Brazilian Journal of Physics (2021) 51:803–812
https://doi.org/10.1007/s13538-021-00882-y

SOCIEDADE BRASILEIRA DE FÍSICA

PARTICLES AND FIELDS

Useful model to understand Schwartz' distributions' approach to non-renormalizable QFTs

M. C. Rocca[1,2,3] · **A. Plastino**[1,3,4]

Received: 22 July 2020 / Accepted: 25 February 2021 / Published online: 9 March 2021

Abstract

Quantum Field Theory (QFT) is a difficult subject, plagued by puzzling infinities. Its most formidable challenge is the existence of many non-renormalizable QFT theories, for which the number of infinities is itself infinite. We will here appeal to a rather non-conventional QFT approach developed in [J. of Phys. Comm. **2** 115029 (2018)] that uses Schwartz' distribution theory (SDT). This technique avoids the need for counterterms. In the SDT approach to QFT, infinities arise due to the presence of products of distributions with coincident point singularities. In the present study, we will carefully discuss a simple QFT-model devised by Bollini and Giambiagi. Because of its simplicity, it makes easy to appreciate just how it is possible to successfully deal with the issue of non-renormalizability via SDT.

Keywords Schwartz' distributions approach to QFT · Dimensional regularization · Lorentz invariant distributions · Convolution of Schwartz' distributions · Non-renormalizable quantum field theories

1 Introduction

Quantum Field Theory (QFT) is regarded as very difficult subject, plagued by puzzling infinities [1]. Its most formidable challenge is the existence of many non-renormalizable QFT theories, for which the number of infinities is itself infinite. In the present study, we will carefully discuss a simple QFT-model devised by Bollini and Giambiagi. Because of its simplicity, it makes it easy to appreciate, via suitable discussion, how it is possible to successfully deal with the issue of non-renormalizability.

In this work, we deal with the non-conventional QFT approach developed in [2]. Such approach appeals to Schwartz' distribution theory (SDT) and is able to entirely avoid the need for counterterms. In the SDT approach to QFT, infinities arise due to the presence of products of distributions with coincident point singularities. Our basic idea consists in appealing to a particular kind of Schwartz's distributions called STDELI, which belong to a ring with divisors of zero. Convolutions between them allow one to deal with both renormalizable and non-renormalizable QFTs on an equal footing.

Bollini and Giambiagi made a great advance for facing these infinities by appealing to Schwartz' Distributions (SD) Theory, a powerful mathematical machinery developed in the midst of the XX-century [3–5]. Regretfully enough, this machinery is not being taken advantage of by many current QFT-researchers. The Schwartz' Distributions' approach to QFT remains rather unconventional today, but it is the one we will be involved with here. There exists a variety of different SD, as we will see below. Among them, we will here highlight the so-called Schwartz Tempered Distributions Explicitly Lorentz Invariant (STDELI) $\mathcal{S}'_L$.

A great advance in facing QFT infinities was made by Bollini and Giambiagi who proposed the so-called Dimensional Regularization [3–5]. However, it cannot be defined for all Schwartz Tempered Distributions Explicitly Lorentz Invariant (STDELI) $\mathcal{S}'_L$. How to overcome this difficulty will also concern us here.

✉ M. C. Rocca
mariocarlosrocca@gmail.com

A. Plastino
angeloplastino@gmail.com

1 Departamento de Física, Universidad Nacional de La Plata, La Plata, Argentina

2 Departamento de Matemática, Universidad Nacional de La Plata, La Plata, Argentina

3 Consejo Nacional de Investigaciones Científicas y Tecnológicas (IFLP-CCT-CONICET), C. C. 727, 1900 La Plata, Argentina

4 SThAR - EPFL, Lausanne, Switzerland

We remind the reader that a ring is a fundamental algebraic structure. It is used in abstract algebra and consists of a set equipped with two binary operations. These operations generalize the arithmetic addition and multiplication ones. Via such generalization, theorems from arithmetic can be applied to non-numerical entities like as polynomials, series, matrices, etc. An element a of a ring is named a right zero divisor if there exists a nonzero y such that $ya = 0$. Similarly for left zero divisors. We have here a partial case of divisibility in rings. An element that is a left and a right zero divisor is simply called a zero divisor. If there are no nontrivial zero divisors in the ring, then it is called a domain. In the ring $\mathbb{Z}/4\mathbb{Z}$, the residue class $\overline{2}$ is a zero divisor. The ring of $n \times n$ matrices over a field has zero divisors if $n \geq 2$.

The main problem of the Schwartz' Distributions' approach to QFT is that of defining the product of two Schwartz' distributions (SD). It is a fundamental fact for us that the product of these distributions happens to be [2] a product in a ring with divisors of zero. Another fundamental fact is that, in quantum field theory, the problem of evaluating the product of SD with coincident point singularities is related to the asymptotic behavior of Feynman's loop integrals of propagators. We need to assume here that the reader is familiar with Feynman diagrams. If this is not the case, we strongly recommend consulting reference [1].

Distributions have formidable sounding names, but the reader should not be intimidated by them. In applications, things will turn out to be of a rather transparent nature, as we will see. From a mathematical point of view, practically all definitions of that distributions' product lead to limitations on the set of Schwartz' distributions that can be multiplied by each other to yield another SD of the same type. Note that Schwartz himself was not able to define a product of distributions regarded as an algebra, instead of as a ring with divisors of zero. In references [7–10], it was demonstrated that it is indeed possible to advance a general convolution between the a special kind of SDs, of impressive names. These are called ultradistributions (UDs) and were invented by Sebastiao e Silva [11], who named them Ultrahyperfunctions (UHF). Among the UDs, we have distributions called Tempered UDs (defined in a very restricted range), and Ultradistributions of Exponential Type.

The excellent news is that the convolution of two UHFs yields another Ultrahyperfunction. This entails that we do have now a product in a ring with zero divisors. Such a ring is called, precisely, the space of Distributions of Exponential Type, or also of Ultradistributions of Exponential Type. They are obtained by applying the anti-Fourier transform to the space of Tempered Ultradistributions or of Ultradistributions of Exponential Type.

We must clarify at this point that the Ultrahyperfunctions are the generalization and extension to the complex plane of the above-mentioned Schwartz Tempered Distributions and of those of Exponential kind. This entails that the Temperate Distributions and those of Exponential Type are a subset of the Ultrahyperfunctions of da Silva.

For treating non-renormalizable QFT theories, the problem we face is that of formulating the convolution between Ultradistributions. This is a complex issue, difficult to manage, even if it has the advantage of allowing one to attempt non-renormalizable QFT's [2].

Fortunately, the present authors have found [2] that a method similar to that employed to obtain the convolution of Ultradistributions can also be used to define the convolution of still another formidable sounding SDs. These are called Lorentz Invariant distributions. For them, the required convolution-definition utilizes the Dimensional Regularization (DR) technique of Bollini–Giambiagi, but effected in momentum space [2]. *As a consequence, Ultradistributions can be avoided (great news!) in the calculations of this paper, which considerably simplifies it* [2]. With our theory, we have been able to perform the Non-relativistic QFT of the emergent gravity of Verlinde [67]. This QFT is clearly non-renormalizable.

Taking advantage of the BG-regularization technique, one can also work in configuration space [5]. Thus, one can obtain a convolution of Lorentz Invariant Tempered Distributions in momentum space and the corresponding product in configuration space. To repeat, the reader should not be intimidated by this complexity and multiplicity of distributions' names. Things will become transparent when we use math-formulas. To end these introductory remarks, we remind that DR is one of the most important advances in theoretical physics and is used in several disciplines of it [12-65].

2 Preliminary Materials

2.1 The concept of Schwartz' distribution

Schwartz-distributions are defined as continuous linear functionals over a space of infinitely differentiable functions such that all continuous functions have derivatives which are themselves generalized functions. The most commonly encountered generalized function is the delta function. These mathematical objects are the main mathematical tools of this work. Distributions (generalized functions) are thus mathematical objects devised with the intent of generalizing the traditional concept of function. They allow one to differentiate functions for whom derivatives may not exist in the conventional sense.

Specifically, any locally integrable function possesses a "distributional" derivative. The distribution-notion is widely employed in the theory of partial differential equations. It may be easier sometimes to ensure the existence of distributional solutions than that of conventional solutions. It might also be the case that classical solutions do not exist. Distribution-theory is very important in physics and engineering. Many problems there lead to differential equations whose solutions or initial conditions are distributions. The Dirac's delta function is a prominent example. There are, of course, distributions of various types, that we will use below. One important example is that of Schwartz Tempered Distributions [11, 66], which permit integration in case where it is not feasible to do so with conventional techniques.

2.2 Main definitions

In this subsection, we give the definitions that we will use in this paper. We consider first the case on the ν-dimensional Minkowskian space $\boldsymbol{M}_\nu$. Let $\boldsymbol{S}^{'}$ be the space of Schwartz Tempered Distributions [11, 66]. Let be $g \in \boldsymbol{S}^{'}$. We say that $g \in \boldsymbol{S}^{'}_L$ if and only if,

$$g(\rho) = \frac{d^l}{d\rho^l} f(\rho) \tag{1}$$

where the derivative is in the sense of distributions, *l is a natural number*, $\rho = k^2 = k_0^2 - k_1^2 - k_2^2 - \cdots - k_{\nu-1}^2$, f satisfies,

$$\int\limits_{-\infty}^{\infty} \frac{|f(\rho)|}{(1+\rho^2)^n} d\rho < \infty, \tag{2}$$

and is continuous in $\boldsymbol{M}_\nu$. *The exponent n is a natural number.* We say then that $f \in \boldsymbol{T}_{1L}$.

In the case of Euclidean space $\boldsymbol{R}_\nu$, let $g \in \boldsymbol{S}^{'}$. We say that $g \in \boldsymbol{S}^{'}_R$ if and only if

$$g(k) = \frac{d^l}{dk^l} f(k), \tag{3}$$

where $k^2 = k_0^2 + k_1^2 + k_2^2 - \cdots + k_{\nu-1}^2$, with $f(k)$ satisfying,

$$\int\limits_{0}^{\infty} \frac{|f(k)|}{(1+k^2)^n} dk < \infty, \tag{4}$$

and $f(k)$ is continuous in $\boldsymbol{R}_\nu$. We say then that $f \in \boldsymbol{T}_{1R}$. We call $\boldsymbol{S}^{'}_{LA}$ and $\boldsymbol{S}^{'}_{RA}$ the Fourier Anti-transformed Spaces of $\boldsymbol{S}^{'}_L$ and $\boldsymbol{S}^{'}_R$, respectively.

2.3 Brief description of our theoretical scheme

We will here appeal to a rather non-conventional QFT approach developed in [2] that uses Schwartz' distribution theory (SDT) [6]. The technique of [2] avoids the need for counterterms. In our SDT approach to QFT, infinities arise due to the presence of products of distributions with coincident point singularities. In [2], the problem is tackled using special Schwartz distributions called STDELI (Schwartz Tempered Distributions Explicitly Lorentz Invariant), discussed at length in [2, 7–10]. We do not need the specific details here. It suffices to stipulate that, in [2], we were able to introduce an adequate process of convolution between two STDELIs. This convolution becomes, as a consequence, a product in a ring with divisors of zero in a very convenient space, that of the anti-Fourier transformed one [2, 7–10]. Again, mathematical precisions are not necessary for understanding what we do here. We only need to know the following fact. Consider two arbitrary STEDELIs f and g. Then, according to [2, 7–10], in a space of ν dimensions the convolution $f * g$ between them can be cast as a Laurent expansion with coefficients a_m [2],

$$f * g = \sum_{m=-n}^{\infty} a_m (\nu - 4)^m, \tag{5}$$

It is of the essence to point out that [2] the term in that expansion that does not depend on the Laurent exponent, namely a_0, yields the value of the convolution in four dimensions. This nice mathematical fact provides the root of our developments.

The overarching fact, whose importance cannot be exaggerated, is the following one [2]: the convolution of *propagators* in QFT, the main ingredient in dealing with QFTs from the Schwartz-distributions perspective, is just a special case of the relation (5).

In more detail, the essential ingredient of our treatment is the following fact. If f and g are now QFT propagators, then [2]

$$f * g = \sum_{m=-1}^{\infty} a_m (\nu - 4)^m, \tag{6}$$

and

$$\lim_{\nu \to 4} \left\{ \frac{\partial}{\partial \nu} \left[(\nu - 4)(f * g) \right] \right\} = a_0, \tag{7}$$

an identification result that removes most QFT-obstacles from our path.

As stated in the Introduction, a great advance in facing QFT infinities was made by Bollini and Giambiagi (BG) who proposed the so-called Dimensional Regularization (DR) technique [3–5]. We remark at this point now that (7) is precisely the prescription of the BG DR to calculate the finite part of the convolution of two propagators. However, this prescription is not suitable for the convolution of two general STDELIs, since for them it is $n \geq 1$ in (ref ep2.5). This fact explains to us why BG DR is not suitable for quantifying non-renormalizable QFT theories. The reason lies in the fact that, with the BG DR, we cannot define a product in a ring with zero divisors when considering STDELI. It is then necessary to resort to Laurent serial expansions for such products, and as a consequence, to products in a ring with zero divisors.

If the dimension is ν_0 instead of four, we have, more generally,

$$f * g = \sum_{m=-n}^{\infty} a_m(\nu - \nu_0)^m. \tag{8}$$

3 The two fields (Ω and ϕ) model we use to illustrate our theory

This model was advanced by Bollini–Giambiagi (BG) en [3]. They proposed a Lagrangian in which two fields enter (ψ and ϕ)

$$\mathcal{L} = -\frac{1}{2}\left[\partial_\mu\psi\partial^\mu\psi + m^2\psi^2 + \partial_\mu\phi\partial^\mu\phi - g\phi\psi^2\right], \tag{9}$$

whose free part reads

$$\mathcal{L}_F = -\frac{1}{2}\left[\partial_\mu\psi\partial^\mu\psi + m^2\psi^2 + \partial_\mu\phi\partial^\mu\phi\right], \tag{10}$$

with the interaction

$$\mathcal{L}_I = -g\phi\psi^2. \tag{11}$$

The solution for the equations of motion (EOM) for the field ψ is [3]

$$\psi = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\left[\frac{a(k)}{\sqrt{2\omega}}e^{ik_\mu x^\mu} + \frac{a^+(k)}{\sqrt{2\omega}}e^{-ik_\mu x^\mu}\right]d^3k, \tag{12}$$

with $\omega = \sqrt{k^2 + m^2}$.

For the field ϕ, the EOM solution reads [3]

$$\phi = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\left[\frac{b(k)}{\sqrt{2k_0}}e^{ik_\mu x^\mu} + \frac{b^+(k)}{\sqrt{2k_0}}e^{-ik_\mu x^\mu}\right]d^3k, \tag{13}$$

with $k_0 = |k|$.

Creation-destruction operators verify [3]

$$[a(k), a^+(k')] = \delta(k - k'). \tag{14}$$

$$[b(k), b^+(k')] = \delta(k - k'), \tag{15}$$

while the pertinent Feynman propagators are [3]

$$\Delta_\psi(x - y) =< 0|T[\psi(x)\psi(y)]|0 >, \tag{16}$$

so that the ψ-field propagator turns out to be [3]

$$\Delta_\psi(x - y) = \frac{i}{(2\pi)^4}\int\frac{e^{ik_\mu(x^\mu - y^\mu)}}{k^2 + m^2 - i0}d^4k. \tag{17}$$

Instead, for the field ϕ we have [3]

$$\Delta_\phi(x - y) =< 0|T[\phi(x)\phi(y)]|0 >, \tag{18}$$

so that the ϕ-field propagator becomes [3]

$$\Delta_\phi(x - y) = \frac{i}{(2\pi)^4}\int\frac{e^{ik_\mu(x^\mu - y^\mu)}}{k^2 - i0}d^4k. \tag{19}$$

4 The ν-dimensional self-energy

The amount of energy that a particle Q acquires as a result of modifications that Q itself generates in its surrounding medium is called its self-energy Σ. It can be thought of as an effective mass caused by interactions between Q and its environment.

4.1 Computing Σ using Feynman's parametrization

This is a well-known methodology [1]. The Feynman parameters are very well known [1]. One calls Feynman's parametrization to a technique devised for the evaluation of loop integrals emerging from Feynman diagrams with one or several loops.

Feynman remarked that, for any complex numbers A and B, as long as 0 is not contained in the line segment connecting A and B, one has [1]

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1 - x)]^2} \tag{20}$$

The result can suitably be extended to cases in which A and B are functions of a variable x [1].

4.2 Σ in Minkowski's space

As it is well known, in Minkowski's ν dimensional space, Σ gets defined as [3]

$$\Sigma_\psi(\rho) = (\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}. \tag{21}$$

where $\rho = k_1^2 + k_2^2 + k_3^2 + ... + k_{\nu-1}^2 - k_0^2$.

In Euclidean space, instead, we have

$$\Sigma_\psi(\rho) = (\rho + m^2)^{-1} * \rho^{-1}. \tag{22}$$

where $\rho = k_0^2 + k_1^2 + k_2^2 + ... + k_{\nu-1}^2$. One has

$$\Sigma_\psi(k) = \int\frac{d^4p}{(p^2 + m^2 - i0)[(p - k)^2 - i0]}, \tag{23}$$

and in ν dimensions we have

$$\Sigma_\psi(k, \nu) = \int\frac{d^\nu p}{(p^2 + m^2 - i0)[(p - k)^2 - i0]}. \tag{24}$$

A Wick rotation is an approach for finding solutions to problems in Minkowski space from solutions to a related issue in Euclidean space. One appeals to a transformation that substitutes an imaginary variable for a real one. This transformation is employed also for problems in quantum mechanics [1]. Using (35), and after a Wick rotation [1], the self-energy becomes

$$\Sigma_\psi(k,\nu) = i\int_0^1\int \frac{d^\nu p\, dx}{[(p-kx)^2+a]^2} \tag{25}$$

where

$$a = (p^2+m^2)x - p^2x^2. \tag{26}$$

Change now variables in the fashion $u = p - kx$. Then,

$$\Sigma_\psi(k,\nu) = i\int_0^1\int \frac{d^\nu u\, dx}{[u^2+a]^2} \tag{27}$$

To integrate over u, we need the result of the celebrated book by Gel'fand and Shilov [66]

$$\int \frac{(u^2)^n}{(u^2+a)^m} d^\nu u = \frac{\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \frac{\Gamma\left(\frac{\nu}{2}+n\right)\Gamma\left(m-n-\frac{\nu}{2}\right)}{\Gamma(m)a^{m-n-\frac{\nu}{2}}}. \tag{28}$$

With it, we find for the self-energy,

$$\Sigma_\psi(k,\nu) = i\pi^{\frac{\nu}{2}}\Gamma\left(2-\frac{\nu}{2}\right)(k^2+m^2)^{\frac{\nu}{2}-2}\int_0^1 x^{\frac{\nu}{2}-2}\left(1-\frac{k^2x}{k^2+m^2}\right)^{\frac{\nu}{2}-2}dx. \tag{29}$$

Appealing again to the book [66], we have, in terms of hypergeometric functions F

$$\int_0^u x^\mu(1+\beta x)^{-\nu}dx = \frac{u^\mu}{\mu}F(\nu,\mu,\mu+1;-\beta u), \tag{30}$$

and

$$\frac{2i\pi^{\frac{\nu}{2}}\Gamma\left(2-\frac{\nu}{2}\right)}{\nu-2}F\left(2-\frac{\nu}{2},\frac{\nu}{2}-1,\frac{\nu}{2},\frac{k^2}{k^2+m^2}\right). \tag{31}$$

Once more, we need to appeal to the book [66] to obtain, in terms of hypergeometric functions F

$$F(\alpha,\beta,\gamma;z) = (1-z)^{-\alpha}F\left(\alpha,\gamma-\beta,\gamma;\frac{z}{z-1}\right). \tag{32}$$

Thus,

$$\Sigma_\psi(k,\nu) = \frac{2i\pi^{\frac{\nu}{2}}m^{\nu-4}}{\nu-2}\Gamma\left(2-\frac{\nu}{2}\right)F\left(1,2-\frac{\nu}{2},\frac{\nu}{2};-\frac{k^2}{m^2}\right). \tag{33}$$

Appeal now to the hypergeometric functions equality

$$F\left(1,2-\frac{\nu}{2},\frac{\nu}{2};-\frac{k^2}{m^2}\right) = \Gamma\left(2-\frac{\nu}{2}\right)-$$

$$\Gamma\left(3-\frac{\nu}{2}\right)\frac{2k^2}{\nu m^2}F\left(1,3-\frac{\nu}{2},1+\frac{\nu}{2};-\frac{k^2}{m^2}\right), \tag{34}$$

entailing

$$\Sigma_\psi(k,\nu) = \frac{2i\pi^{\frac{\nu}{2}}}{\nu-2}m^{\nu-4}\Gamma\left(2-\frac{\nu}{2}\right)-$$

$$\frac{4i\pi^{\frac{\nu}{2}}}{\nu(\nu-2)}m^{\nu-6}k^2\Gamma\left(3-\frac{\nu}{2}\right)F\left(1,3-\frac{\nu}{2},1+\frac{\nu}{2};-\frac{k^2}{m^2}\right), \tag{35}$$

yielding thus our desired self-energy.

4.3 Euclidean case

Here, things work in almost the same fashion as above. Thus, we give only the final result

$$\Sigma_\psi(k,\nu) = \frac{2\pi^{\frac{\nu}{2}}}{\nu-2}m^{\nu-4}\Gamma\left(2-\frac{\nu}{2}\right)-$$

$$\frac{4\pi^{\frac{\nu}{2}}}{\nu(\nu-2)}m^{\nu-6}k^2\Gamma\left(3-\frac{\nu}{2}\right)F\left(1,3-\frac{\nu}{2},1+\frac{\nu}{2};-\frac{k^2}{m^2}\right) \tag{36}$$

Note that the two self-energies here discussed are related via a Wick rotation Rotating the Euclidean Σ yields the Minkowskian one.

5 The four-dimensional self-energy, a crucial QFT observable

5.1 Minkowskian case

To be able to evaluate self-energy in four dimensions, we must Laurent-expand it in powers of $\nu - 4$:

$$\Sigma_\psi(k,\nu) = \frac{2i\pi^2}{4-\nu} + i\pi^2(1-C-\ln m^2-\ln\pi)-$$

$$\frac{i\pi^2k^2}{2m^2}F\left(1,1,3;-\frac{k^2}{m^2}\right)+\sum_{m=1}^{\infty}a_m(\nu-4)^m. \tag{37}$$

According to our theory [2], the self-energy in four dimensions is given by the independent term (of $\nu - 4$) in Laurent's expansion of it. In terms of the hypergeometric function F, we have

$$\Sigma_\psi(k) = i\pi^2(1 - C - \ln m^2 - \ln \pi) - \frac{i\pi^2 k^2}{2m^2} F\left(1, 1, 3; -\frac{k^2}{m^2}\right). \tag{38}$$

5.2 Bollini and Giambiagi's DR technique [3, 4, 5] and self-energies

Let us now study the above issue by appeal to the DR of BG. According to it, the finite part of self-energy is given by

$$FP\left[\Sigma_\psi(\rho, \nu)\right] = \lim_{\nu \to 4} \left\{ \frac{\partial}{\partial \nu} \left[(\nu - 4)\Sigma_\psi(\rho, \nu)\right] \right\} =$$

$$i\pi^2(1 - C - \ln m^2 - \ln \pi) - \frac{i\pi^2 k^2}{2m^2} F\left(1, 1, 3; -\frac{k^2}{m^2}\right). \tag{39}$$

It is known then that Laurent's expansion can then be cast in the form:

$$\Sigma_\psi(k, \nu) = \left[\frac{2i\pi^2}{4 - \nu} + A\right] + [FP\Sigma_\psi(k, \nu) - A] + \sum_{m=1}^{\infty} a_m(\nu - 4)^m. \tag{40}$$

According to the prescription of BG, the self-energy in four dimensions becomes (A is an arbitrary constant)

$$\Sigma_\psi(k) = i\pi^2(1 - C - \ln m^2 - \ln \pi) - A - \frac{i\pi^2 k^2}{2m^2} F\left(1, 1, 3; -\frac{k^2}{m^2}\right), \tag{41}$$

and we see that, in BG's DR approach, $\Sigma_\psi(k)$is not uniquely defined because of the presence of an arbitrary constant. It is clear that one needs to generalize the BG approach, as we did in [2]. The generalization yields results without the constant A.

5.3 Euclidean case with the treatment of [2]

The Euclidean case is quite similar to the precedent one, i.e.,

$$\Sigma_\psi(k, \nu) = \frac{2\pi^2}{4 - \nu} + \pi^2(1 - C - \ln m^2 - \ln \pi) -$$

$$\frac{\pi^2 k^2}{2m^2} F\left(1, 1, 3; -\frac{k^2}{m^2}\right) + \sum_{m=1}^{\infty} a_m(\nu - 4)^m. \tag{42}$$

The self-energy in four dimensions is then

$$\Sigma_\psi(k) = \pi^2(1 - C - \ln m^2 - \ln \pi) - \frac{\pi^2 k^2}{2m^2} F\left(1, 1, 3; -\frac{k^2}{m^2}\right). \tag{43}$$

5.4 The old BG-DR treatment

The prescription of the DR of BG follows almost exactly the Minkowski's one discussed above. One has

$$FP\left[\Sigma_\psi(\rho, \nu)\right] = \lim_{\nu \to 4} \left\{ \frac{\partial}{\partial \nu} \left[(\nu - 4)\Sigma_\psi(\rho, \nu)\right] \right\} =$$

$$\pi^2(1 - C - \ln m^2 - \ln \pi) - \frac{i\pi^2 k^2}{2m^2} F\left(1, 1, 3; -\frac{k^2}{m^2}\right). \tag{44}$$

Laurent's expansion reads

$$\Sigma_\psi(k, \nu) = \left[\frac{2\pi^2}{4 - \nu} + A\right] + [FP\Sigma_\psi(k, \nu) - A] + \sum_{m=1}^{\infty} a_m(\nu - 4)^m \tag{45}$$

and the self-energy in four dimensions becomes

$$\Sigma_\psi(k) = \pi^2(1 - C - \ln m^2 - \ln \pi) - A - \frac{i\pi^2 k^2}{2m^2} F\left(1, 1, 3; -\frac{k^2}{m^2}\right), \tag{46}$$

exhibiting again the undesired arbitrary constant A.

6 The ν-dimensional vacuum polarization (VP)

In quantum field theory, the vacuum polarization describes a process in which a background field generates virtual particle–antiparticle pairs that change the distribution of charges and currents that produced the original field.

6.1 Minkowskian case

The VP for the field ψ is given by [1]

$$\Pi_\psi(k) = \int \frac{d^4 p}{(p^2 + m^2 - i0)[(p - k)^2 + m^2 - i0]}, \tag{47}$$

so that in ν dimensions one has

$$\Pi_\psi(k, \nu) = \int \frac{d^\nu p}{(p^2 + m^2 - i0)[(p - k)^2 + m^2 - i0]}. \tag{48}$$

Appeal now to Feynman parametrization to find

$$\Pi_\psi(k, \nu) = i \int_0^1 \int \frac{d^\nu p \; dx}{[(p - kx)^2 + a]^2}, \tag{49}$$

where

$$a = p^2 x(1 - x) + m^2. \tag{50}$$

Once again, we change variables in the fashion $u = p - kx$ and get

$$\Sigma_\psi(k,\nu) = i\int_0^1\int \frac{d^\nu u\, dx}{[u^2+a]^2}. \tag{51}$$

Evaluating the integral over u yields

$$\Pi_\psi(k,\nu) = i\pi^{\frac{\nu}{2}}\Gamma\left(2-\frac{\nu}{2}\right)\left(\frac{k^2+4m^2}{4}\right)^{\frac{\nu}{2}-2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(1-\frac{4k^2x^2}{k^2+4m^2}\right)^{\frac{\nu}{2}-2}dx. \tag{52}$$

Feynman parametrization then gives

$$\Pi_\psi(k,\nu) = i\pi^{\frac{\nu}{2}}\Gamma\left(2-\frac{\nu}{2}\right)\left(\frac{k^2+4m^2}{4}\right)^{\frac{\nu}{2}-2}F\left(2-\frac{\nu}{2},\frac{1}{2},\frac{3}{2};\frac{k^2}{k^2+4m^2}\right), \tag{53}$$

that can be recast as

$$\Pi_\psi(k,\nu) = i\pi^{\frac{\nu}{2}}m^{\nu-4}\Gamma\left(2-\frac{\nu}{2}\right)F\left(1,2-\frac{\nu}{2},,\frac{3}{2};-\frac{k^2}{4m^2}\right). \tag{54}$$

6.2 Euclidean case

Since it is quite similar to the above-discussed Minkowski one, we give just the final result.

$$\Pi_\psi(k,\nu) = \pi^{\frac{\nu}{2}}m^{\nu-4}\Gamma\left(2-\frac{\nu}{2}\right)F\left(1,2-\frac{\nu}{2},,\frac{3}{2};-\frac{k^2}{4m^2}\right). \tag{55}$$

7 The four-dimensional vacuum polarization

7.1 Minkowskian case

In order to evaluate the vacuum polarization in four dimensions, we again Laurent-expand around $\nu = 4$,

$$\Pi_\psi(k,\nu) = \frac{2i\pi^2}{4-\nu} - i\pi^2(\ln m^2 + \ln\pi + C) -$$

$$\frac{i\pi^2k^2}{6m^2}F\left(1,1,\frac{5}{2};-\frac{k^2}{4m^2}\right) + \sum_{m=1}^{\infty}a_m(\nu-4)^m. \tag{56}$$

Therefore, the vacuum polarization in four dimensions is

$$\Pi_\psi(k) = -i\pi^2(\ln m^2 + \ln\pi + C) - \frac{i\pi^2k^2}{6m^2}F\left(1,1,\frac{5}{2};-\frac{k^2}{4m^2}\right). \tag{57}$$

Instead, if we use the old BG-DR recipe discussed above, we have

$$\Pi_\psi(k,\nu) = \left(\frac{2i\pi^2}{4-\nu}+A\right) - i\pi^2(\ln m^2 + \ln\pi + C) - A$$

$$\frac{i\pi^2k^2}{6m^2}F\left(1,1,\frac{5}{2};-\frac{k^2}{4m^2}\right) + \sum_{m=1}^{\infty}a_m(\nu-4)^m, \tag{58}$$

so that the four-dimensional vacuum polarization is given by

$$\Pi_\psi(k) = -i\pi^2(\ln m^2 + \ln\pi + C) - A - \frac{i\pi^2k^2}{6m^2}F\left(1,1,\frac{5}{2};-\frac{k^2}{4m^2}\right), \tag{59}$$

displaying once again the unwelcome constant A.

7.2 Euclidean case

In the Euclidean space, and using our [2]-recipe, things do not change much from the Minkowski instance. Thus, we give only the final result. One has

$$\Pi_\psi(k) = -\pi^2(\ln m^2 + \ln\pi + C) - \frac{\pi^2k^2}{6m^2}F\left(1,1,\frac{5}{2};-\frac{k^2}{4m^2}\right). \tag{60}$$

Instead, with the old BG-DR recipe, we get a result dependent on an undesired arbitrary constant A:

$$\Pi_\psi(k) = -\pi^2(\ln m^2 + \ln\pi + C) - A\frac{\pi^2k^2}{6m^2}F\left(1,1,\frac{5}{2};-\frac{k^2}{4m^2}\right). \tag{61}$$

8 The non-renormalizable eight-dimensional case for the self-energy

8.1 Minkowskian case

Laurent's expansion for Σ in 8-dimensions is

$$\Sigma(k,\nu) = \frac{i\pi^4m^4}{3(8-\nu)}\left[1+\frac{1}{2}\frac{k^2}{m^2}+\frac{1}{10}\left(\frac{k^2}{m^2}\right)^2\right]+$$

$$\frac{i\pi^4m^4}{6}\left(\frac{11}{6}-C-\ln m^2-\ln\pi\right)+$$

$$\frac{i\pi^4m^4}{12}\left(\frac{11}{12}-C-\ln m^2-\ln\pi\right)\frac{k^2}{m^2}+$$

$$\frac{i\pi^4 m^4}{60}\left(\frac{29}{30}-C-\ln m^2-\ln\pi\right)\left(\frac{k^2}{m^2}\right)^2-$$

$$\frac{i\pi^4 m^4}{280}F\left(1,1,7;-\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3+\sum_{m=1}^{\infty}a_m(\nu-4)^m. \qquad (62)$$

Note the presence of k^2 powers in the expansion. The eight-dimensional self-energy is then

$$\Sigma(k)=\frac{i\pi^4 m^4}{6}\left(\frac{11}{6}-C-\ln m^2-\ln\pi\right)+$$

$$\frac{i\pi^4 m^4}{12}\left(\frac{11}{12}-C-\ln m^2-\ln\pi\right)\frac{k^2}{m^2}+$$

$$\frac{i\pi^4 m^4}{60}\left(\frac{29}{30}-C-\ln m^2-\ln\pi\right)\left(\frac{k^2}{m^2}\right)^2-$$

$$\frac{i\pi^4 m^4}{280}F\left(1,1,7;-\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3. \qquad (63)$$

8.2 Euclidean case

Here, Laurent's expansion becomes

$$\Sigma(k,\nu)=\frac{\pi^4 m^4}{3(8-\nu)}\left[1+\frac{1}{2}\frac{k^2}{m^2}+\frac{1}{10}\left(\frac{k^2}{m^2}\right)^2\right]+$$

$$\frac{\pi^4 m^4}{6}\left(\frac{11}{6}-C-\ln m^2-\ln\pi\right)+$$

$$\frac{\pi^4 m^4}{12}\left(\frac{11}{12}-C-\ln m^2-\ln\pi\right)\frac{k^2}{m^2}+$$

$$\frac{\pi^4 m^4}{60}\left(\frac{29}{30}-C-\ln m^2-\ln\pi\right)\left(\frac{k^2}{m^2}\right)^2-$$

$$\frac{\pi^4 m^4}{280}F\left(1,1,7;-\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3+\sum_{m=1}^{\infty}a_m(\nu-4)^m, \qquad (64)$$

so that

$$\Sigma(k)=\frac{\pi^4 m^4}{6}\left(\frac{11}{6}-C-\ln m^2-\ln\pi\right)+$$

$$\frac{\pi^4 m^4}{12}\left(\frac{11}{12}-C-\ln m^2-\ln\pi\right)\frac{k^2}{m^2}+$$

$$\frac{\pi^4 m^4}{60}\left(\frac{29}{30}-C-\ln m^2-\ln\pi\right)\left(\frac{k^2}{m^2}\right)^2-$$

$$\frac{\pi^4 m^4}{280}F\left(1,1,7;-\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3. \qquad (65)$$

9 The eight-dimensional vacuum polarization

9.1 Minkowskian case

We proceed as for the eight-dimensional self-energy's instance. Laurent's expansion reads

$$\Pi(k,\nu)=\frac{i\pi^4 m^4}{(8-\nu)}\left[1+4\frac{k^2}{m^2}+8\left(\frac{k^2}{m^2}\right)^2\right]+$$

$$\frac{i\pi^4 m^4}{2}\left(\frac{3}{2}-C-\ln m^2-\ln\pi\right)+$$

$$\frac{2i\pi^4 m^4}{3}\left(1-C-\ln m^2-\ln\pi\right)\frac{k^2}{m^2}-$$

$$\frac{4i\pi^4 m^4}{15}\left(C+\ln m^2+\ln\pi\right)\left(\frac{k^2}{m^2}\right)^2-$$

$$\frac{8i\pi^4 m^4}{105}F\left(1,1,\frac{9}{2};-\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3+\sum_{m=1}^{\infty}a_m(\nu-4)^m, \qquad (66)$$

and then

$$\Pi(k)=\frac{i\pi^4 m^4}{2}\left(\frac{3}{2}-C-\ln m^2-\ln\pi\right)+$$

$$\frac{2i\pi^4 m^4}{3}\left(1-C-\ln m^2-\ln\pi\right)\frac{k^2}{m^2}-$$

$$\frac{4i\pi^4 m^4}{15}\left(C+\ln m^2+\ln\pi\right)\left(\frac{k^2}{m^2}\right)^2-$$

$$\frac{8i\pi^4 m^4}{105}F\left(1,1,\frac{9}{2};-\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3. \qquad (67)$$

9.2 Euclidean case

Similarly, here the Laurent's expansion is

$$\Pi(k,\nu) = \frac{\pi^4 m^4}{(8-\nu)}\left[1 + 4\frac{k^2}{m^2} + 8\left(\frac{k^2}{m^2}\right)^2\right] +$$

$$\frac{\pi^4 m^4}{2}\left(\frac{3}{2} - C - \ln m^2 - \ln \pi\right) +$$

$$\frac{2\pi^4 m^4}{3}\left(1 - C - \ln m^2 - \ln \pi\right)\frac{k^2}{m^2} -$$

$$\frac{4\pi^4 m^4}{15}\left(C + \ln m^2 + \ln \pi\right)\left(\frac{k^2}{m^2}\right)^2 -$$

$$\frac{8\pi^4 m^4}{105} F\left(1, 1, \frac{9}{2}; -\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3 + \sum_{m=1}^{\infty} a_m (\nu - 4)^m, \qquad (68)$$

so that

$$\Pi(k) = \frac{\pi^4 m^4}{2}\left(\frac{3}{2} - C - \ln m^2 - \ln \pi\right) +$$

$$\frac{2\pi^4 m^4}{3}\left(1 - C - \ln m^2 - \ln \pi\right)\frac{k^2}{m^2} -$$

$$\frac{4\pi^4 m^4}{15}\left(C + \ln m^2 + \ln \pi\right)\left(\frac{k^2}{m^2}\right)^2 -$$

$$\frac{8\pi^4 m^4}{105} F\left(1, 1, \frac{9}{2}; -\frac{k^2}{m^2}\right)\left(\frac{k^2}{m^2}\right)^3. \qquad (69)$$

10 Conclusions

In QFT, when we use perturbative expansions, we deal either with products of distributions in configuration space or with convolutions of Schwartz' distributions in momentum space [2].

We have here appealed to a rather non-conventional QFT approach developed in [2] that uses Schwartz' distribution theory (SDT) [6]. The technique of [2] avoids the need for counterterms. In this SDT approach to QFT, infinities arise due to the presence of products of distributions with coincident point singularities. In [2], the problem is tackled and solved using special Schwartz distributions called STDELI, discussed at length in [2]. The ensuing considerations were of a very complex nature, so that a simple model became absolutely necessary to illustrate these novel techniques.

We have done precisely this in this work. We appealed to a simple two-fields (ψ and ϕ) model with interaction $g\phi\psi^2$. The model is renormalizable in 4-dimensions but NOT in 8-dimensions. We were able to calculate for such model two important QFT quantities: the self-energy and the vacuum polarization for the field ϕ.

The present works add then "flesh" to the rather abstract and involved mathematical discussion of [2], showing conclusively that we can rather easily obtain observables for non-renormalizable QFT models.

References

1. Riccardo D'Auria, Mario Trigiante, From Special Relativity to Feynman Diagrams, A Course in Theoretical Particle Physics for Beginners (Springer, Belin, 2016); Alexander L. Fetter, John Dirk Walecka, Quantum Theory of Many-Particle Systems (Dover, NY, 2003); Richard D. Mattuck, A Guide to Feynman Diagrams in the Many-Body Problem (Dover, NY, 1992); A. Zee, Quantum Field Theory in a Nutshell, Second Edition (Amazon Books,2010)
2. A. Plastino, M.C. Rocca, J. of Phys. Comm. **2**, 115029 (2018)
3. C.G. Bollini, J.J. Giambiagi, Phys. Lett. B **40**, 566 (1972a)
4. C.G. Bollini, J.J. Giambiagi, Il Nuovo Cim. B **12**, 20 (1972b)
5. C. G. Bollini and J.J Giambiagi , Phys. Rev. **D 53**, 5761 (1996)
6. L. Schwartz, *Théorie des distributions* (Hermann, Paris, 1966)
7. C.G. Bollini, T. Escobar, M.C. Rocca, Int. J. of Theor. Phys. **38**, 2315 (1999)
8. C.G. Bollini, M.C. Rocca, Int. J. of Theor. Phys. **43**, 1019 (2004a)
9. C.G. Bollini, M.C. Rocca, Int. J. of Theor. Phys. **43**, 59 (2004b)
10. C.G. Bollini, P. Marchiano, M.C. Rocca, Int. J. of Theor. Phys. **46**, 3030 (2007)
11. J. Sebastiao e Silva , Math. Ann. **136**, 38 (1958)
12. D. Berenstein, A. Miller, Phys. Rev. D **90**, 086011 (2014)
13. D. Anselmi, Phys. Rev. D **89**, 125024 (2014)
14. P. Jaranowski and G. Schfer, Phys. Rev. D **87**, 081503(R) (2013)
15. T. Inagaki, D. Kimura, H. Kohyama, A. Kvinikhidze, Phys. Rev. D **86**, 116013 (2012)
16. J. Qiu, Phys. Rev. D **77**, 125032 (2008)
17. L. Blanchet, T. Damour, G. Esposito-Farse, and B. R. Iyer, Phys. Rev. D **71**, 124004 (2005)
18. F. Bastianelli, O. Corradini, A. Zirotti, Phys. Rev. D **67**, 104009 (2003)
19. D. Lehmann and G. Przeau, Phys. Rev. D **65**, 016001 (2001)
20. A. P. Bata Scarpelli, M. Sampaio, and M. C. Nemes, Phys. Rev. D **63**, 046004 (2001)
21. E. Braaten and Yu-Qi Chen, Phys. Rev. D **55**, 7152 (1997)
22. J. Smith and W. L. van Neerven EPJ C **40**, 199 (2005)
23. J. F. Schonfeld EPJ C **76**, 710 (2016)
24. C. Gnendiger et al., EPJ C **77**, 471 (2017)
25. P. Arnold, Han-Chih Chang and S. Iqbal, JHEP 100 (2016)
26. I. AravE, Y. Oz, A. Raviv-Moshe, JHEP **88**, (2017)
27. C. Anastasiou, S. Buehler, C. Duhr, F. Herzog, JHEP **62**, (2012)
28. F. Niedermayer, P. Weisz, JHEP **110**, (2016)
29. C. Coriano, L. Delle Rose, E. Mottolaand M. Serino, JHEP 147 (2012)
30. F. Dulat, S. Lionetti, B. Mistlberger, A. Pelloni, C. Specchia, JHEP **17**, (2017)
31. T. Gehrmann, N. Greiner, JHEP **50**, (2010)
32. T. Lappia, R. Paatelainena, Ann. of Phys. **379**, 34 (2017)
33. S.Grooteab, J.G.Krner and A.A.Pivovarov, Ann. of Phys. **322**, 2374 (2007)
34. N.C. Tsamis, R.P. Woodard, Ann. of Phys. **321**, 875 (2006)
35. S. Krewaland, K. Nakayama, Ann. of Phys. **216**, 210 (1992)
36. L. Rosen and J. D. Wright Comm. Math. Phys. **134**, 433 (1990)
37. F. David Comm. Math. Phys. **81**, 149 (1981)

38. P. Breitenlohner and D. Maison Comm. Math. Phys. **52**, 11 (1977)
39. S. Teber, A.V. Kotikov, EPL **107**, 57001 (2014)
40. H. Fujisaki, EPL **28**, 623 (1994)
41. M. W. Kalinowski, M. Seweryski and L. Szymanowski, JMP **24**, 375 (1983)
42. R. Contino, A. Gambassi, JMP **44**, 570 (2003)
43. M. Dutsch, K. Fredenhagen, K. J. Keller and K. Rejzner3, JMP **55**, 122303 (2014)
44. T. Nguyena, JMP **57**, 092301 (2016)
45. J. Ben Geloun and R. Toriumi, JMP **56**, 093503 (2015)
46. J. Ben Geloun and R. Toriumi, J. Phys. A **45**, 374026 (2012)
47. B. Mutet, P. Grange, E. Werner, J. Phys. A **45**, 315401 (2012)
48. M.C. Abbott, P. Sundin, J. Phys. A **45**, 025401 (2012)
49. T. Fujihara et al., J. Phys. A **39**, 6371 (2008)
50. Silke Falk et al., J. Phys. A **43**, 035401 (2010)
51. Germn Rodrigo et al., J. Phys. G **25**, 1593 (1999)
52. B.M. Pimentel, J.L. Tomazelli, J. Phys. G **20**, 845 (1994)
53. A. Khare, J. Phys. G **3**, 1019 (1977)
54. J.C. D'Cruz, J. Phys. G **1**, 151 (1975)
55. R. Sepahv, S. Dadfar, Nuc. Phys. A **960**, 36 (2017)
56. J.V. Steele, R.J. Furnstahl, Nuc. Phys. A **630**, 46 (1998)
57. D.R. Phillips, S.R. Beane, T.D. Cohena, Nuc. Phys. A **631**, 447 (1998)
58. A.J. Stoddart, R.D. Viollier, Nuc. Phys. A **532**, 657 (1991)
59. E. Panzer, Nuc. Phys. B **874**, 567 (2013)
60. R. N. Lee, A. V. Smirnov and and V. A. Smirnov, Nuc. Phys. B **856**, 95 (2012)
61. A.P. Isaev, Nuc. Phys. B **662**, 461 (2003)
62. J.M. Campbell, E.W.N. Glover, D.J. Miller, Nuc. Phys. B **498**, 397 (1997)
63. C.J. Yang, M. Grasso, X. Roca-Maza, G. Colo, K. Moghrabi, Phys. Rev. C **94**, 034311 (2016)
64. K. Moghrabi, M. Grasso, Phys. Rev. C **86**, 044319 (2012)
65. D.R. Phillips, I.R. Afnan, A.G. Henry-Edwards, Phys. Rev. C **61**, 044002 (2000)
66. I.M. Gel'fand, G.E. Shilov, Generalized Functions, vol. 1 (Academic Press, 1964)
67. A. Plastino, M.C. Rocca, Ann. of Phys. **412**, 168013 (2020)