3\pi/4}\,, \tag{2.6}$$

the following analysis will be carried out only for Q_+, and it will extend immediately to Q_-.

The Domain of $Q_+^\dagger$

It can be seen that the domain of $Q_+^\dagger$ is

$$\mathcal{D}(Q_+^\dagger) = \{\Phi \in AC(\mathbf{R}^+\backslash\{0\}) \cap \mathbf{L_2}(\mathbf{R}^+) : A\,\phi_1,\, \widetilde{A}\,\phi_2 \in \mathbf{L_2}(\mathbf{R}^+)\}\,, \tag{2.7}$$

where the action of $Q_+^\dagger$ on $\Phi \in \mathcal{D}(Q_+^\dagger)$ reduces also to the application of the differential operator

$$Q_+^\dagger \Phi = \begin{pmatrix} 0 & \widetilde{A} \\ A & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2.8}$$

The Spectrum of $Q_+^\dagger$

The eigenvalue problem for the adjoint, $Q_+^\dagger \Phi_\lambda = \lambda \Phi_\lambda$, reduces to the system of ordinary differential equations

$$A\,\phi_{\lambda,1} = \lambda\,\phi_{\lambda,2}, \quad \widetilde{A}\,\phi_{\lambda,2} = \lambda\,\phi_{\lambda,1}\,, \tag{2.9}$$

with $\Phi_\lambda \in \mathcal{D}(Q_+^\dagger)$ and $\lambda \in \mathbf{C}$.

The substitution $\phi_{\lambda,1}(x) = x^g\, e^{-x^2/2}\, F(x^2)$ leads to the Kummer's (Confluent Hypergeometric) equation [31] for $F(z)$,

$$z\, F''(z) + (b - z)\, F'(z) - a\, F(z) = 0\,, \tag{2.10}$$

with $a = -\frac{\lambda^2}{2}$ and $b = g + \frac{1}{2}$.

This equation has two linearly independent solutions given by the Kummer's function

$$y_1(z) = U(a,b,z) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\, \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right\}, \tag{2.11}$$

and

$$y_2(z) = e^z\, U(b-a, b, -z)\,, \tag{2.12}$$

where $M(a,b,z) = {}_1F_1(a;b;z)$.

Since, for large values of the argument,

$$U(a,b,z) = z^{-a} \left\{ 1 + \mathcal{O}(|z|^{-1}) \right\}, \tag{2.13}$$

only $y_1(x^2)$ leads to a function $\phi_{\lambda,1}(x) \in \mathbf{L_2}(1,\infty)$, while $y_2(x^2)$ should be discarded.

Therefore, for any $\lambda \in \mathbf{C}$ we get

$$\phi_{\lambda,1}(x) = x^g\, e^{-x^2/2}\, U\left(-\frac{\lambda^2}{2}, g + \frac{1}{2}, x^2 \right) \tag{2.14}$$

and

$$\phi_{\lambda,2}(x) = -\frac{\lambda}{\sqrt{2}}\, x^{g+1}\, e^{-x^2/2}\, U\left(1 - \frac{\lambda^2}{2}, g + \frac{3}{2}, x^2 \right), \tag{2.15}$$

which is also in $\mathbf{L_2}(1,\infty)$.

In order to determine the spectrum of $Q_+^\dagger$, we must now consider the behavior of $\Phi_\lambda(x)$ near the origin. From the small argument expansion of Kummer's functions [31] one can straightforwardly show that three cases should be distinguished, according to the values of the coupling g.

- If $g \geq 1/2$, it can be seen that $\Phi_\lambda(x) \notin \mathbf{L_2}(0,1)$ unless $\lambda^2 = 2n$, with $n = 0,1,2,\ldots$. In this case, the Kummer's function reduces to a Laguerre polynomial, $U(-n,b,z) = (-1)^n\, n!\, L_n^{(b-1)}(z)$ (of degree n in z), and we have $\phi_{\lambda,1}(x) \sim x^g$ and $\phi_{\lambda,2}(x) \sim x^{g+1}$ for $0 < x \ll 1$. Therefore, in this region $Q_+^\dagger$ has a discrete real spectrum, symmetric with respect to the origin, given by the (degeneracy one) eigenvalues

$$\lambda_0 = 0\,, \quad \lambda_{\pm,n} = \pm\sqrt{2n}\,, \quad n = 1,2,3,\ldots, \tag{2.16}$$

corresponding to the eigenfunctions $\Phi_0 = x^g\, e^{-x^2/2} \begin{pmatrix} 1 & 0 \end{pmatrix}^t$ and

$$\Phi_{\pm,n} = (-1)^n\, n!\, x^g\, e^{-x^2/2} \begin{pmatrix} L_n^{(g-\frac{1}{2})}(x^2) \\ \mp\frac{x}{\sqrt{n}}\, L_{n-1}^{(g+\frac{1}{2})}(x^2) \end{pmatrix}. \tag{2.17}$$

- For $g \leq -1/2$, it can be seen that $\Phi_\lambda(x) \notin \mathbf{L_2}(0,1)$ unless $\lambda^2 = 2(n - g + \frac{1}{2})$, with $n = 0, 1, 2, \ldots$. In this case, we have $\phi_{\lambda,1}(x) \sim x^{1-g}$ and $\phi_{\lambda,2}(x) \sim x^{-g}$ for $0 < x \ll 1$. Therefore, in this region $Q_+^\dagger$ has a discrete real spectrum, symmetric with respect to the origin, given by the (degeneracy one) eigenvalues
$$\lambda_{\pm,n} = \pm\sqrt{2n + 1 - 2g}\,, \quad n = 0, 1, 2, \ldots \tag{2.18}$$
corresponding to the eigenfunctions
$$\Phi_{\pm,n} = (-1)^n\, n! x^{-g}\, e^{-x^2/2} \begin{pmatrix} x\, L_n^{(\frac{1}{2}-g)}(x^2) \\ \mp\sqrt{n + \frac{1}{2} - g}\, L_n^{(-g-\frac{1}{2})}(x^2) \end{pmatrix}. \tag{2.19}$$
Notice that no eigenvalue vanishes for these values of the coupling.
- For $-1/2 < g < 1/2$, it can be seen that $\Phi_\lambda(x) \in \mathbf{L_2}(0,1)$, $\forall \lambda \in \mathbf{C}$. This means that, for these values of g, every complex number is an eigenvalue of $Q_+^\dagger$ with degeneracy one. In particular, the eigenfunction of $Q_+^\dagger$ corresponding to $\lambda = +i$ is given by
$$\Phi_{+\imath}(x) = \begin{pmatrix} \phi_{+\imath,1} \\ \phi_{+\imath,2} \end{pmatrix} = x^g\, e^{-x^2/2} \begin{pmatrix} U\left(\frac{1}{2}, g + \frac{1}{2}, x^2\right) \\ -\frac{i}{\sqrt{2}}\, x\, U\left(\frac{3}{2}, g + \frac{3}{2}, x^2\right) \end{pmatrix}, \tag{2.20}$$
while the eigenfunction corresponding to $\lambda = -i$ is given by its complex conjugate,
$$\Phi_{-\imath}(x) = \Phi_{+\imath}(x)^* = \begin{pmatrix} \phi_{+\imath,1} \\ -\phi_{+\imath,2} \end{pmatrix}. \tag{2.21}$$

For $|g| \geq 1/2$ the operator Q_+ is essentially self-adjoint

As previously seen, the *deficiency indices* of Q_+, defined as the dimensions of the deficiency subspaces
$$n_\pm := \dim\, \mathrm{Ker}(Q_+^\dagger \mp i)\,, \tag{2.22}$$
vanish for $|g| \geq 1/2$.

According to von Neumann's theory, this means that Q_+ is *essentially self-adjoint* for these values of the coupling, admitting then a unique self-adjoint extension given by $Q_+^\dagger$ (which, in this case, is itself a self-adjoint operator).

The corresponding self-adjoint extension of the Hamiltonian for $|g| \geq 1/2$ is given by
$$\bar{H} = (Q_+^\dagger)^2, \tag{2.23}$$
where the operator composition in the right-hand side is possible in the dense domain
$$\mathcal{D}(\bar{H}) = \left\{\psi \in \mathcal{D}(Q_+^\dagger) : Q_+^\dagger \psi \in \mathcal{D}(Q_+^\dagger)\right\}. \tag{2.24}$$
In particular, every eigenfunctions of $Q_+^\dagger$ belongs to $\mathcal{D}\left(\bar{H}\right)$. Therefore, it is also an eigenfunction of $\bar{H}$ with eigenvalue $E = \lambda^2$.

Therefore, we have for the spectrum of $\bar{H}$

- For $g \geq 1/2$, there is a unique zero mode, while the positive eigenvalues of $\bar{H}$, $E_n = 2\,n\,,\ n = 1, 2, 3, \ldots$, have degeneracy two (since the eigenvalues of $Q_+^\dagger$ are $\lambda_{\pm,n} = \pm\sqrt{2n}$). One can combine $\Phi_{\pm,n}$ to get energy eigenfunctions with only the upper (bosonic) or the lower (fermionic) component non vanishing. In this case the SUSY is exact.
- For $g \leq -1/2$, there is no zero mode, the eigenvalues of $\bar{H}$ are positive, $E_n = 2\,n + 1 - 2\,g \geq 2\,,\ n = 0, 1, 2, \ldots$, and have degeneracy two (since of $Q_+^\dagger$ are $\lambda_{\pm,n} = \pm\sqrt{2n + 1 - 2g}$). Once again, the eigenfunctions $\Phi_{\pm,n}$ can be combined to get energy eigenfunctions with only one non-vanishing component, *i.e.*, also eigenfunctions of the grading operator. In this case the SUSY is spontaneously broken.

For $|g| < 1/2$ the operator Q_+ is not essentially self-adjoint

As we have seen, for $|g| < 1/2$ the deficiency indices are $n_\pm = 1$. According to von Neumann's theory, in this region Q_+ admits a one parameter family of SAE, Q_+^γ, which are in a one-to-one correspondence with the isometries from $\mathcal{K}_+$ onto $\mathcal{K}_-$, characterized by

$$\mathcal{U}(\gamma)\Phi_{+\imath}(x) := e^{2i\gamma}\Phi_{-\imath}\,, \quad \gamma \in [0, \pi)\,, \tag{2.25}$$

with $\Phi_{+\imath}$ and $\Phi_{-\imath}$ eigenvectors of $Q_+^\dagger$ with eigenvalues $+\imath$ and $-\imath$ respectively.

Let us call $\bar{Q}_+$ the *closure* of Q_+, which is given by $\bar{Q}_+ := Q_+^{\dagger\dagger}$. Its domain contains those functions $\overline{\Phi}$ for which $\left(\overline{\Phi}, Q_+^\dagger \Phi\right)$ is a continuous linear functional of $\Phi \in \mathcal{D}\left(Q_+^\dagger\right)$. In particular, since Q_+ is symmetric, $\mathcal{D}\left(Q_+\right) \subset \mathcal{D}\left(\bar{Q}_+\right) \subset \mathcal{D}\left(Q_+^\dagger\right)$.

The *self-adjoint operator* Q_+^γ is defined as the *restriction* of $Q_+^\dagger$ to the dense subspace $\mathcal{D}(Q_+^\gamma) \subset \mathcal{D}(Q_+^\dagger)$ composed by those functions which can be written as

$$\Phi = \overline{\Phi} + c\left(\Phi_{+\imath} + e^{2i\gamma}\Phi_{-\imath}\right)\,, \tag{2.26}$$

with $\overline{\Phi} \in \mathcal{D}(\bar{Q}_+)$, and $c \in \mathbf{C}$. The action of Q_+^γ is given by

$$Q_+^\gamma \Phi = Q_+^\dagger \overline{\Phi} + ic\left(\Phi_{+\imath} - e^{2i\gamma}\Phi_{-\imath}\right)\,. \tag{2.27}$$

This structure of the functions in $\mathcal{D}\left(Q_+^\gamma\right)$ completely characterizes its behavior near the origin and allows to determine the spectrum of Q_+^γ. Indeed, it can be shown [20] that the functions in the domain of the closure of Q_+, $\mathcal{D}(\bar{Q}_+)$, behave as

$$\overline{\phi}_1(x) = o(x^g)\,, \quad \overline{\phi}_2(x) = o(x^{-g})\,, \tag{2.28}$$

for $x \to 0^+$.

On the other side, from the expression of the eigenfunctions Φ_λ of $Q_+^\dagger$ in terms of $U(a,b,z)$ one can easily see that its components behave as

$$\begin{aligned} \phi_{\lambda,1}(x) &= \frac{\Gamma\left(\frac{1}{2}-g\right)}{\Gamma\left(\frac{1-\lambda^2}{2}-g\right)}\, x^{g} + O(x^{1-g})\,, \\ \phi_{\lambda,2}(x) &= \frac{\sqrt{2}}{\lambda}\, \frac{\Gamma\left(\frac{1}{2}+g\right)}{\Gamma\left(-\frac{\lambda^2}{2}\right)}\, x^{-g} + O(x^{1+g})\,, \end{aligned} \tag{2.29}$$

and then they are dominant near the origin.

Therefore, no eigenfunction of $Q_+^\dagger$ belongs to $\mathcal{D}(\bar{Q}_+)$, and it is the contributions of $\Phi_\pm$ in $\Phi_\lambda = \overline{\Phi} + c\big(\Phi_+ + e^{2i\gamma}\Phi_-\big)$ which determine the spectrum of Q_+^γ. Indeed, for a non vanishing c, the limit

$$\lim_{x\to 0+} \frac{x^{-g}\,\phi_{\lambda,1}(x)}{x^{g}\,\phi_{\lambda,2}(x)} = \frac{\lambda}{\sqrt{2}}\, \frac{\Gamma\left(-\frac{\lambda^2}{2}\right)}{\Gamma\left(\frac{1-\lambda^2}{2}-g\right)} \frac{\Gamma\left(\frac{1}{2}-g\right)}{\Gamma\left(\frac{1}{2}+g\right)} \tag{2.30}$$

must coincide with

$$\lim_{x\to 0+} \frac{\Re\left\{e^{-i\gamma}\, x^{-g}\, \phi_{+\imath,1}(x)\right\}}{\Re\left\{e^{-i\gamma}\, x^{g}\, \phi_{+\imath,2}(x)\right\}} = -\sqrt{\frac{\pi}{2}}\, \frac{\cot(\gamma)}{\Gamma\left(1-g\right)} \frac{\Gamma\left(\frac{1}{2}-g\right)}{\Gamma\left(\frac{1}{2}+g\right)}\,. \tag{2.31}$$

Consequently, the eigenvalues of Q_+^γ (which are real) are the solutions of the transcendental equation

$$f(\lambda) := \frac{\lambda\, \Gamma\left(-\frac{\lambda^2}{2}\right)}{\Gamma\left(\frac{1-\lambda^2}{2}-g\right)} = -\frac{\sqrt{\pi}\, \cot(\gamma)}{\Gamma\left(1-g\right)} =: \beta(\gamma)\,. \tag{2.32}$$

Notice that $f(\lambda)$ is an odd function of λ, and $-\infty \leq \beta(\gamma) < \infty$ for $0 \leq \gamma < \pi$. These eigenvalues of Q_+^γ are determined by the intersections of the graphic of $f(\lambda)$ with the horizontal line corresponding to the constant $\beta(\gamma)$. The eigenvalues are non degenerate, as shown in the figure 1.

Notice that these spectra are, in general, non symmetric with respect to the origin. The exceptions are the self-adjoint extensions corresponding to $\gamma = 0$ ($\beta = -\infty$) and $\gamma = \pi/2$ ($\beta = 0$). Indeed, the condition $f(-\lambda) = f(\lambda)$ for a non vanishing λ requires that

$$\frac{1}{\Gamma\left(-\frac{\lambda^2}{2}\right)\Gamma\left(\frac{1-\lambda^2}{2}-g\right)} = 0\,, \tag{2.33}$$

whose solutions correspond to the intersections with the constant $\beta = -\infty$ ($\gamma = 0$),

$$-\frac{\lambda^2}{2} = -n \Rightarrow \lambda_{\pm,n} = \pm\sqrt{2n}, \quad n = 1,2,3,\ldots \tag{2.34}$$

or the constant $\beta = 0$ ($\gamma = \pi/2$),

$$\frac{1-\lambda^2}{2} - g = -n \Rightarrow \lambda_{\pm,n} = \pm\sqrt{2n+1-2g}, \quad n = 0,1,2,\ldots. \tag{2.35}$$

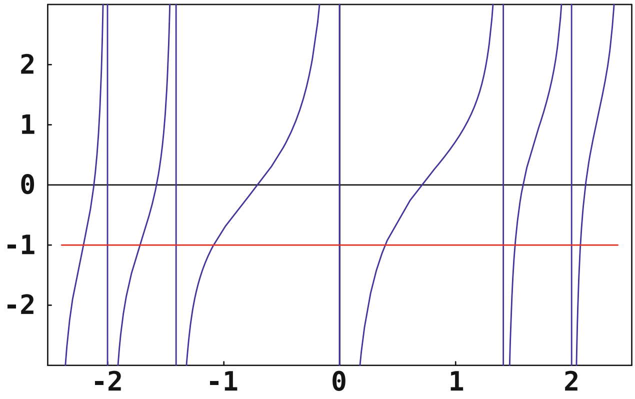

FIGURE 1. $f(\lambda) := \dfrac{\lambda\,\Gamma\left(-\frac{\lambda^2}{2}\right)}{\Gamma\left(\frac{1-\lambda^2}{2} - g\right)}$ for $g = 1/4$, and $\beta(\gamma) \equiv -1$.

Notice also that $Q_+^{\gamma=0}$ is the only self-adjoint extension having a zero mode, and for $0 < \gamma < \pi$ the eigenvalues are contained between contiguous asymptotes of the function $\Gamma\left(-\frac{\lambda^2}{2}\right)$,

$$\sqrt{2n} < |\lambda_{\pm,n}| < \sqrt{2(n+1)}\,. \tag{2.36}$$

3. The Hamiltonian

As previously stated, for each SAE Q_+^γ, with $\gamma \in [0,\pi)$, we get a self-adjoint extension of the Hamiltonian defined by

$$H_\gamma = (Q_+^\gamma)^2 \equiv (Q_+^\dagger)^2\Big|_{\mathcal{D}(H_\gamma)}\,, \tag{3.1}$$

where the operator composition on the right-hand side is defined as the restriction of $(Q_+^\dagger)^2$ to the dense subspace

$$\mathcal{D}\left(H_\gamma\right) = \left\{\psi \in \mathcal{D}\left(Q_+^\gamma\right) : Q_+^\dagger\psi \in \mathcal{D}\left(Q_+^\gamma\right)\right\}\,. \tag{3.2}$$

This domain includes, in particular, all the eigenfunctions of Q_+^γ, which are then also eigenvectors of H_γ:

$$Q_+^\gamma\Phi_\lambda = \lambda\Phi_\lambda \Rightarrow H_\gamma\Phi_\lambda = \lambda^2\Phi_\lambda\,. \tag{3.3}$$

Notice that, except for the special values $\gamma = 0, \pi/2$, the spectrum of H_γ is non degenerate.

Three cases can be distinguished for $|g| < 1/2$:

- For $\gamma = 0$ ($\beta = -\infty$) we get the only self-adjoint extension of H having a non degenerate zero mode and doubly degenerate positive eigenvalues $E_{\pm,n} = (\lambda_{\pm,n})^2 = 2n,\ n = 1, 2, 3, \ldots$. The corresponding eigenvectors can be chosen to have a definite character with respecto to the grading σ_3. This SAE of the Hamiltonian corresponds to an exactly realized SUSY.
- For $\gamma = \pi/2$ ($\beta = 0$) we get a SAE of H with no zero modes and a doubly degenerate non-vanishing eigenvalues, $E_{\pm,n} = (\lambda_{\pm,n})^2 = 2n + 1 - 2g$, with $n \geq 0$. They are all positive, since $1 - 2g > 0$. The eigenvectors can also be chosen with a definite σ_3 eigenvalue. In the present case, the condition imposed on the functions in $\mathcal{D}(Q_+^{\gamma=\pi/2})$ breaks the SUSY preserving the doubly degeneracy of the spectrum. This case corresponds to a SAE of H with spontaneously broken SUSY.
- Finally, for $\gamma \neq 0, \pi/2$ we get SAE of H with no zero modes and non degenerate positive eigenvalues (the square of the solutions of $f(\lambda) = \beta(\gamma)$). The eigenfunctions of H_γ (those of Q_+^γ) have both components non-vanishing (they are not eigenvectors of the grading). Then, they have not bosonic or fermionic character. In this case, the condition imposed on the functions belonging to $\mathcal{D}\left(Q_+^\gamma\right)$ breaks not only the SUSY, but also the degeneracy of the spectrum.

4. On the existence of a second supercharge

As previously mentioned, the differential operators Q_+ and Q_- are related by a unitary transformation, $Q_- = e^{i\sigma_3\pi/4} Q_+ e^{-i\sigma_3\pi/4}$. Then, each SAE of the first, Q_+^γ, determines a SAE of the second, Q_-^γ, whose domain is obtained from $\mathcal{D}(Q_+^\gamma)$ through this unitary transformation,

$$\mathcal{D}(Q_-^\gamma) = \left\{\Psi : e^{-i\pi\sigma_3/4}\Psi \in \mathcal{D}(Q_+^\gamma)\right\} = e^{i\pi\sigma_3/4}\left(\mathcal{D}(Q_+^\gamma)\right) . \tag{4.1}$$

Consequently, Q_-^γ is an equivalent representation of the self-adjoint supercharge Q_+^γ, sharing both operators the same spectrum.

Similarly, its square $(Q_-^\gamma)^2$, defined on the dense subspace

$$\mathcal{D}\left((Q_-^\gamma)^2\right) = \left\{\Psi \in \mathcal{D}(Q_-^\gamma) : Q_-^\gamma \Psi \in \mathcal{D}(Q_-^\gamma)\right\} = e^{i\pi\sigma_3/4}\left(\mathcal{D}(H_\gamma)\right) , \tag{4.2}$$

is an equivalent representation of the SAE $H_\gamma = (Q_+^\gamma)^2$ of the Hamiltonian H, initially defined on $\mathcal{C}_0^\infty(\mathbf{R}^+\backslash\{0\})$.

These equivalent representations of the Hamiltonian coincide only if the domain $\mathcal{D}(Q_+^\gamma)$ is left invariant by the grading operator σ_3 and, consequently, by the unitary transformation $e^{i\pi\sigma_3/4}$. And this occurs only for the particular self-adjoint extensions corresponding to $\gamma = 0$ and $\gamma = \pi/2$ since

$$e^{i\pi\sigma_3/4}\left(\Phi_{+\imath}(x) + e^{2i\gamma}\Phi_{-\imath}(x)\right) = \begin{pmatrix} e^{i\frac{\pi}{4}}\left(1 + e^{2i\gamma}\right)\phi_{+\imath,1}(x) \\ \\ e^{-i\frac{\pi}{4}}\left(1 - e^{2i\gamma}\right)\phi_{+\imath,2}(x) \end{pmatrix}. \tag{4.3}$$

Consequently, the operator compositions

$$(Q_+^\gamma)^2, \quad (Q_-^\gamma)^2, \quad Q_+^\gamma Q_-^\gamma \quad \text{and} \quad Q_-^\gamma Q_+^\gamma \tag{4.4}$$

make sense in the same (dense) domain $\mathcal{D}(H_\gamma)$ only for $\gamma = 0, \pi/2$, values of the parameter characterizing self-adjoint extensions for which the $N = 2$ SUSY algebra is realized,

$$\{Q_+^\gamma, Q_-^\gamma\} = 0\,, \qquad H_\gamma = (Q_+^\gamma)^2 = (Q_-^\gamma)^2\,. \tag{4.5}$$

For other values of the parameter γ, $\mathcal{D}(Q_+^\gamma)$ is not left invariant by $e^{i\pi\sigma_3/4}$, and there is no dense domain in the Hilbert space where the $N = 2$ SUSY algebra could be realized in terms of self-adjoint operator compositions.

Therefore, for $\gamma \neq 0, \pi/2$ only one self-adjoint supercharge can be defined in the domain of the Hamiltonian, and the SUSY algebra reduces to the $N = 1$ case,

$$H_\gamma = (Q_+^\gamma)^2 \tag{4.6}$$

(or, equivalently, $H_\gamma = (Q_-^\gamma)^2$).

It is worthwhile to remark that the double degeneracy of the non vanishing eigenvalues of H_γ with $\gamma = 0, \pi/2$ is a consequence of the existence of a second supercharge. Indeed, if

$$Q_+^\gamma \Phi_\lambda = \lambda \Phi_\lambda\,, \tag{4.7}$$

with $\Phi_\lambda \in \mathcal{D}(H_\gamma)$ and $\lambda \neq 0$, then $\{Q_+^\gamma, Q_-^\gamma\} = 0$ imply that

$$Q_+^\gamma(Q_-^\gamma\Phi_\lambda) = -Q_-^\gamma(Q_+^\gamma\Phi_\lambda) = -\lambda(Q_-^\gamma\Phi_\lambda)\,. \tag{4.8}$$

Therefore, $Q_-^\gamma\Phi_\lambda$ $\left(\in \mathcal{D}(Q_-^\gamma) \equiv \mathcal{D}(Q_+^\gamma)\right)$ is a linearly independent eigenvector of Q_+^γ corresponding to the eigenvalue $-\lambda$, since $Q_-^\gamma\Phi_\lambda \perp \Phi_\lambda$ and

$$\parallel Q_-^\gamma\Phi_\lambda \parallel^2 = (\Phi_\lambda, (Q_-^\gamma)^2\Phi_\lambda) = (\Phi_\lambda, H_\gamma\Phi_\lambda) = \lambda^2 \parallel \Phi_\lambda \parallel^2 \neq 0\,. \tag{4.9}$$

This explains why it is not possible to construct a second supercharge when the spectrum of the first one is not symmetric with respect to the origin.

5. Conclusions

Then, we have the following situation:

- For a general SAE Q_+^γ, the conditions the functions in $\mathcal{D}(H_\gamma)$ satisfy near the origin prevent the $N = 2$ SUSY. Only the $N = 1$ SUSY algebra is realized, with a non symmetric spectrum for Q_+^γ and a non degenerate spectrum for H_γ. This SUSY is spontaneously broken, since there are no zero modes.

- The only exceptions are the $\gamma = 0$ and $\gamma = \pi/2$ SAE, for which the $N = 2$ SUSY algebra can be realized. In these two cases the supercharges have a common symmetric spectrum and the positive eigenvalues of the Hamiltonian are doubly degenerate.
- For $\gamma = 0$, the (non degenerate) ground state of H_0 has a vanishing energy and the SUSY is exact, while for $\gamma = \pi/2$ the (doubly degenerate) ground state of $H_{\pi/2}$ has positive energy and the SUSY is spontaneously broken.

It is interesting to remark that a similar supersymmetric structure is found in the case of particles living in a plane and subject to the presence of an Aharonov-Bohm singular magnetic flux [28, 29].

Finally, it is worthwhile to point out that the $N = 2$ SUSY can be realized only when the supercharge domain $\mathcal{D}(Q_+^\gamma)$ is scale invariant. Indeed, given a function $\Phi(x) \in \mathcal{D}(Q_+^\gamma)$, under the scaling isometry

$$T_a\Phi(x) := a^{1/2}\Phi(ax) \tag{5.1}$$

with $a > 0$, we get

$$\lim_{x\to 0+} \frac{x^{-g}\,(T_a\Phi)_1(x)}{x^g\,(T_a\Phi)_2(x)} = -\sqrt{\frac{\pi}{2}}\,\frac{a^{2g}\cot(\gamma)}{\Gamma(1-g)}\,\frac{\Gamma\left(\frac{1}{2}-g\right)}{\Gamma\left(\frac{1}{2}+g\right)}\,, \tag{5.2}$$

which means that $T_a\Phi(x) \in \mathcal{D}(Q_+^{\gamma_a})$ with γ_a defined by

$$\cot(\gamma_a) = a^{2g}\cot(\gamma)\,. \tag{5.3}$$

Then, $\gamma_a = \gamma$ only for $\gamma = 0, \pi/2$. For other values of γ the domain $\mathcal{D}(Q_+^\gamma)$ is not scale invariant.

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