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ON THE UNIQUENESS OF GIBBS STATES IN SOME DYNAMICAL SYSTEMS

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Abstract. By applying Grothendieck theory and Ruelle thermodynamic formalism, we prove that, for expansive dynamical systems and interaction potentials satisfying certain conditions of analyticity, the associated Gibbs states are unique. This allows us to draw an analogy between some quantities in classical thermodynamics and abstract dynamics in the spirit of the previous work of the authors [13].

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1. Introduction

The phenomenon of phase transition is manifested by the coexistence of two or more pure phases for "physically acceptable" interactions. To investigate the coexistence or not of phases it is important to study the structure of the space of all Gibbs states. This can be done by analyzing the variation of some thermodynamical quantities, associated with these states, when internal parameters, such as the temperature, are changed.

This problem was widely studied in the context of the classical statistical mechanics of lattices. In this case, one considers a finite set Ω (whose elements are called *spins*) and a countable infinite set $\mathbb{L}$ (the lattice, whose elements are the *sites*). The *configuration space* is $\Omega^{\mathbb{L}}$, i.e., a *configuration* is a sequence $(\sigma(i))_{i\in\mathbb{L}}$, where $\sigma : \mathbb{L} \to \Omega$. The model is completed by given a function $\rho : \Omega^{\mathbb{L}} \to \mathbb{R}$ called the *interaction* and a $(\mathrm{card}(\Omega) \times \mathrm{card}(\Omega))$-matrix (the *transition matrix*), which defines allowed configurations of the system.

Our purpose is to apply a dynamical approach to subspaces $X \subseteq \mathbb{R}^d$. This makes a difference with the standard treatments in classical thermodynamics and even in more mathematical fields like theoretical probability.

Let $\mathcal{G}(q)$ be the set of Gibbs states associated with "interaction" φ with a free energy $T(q) = T_\varphi(q)$, where q is the inverse temperature. This set is convex [20], extremal Gibbs states are interpreted as *pure homogeneous phases*, and any Gibbs state admits a unique integral decomposition in terms of pure phases.

It is known from the Ruelle thermodynamic formalism that the Gibbs states are tangent to $T(q)$. The name "tangent" arises since $T(q) = T_\varphi(q)$ can be considered as a functional on the space of interactions. Hence a phase transition is detected when this function has a singularity at some q. We recall that an f-invariant measure μ is said to be *tangent to* $T(q)$ at q with respect to an interaction φ if for every q', we have

$$T(q + q') - T(q') \geq \int q'\varphi \, d\mu.$$

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This kind of analysis, for one-dimensional lattices, was done primarily by Ruelle [19, 20] and Sinai [23]. Accordingly, the absence of phase transition is proved by showing that $\exp(T(q))$ is an isolated eigenvalue of the transfer operator associated with $q\varphi$ for interactions φ belonging to some special classes. In [20], Ruelle also applied this formalism to more abstract spaces, namely, Smale spaces.

In this work, we extend the previous analysis in some directions. Instead of a symbolic space (a one-dimensional lattice in the terminology of statistical mechanics), we consider, as was earlier mentioned, a compact submanifold $X \subset \mathbb{R}^d$. More generally, the dynamics are given by continuous mappings $f : X \to X$ and we introduce a free energy $T(q)$ adequate to this context. In this way, a contact with thermodynamics can be made, as was pointed out in [5, 20]. Following Mayer [12], we introduce certain "transfer operators" $\mathcal{L}_q$ and establish for any q a relationship between free energy $T(q)$ and the spectral radius of $\mathcal{L}_q$. This and other spectral properties that we study in this work lead to the proof of absence of phase transition for the systems under consideration.

In a recent work [13] we have obtained formal relationships among statistical mechanics, multifractal analysis, and abstract dynamical systems. Here we will prove nonexistence of phase transitions under a substantially different approach and by imposing another class of conditions. Thus, we may reproduce the results of [13] within the framework of this paper.

The plan is as follows: in the next section, we recall the concept of Gibbs states in any dimension or, more generally, for abstract dynamics and introduce the formalism to define the partition functions and free energy functions. In Sec. 3, we use the transfer operators and prove the absence of a phase transition in the models considered.

2. Gibbs States and the Free Energy Function

Let X be a compact subset of $\mathbb{R}^d$, $d > 0$, a dynamical mapping $f : X \to X$ be continuous, and a potential $\varphi \in C(X)$. The *statistical sum* for $x \in X$

$$S_n(\varphi(x)) := \sum_{i=0}^{n-1} \varphi(f^i(x)). \tag{1}$$

The set of "microstates" under consideration will be the whole set of periodic points $P_n(f) = \{x : f^n x = x\}$. The "Hamiltonian of n particles" will be given by the statistical sum $\mathcal{H}_n(x) = S_n(\varphi(x))$.

By analogy with classical statistical mechanics, we introduce the canonical partition function (q is interpreted as the inverse of the temperature):

$$Z_n(q) = \sum_{x \in P_n(f)} \exp(-q\mathcal{H}_n(x)). \tag{2}$$

Therefore, the *function free energy* is the limit

$$T(q) = T_{\varphi,f}(q) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(q) \tag{3}$$

if it exists.

Given a metric d in X, we consider the associated distance

$$d_n(x, y) = \max_{i=0,1,\dots,n-1} d(f^i(x), f^i(y)). \tag{4}$$

The ball with center x of radius ε in this metric will be denoted by $B_{n,\varepsilon}(x)$.

The space $\mathcal{G}(q)$ of Gibbs states associated with $q\varphi$ consists of f-invariant measures μ_q such that [8, 20] for sufficiently small $\varepsilon > 0$, there exist constants $A_\varepsilon, B_\varepsilon > 0$ such that for any $x \in X$ and any positive integer n

$$A_\varepsilon(\exp(S_n(q\varphi(x))) - nT(q)) \le \mu_q(B_{n,\varepsilon}(x)) \le B_\varepsilon(\exp(S_n(q\varphi(x))) - nT(q)). \tag{5}$$

We make the following assumptions on the systems to be studied. Any dynamical system (X, f) will be endowed with a $(\mathrm{card}(\Omega) \times \mathrm{card}(\Omega))$-matrix A (where Ω is some finite set). This matrix plays a role similar to that of transfer matrices in statistical mechanics of lattices. Its entries are either 0 or 1, and for $\kappa, \lambda \in \Omega$, the number $A_{\kappa,\lambda}$ indicates the "admissibility" of κ and λ with respect to the system considered. For example, if the system admits a Markov partition, i.e., a set $\{W_1, W_2, \dots, W_k\}$, where $W_i \cap W_i = \emptyset$, $i \neq j$, $f(W_j) = \bigcup_{l=0}^{k} W_{k_l}$, and $W_i = \overline{\mathrm{int}(W_i)}$, then admissible pairs κ, λ are those for which $A_{\kappa,\lambda} = 1$ whenever $f(W_j) \cap W_i \neq \emptyset$ and $A_{\kappa,\lambda} = 0$ otherwise. The elements of Ω can be assimilated to the spins in classical statistical mechanics. In order to ensure that $\mathrm{card}\, P_n(f) < \infty$, we may assume that the dynamical mapping f is *expansive*, i.e., there exists a constant $\delta > 0$ such that $d(f^n(x), f^n(y)) < \delta$ for any integer n implies $x = y$. Since $P_n(f)$ is (n, δ)-separated and X is compact, it follows that $P_n(f)$ is finite (this is a standard fact in topological dynamics).

On dynamical systems as above, the following conditions are imposed:

(C1) There exists a finite covering $(W_\kappa)_{\kappa\in\Omega} \subset X$ such that any W_κ has a complex neighborhoods $U_\kappa \subset \mathbb{C}^d$ such that f can be holomorphically extended to $\widehat{f} : \bigcup_{\varkappa\in\Omega} U_\varkappa \to \bigcup_{\varkappa\in\Omega} U_\varkappa$.

(C2) The mapping $\widehat{f}$ has holomorphic inverse branches

$$\psi_\kappa : \bigcup_{\lambda\in\Omega_\kappa} U_\lambda \to U_\kappa,$$

i.e., $\widehat{f} \circ \psi_\kappa = \mathrm{id}\,|_{\bigcup_{\lambda\in\Omega_\kappa} U_\lambda}$ for any $\kappa \in \Omega$, where $\widehat{f}(U_\kappa) = \bigcup_{\lambda\in\Omega_\kappa} U_\lambda$. Here $\Omega_\kappa = \{\lambda \in \Omega : A_{\kappa,\lambda} = 1\}$. Moreover, the functions ψ_κ map $\bigcup_{\lambda\in\Omega_\kappa} U_\lambda$ strictly inside U_κ.

Since the mapping is expansive, the entropy mapping $\mu \longmapsto h_\mu(f)$ (where $h_\mu(f)$ is the classical Kolmogorov–Sinai measure-theoretic entropy) is upper semi-continuous. In the space of invariant measures, the weak topology is considered. Under this hypothesis, the following result holds: the free energy mapping $T(q) = T_{\varphi,f}(q)$ is differentiable in q if and only if there exists a unique equilibrium state for $q\varphi$ (see [9, 25]).

3. Study of Phase Transitions by Transfer Operators

By $\mathcal{A}_\infty(U)$, $U \subset \mathbb{C}^d$, we denote the space of functions holomorphic in U and bounded on the closure of U (with the supremum norm).

For dynamical systems satisfying conditions (C1)–(C2) and potentials $\varphi \in \mathcal{A}_\infty(U)$, $U = \bigcup_{\varkappa\in\Omega} U_\varkappa$, transfer operators acting on $\bigoplus_{\kappa\in\Omega} \mathcal{A}_\infty(U\kappa)$ can be defined by

$$(\mathcal{L}_\varphi(\chi))_\kappa(z) = \sum_{\lambda\in\Omega_\kappa} A_{\kappa,\lambda} \exp(\varphi_\lambda(z))\chi(\psi_\lambda(z)), \tag{6}$$

where $\varphi_\lambda(z) := \varphi(\psi_\lambda(z))$.

The potentials are originally defined on X and take real values. We assume that, in the sense of conditions (C1)–(C2), the potentials which characterize the systems can be extended to

$$\varphi : U = \bigcup_{\varkappa\in\Omega} U_\varkappa \subset \mathbb{C}^d \to \mathbb{C}.$$

The following fixed point theorem is useful to compute the trace of the transfer operators.

Theorem (Earle–Hamilton [4])**.** *Let D be a bounded connected subspace of a Banach space B and ψ be a holomorphic mapping on D applying it strictly inside itself. Then ψ has exactly one fixed point $\overline{z} \in D$, and $\|D\psi(\overline{z})\| < 1$.*

The meaning of "strictly inside itself" is the following: let D be a bounded connected subspace of a Banach space B and ψ be a holomorphic mapping on D. We say that ψ applies D strictly inside itself if

$$\inf_{\substack{z\in D\\ z'\in B-D}} \|\psi(z)-z'\| \geq \delta > 0.$$

Here $D\psi$ is the differential mapping of ψ considered as a linear operator on B.

The trace of the operator $\mathcal{L}_\varphi$ is given by

$$\operatorname{Tr}(\mathcal{L}_\varphi) = \sum_{\kappa\in\Omega} A_{\kappa,\kappa} \exp(\varphi_\kappa(\overline{z}_\kappa)) \frac{1}{\det(1 - D\psi_\kappa(\overline{z}_\kappa))}, \tag{7}$$

where $\varphi_\kappa(z) := \varphi(\psi_\kappa(z))$ and $\overline{z}_\kappa$ is the fixed point of ψ_κ. This trace formula was obtained by Mayer [12] and yields an expression in the style of the Atiyah–Bott formula on Lefschetz fixed point.

We set $\mathcal{L}_q = \mathcal{L}_{q\varphi}$. One main observation is that the operators of this class are *nuclear*. Let us recall that an operator $\mathcal{L}$ acting on a Banach space B is nuclear if there exist sequences $(x_n) \subset B$, $(f_n) \subset B^*$ (the dual space of B) such that $\|x_n\| = 1$, $\|f_n\| = 1$ and numbers (ρ_n) with $\sum_{n=0}^{\infty} |\rho_n| < \infty$ such that $\mathcal{L}(x) = \sum_{n=0}^{\infty} \rho_n f_n(x) x_n$ for every $x \in B$.

Example. Let

$$\mathcal{L}(\varkappa)(z) = \sum_{\kappa\in\{\pm1\}} \exp(\kappa\mathcal{J}z)\chi(\beta(\kappa+z)),$$

where $\mathcal{J}$ and β are some real parameters. This operator has the form (6) for $\psi_\kappa(z) = \beta(\kappa+z)$, $A_{\kappa,\lambda} = 1$, for any $\kappa,\lambda \in \Omega = \{\pm1\}$ and the interactions $\varphi_1(z) = \mathcal{J}z$ and $\varphi_{-1}(z) = -\mathcal{J}z$.

For $R > \dfrac{\beta}{1-\beta}$, the function $\exp(\kappa\mathcal{J}z)\chi(\beta(\kappa+z))$ belongs to $\mathcal{A}_\infty(D_R)$, where

$$D_R = \{z \in \mathbb{C} : |z| < R\}$$

and, therefore, $\mathcal{L}$ is nuclear for this choice of the parameters.

The operator from the example has an interesting analogy with the following physical model of system of many particles. Let us consider the following one-dimensional spin model: particles at positions $i \in \mathbb{N}$ have spins x_i, which can take the values $+1$ (spins up) or -1 (spins down). A configuraction is a symbolic sequence $C = x_0\ x_1, \ldots$, where $x_i \in \{\pm1\}$. Thus, the set of configurations can be considered as a Markov system

$$\sum_A = \{C = x_0 x_1, \ldots, x_i \in \{\pm1\},\ A_{x_i,x_i,+1} = 1\},$$

where A is the transition matrix.

The potential interaction is given by the mapping $\phi(C) = \sum_{n=1}^{\infty} \mathcal{J} x_0 x_n \beta^n$, where $\mathcal{J}$ is interpreted as a coupling parameter and β as a number which describes the asymptotic dependence of the interaction ϕ on the particles x_n. If we add an extra particle x to the configuration $C = x_0, x_1, \ldots$, we denote by $\langle x, C\rangle$ the configuration $x, x_0, x_1, \ldots$, i.e., the position of x_0 is now occupied by x and any particle that was originally at the ith site in C is moved to the $i+1$th site in $\langle x, C\rangle$. Thus, the interaction in $\langle x, C\rangle$ has now the potential

$$\phi(\langle x, C\rangle) = \sum_{n=1}^{\infty} \mathcal{J} x x_{n-1} \beta^n.$$

The transfer operator for this model is defined as

$$\mathcal{S}_\phi(\omega)(C) = \sum_{\kappa\in\{\pm1\}} \exp\left(\kappa\mathcal{J}\sum_{n=1}^{\infty} x_{n-1}\beta^n\right)\phi(\langle x, C\rangle)$$

(see [12]).

Consider the space of functions

$$\mathcal{F}\left(\sum_A\right) = \left\{\omega \in C\left(\sum_A\right) : \text{ there exists } \chi \in \mathcal{A}_\infty(D_R),\ \omega(C) = \chi(\pi(C))\right\},$$

where

$$\pi : \sum_A \to D_R, \qquad \pi(C) = \sum_{n=1}^{\infty} x_{n-1}\beta^n, \quad \text{and} \quad R > \frac{\beta}{1-\beta}.$$

On $\mathcal{F}(\sum_A)$, the transfer operator $\mathcal{S}_\phi$ acts as

$$\mathcal{S}_\phi(\omega)(C) = \sum_{\kappa\in\{\pm1\}} \exp\left(\kappa\mathcal{J}\sum_{n=1}^{\infty} x_{n-1}\beta^n\right)\chi(\psi_\kappa(\pi(C))),$$

where $\psi_\kappa(z) = \beta(\kappa + z)$ and $\omega = \chi \circ \pi$ (see [12]).

By making the change of variables $z = \sum\limits_{n=1}^{\infty} x_{n-1}\beta^n$, the operator $\mathcal{S}_\phi$ induces an operator acting on $\mathcal{A}_\infty(D_R)$ as follows:

$$\mathcal{L}(\varkappa)(z) = \sum_{\kappa\in\{\pm1\}} \exp(\kappa\mathcal{J}z)\chi(\beta(\kappa + z)).$$

Next, we state our main result.

Theorem 1. *Let (X, f), $X \subset \mathbb{R}^d$, be a dynamical system and $\varphi : X \to \mathbb{R}$ be an interaction potential, for which conditions* (C1)–(C2) *are satisfied. Then, for a such systems, there is no phase transition.*

To prove the theorem, we establish before the following two results.

Lemma 2. *The spectral radius $\rho(\mathcal{L}_q)$ of any operator $\mathcal{L}_q$ is equal to* $\exp(T(q))$, *provided that conditions* (C1)–(C2) *are fulfilled.*

Proof. For any admissible string $(\kappa_0, \kappa_1, \ldots, \kappa_{n-1}) \in \Omega^n$, we denote

$$\psi_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})} := \psi_{\kappa_{n-1}} \circ \cdots \circ \psi_{\kappa_0}.$$

By "admissible" we mean that $A_{\kappa_s,\kappa_{s+1}} = 1$ for every $s = 1, \ldots, n$. Let $\overline{z}_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})}$ be a fixed point of $\psi_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})}$, and recall that by condition (C2), the mappings ψ_κ are inverse branches of $\widehat{f}$. By this fact, we obtain that if

$$\psi_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})}(\overline{z}_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})}) = \overline{z}_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})},$$

then

$$\widehat{f}^{(n)}(\overline{z}_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})}) = \overline{z}_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})}.$$

Thus, there exists a one-to-one correspondence between periodic points of $\widehat{f}$ and the set of admissible strings. More precisely, the set $\{\overline{z}_{(\kappa_0,\kappa_1,\ldots,\kappa_{n-1})}\}$ is equal to $P_n(\widehat{f})$.

Now, if E_q is the maximum in modulus eigenvalue of $\mathcal{L}_q$, then

$$\log E_q = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{L}_q^n 1 = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in P_n(f)} \exp(-S_n(q\varphi(x))),$$

because of the commented correspondence between the configurations and periodic points. Therefore,

$$\rho(\mathcal{L}_q) = \exp(T(q)).$$

□

The concept of nuclearity can be extended to mappings from complete metric topological spaces (Frechet spaces) to Banach spaces (for details, see [6, 12]). There existss a particular class $\mathcal{F}$ of Frechet spaces with the property that any bounded mapping $\mathcal{L} : \mathcal{F} \to B$, where B is an arbitrary Banach space, is nuclear. Such spaces are also said to be nuclear.

Proposition 3. *The transfer operators* $\mathcal{L}_q : \bigoplus_{\kappa\in\Omega} \mathcal{A}_\infty(U_\kappa) \to \bigoplus_{\kappa\in\Omega} \mathcal{A}_\infty(U_\kappa)$, $U_\kappa \subset \mathbb{C}^d$, *for systems which satisfy conditions* (C1)–(C2) *are nuclear for any* q.

Proof. The demonstration is a direct application of the Grothendieck theory [6, 7]. First of all, we note that the operators are sums of operators of the form ϕC_ψ, where C_ψ is the composition operator $C_\psi(\chi)(z) = (\chi\circ\psi)(z)$. Thus, for studying the spectral properties of $\mathcal{L}$, it suffices to analyze composition operators. For this, we consider a suitable nuclear space $\mathcal{F}$ and prove that C_ψ defined on $\mathcal{F}$ is bounded. The space $\mathcal{H}(D)$ will be the space of holomorphic functions in a domain $D \subset \mathbb{C}^d$ equipped with the seminorm $\|\chi\|_K = \sup_{z\in K} |\chi(z)|$, where K is a compact subset of $\mathbb{C}^d$. It is known that the space $\mathcal{H}(D)$ with the topology of the seminorms $\| \ \|_K$ is nuclear [12]. Now, proving that composition operators are bounded in the space $\mathcal{H}(D)$, we prove that they are nuclear.

Let K be a compact subset of D such that $\psi(\overline{D}) \subset D \subset K$. We set

$$B_M := \{\chi \in \mathcal{H}(D) : \|\chi\|_K < M\}.$$

Thus,

$$\|C_\psi(\chi)\| = \sup_{z\in\overline{D}} \{|(\chi \circ \psi)(z)|\} < M.$$

Therefore, the set B_M is carried by C_ψ to a bounded set in $\mathcal{A}_\infty(D)$. To guarantee that C_ψ is defined on $\mathcal{A}_\infty(D)$, we just take the composition of C_ψ with the canonical injection $\iota : \mathcal{A}_\infty(D) \hookrightarrow \mathcal{H}(D)$. Thus, $C_\psi \circ \iota : \mathcal{A}_\infty(D) \to \mathcal{A}_\infty(D)$ is nuclear and, therefore, the transfer operators $\mathcal{L}_q$ are also nuclear. □

Proof of Theorem 1. The *Fredholm determinant* of $\mathcal{L}_q$ is

$$\det(1 - z\mathcal{L}_q) = \exp\left(- \sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{Tr}(\mathcal{L}_q^n) \right),$$

$z \in \mathbb{C}$. The fact of $\mathcal{L}_q$ is nuclear implies that the function $\det(1 - z\mathcal{L}_q)$ is entire in both variables z, q. Moreover, the set of zeros z of the Fredholm determinant agrees with the set of nonzero eigenvalues of $\mathcal{L}_q$.

To obtain from Eq. (7) a development of $\operatorname{Tr}(\mathcal{L}_q^n)$, we use the relationship

$$\det(1 - \mathcal{L}) = \sum_{p=0}^{d} (-1)^p \operatorname{Tr}\Big(\bigwedge_p \mathcal{L} \Big),$$

where $\bigwedge_p \mathcal{L}$ is the p-fold exterior product [12]. From this, a broader class of transfer operators can be obtained. If $\bigwedge_p \mathcal{B}(U_\varkappa)$ denotes the Banach space of the differential p-forms holomorphic on U_κ, then we define

$$\mathcal{L}_\varphi^{(p)} : \bigoplus_{\kappa\in\Omega} \bigwedge_p \mathcal{B}(U_\varkappa) \to \bigoplus_{\kappa\in\Omega} \bigwedge_p \mathcal{B}(U_\varkappa), U_\kappa \subset \mathbb{C}^d,$$

$$(\mathcal{L}_\varphi^{(p)}(w_p))_\kappa(z) = \sum_{\lambda\in\Omega_\kappa} A_{\kappa,\lambda} \exp(\varphi_\lambda(z)) \bigwedge_p D\psi_\lambda(z)(w_p)(\psi_\lambda(z)),$$

where $w_p \in \bigwedge_p \mathcal{B}(U_\varkappa)$ and $\bigwedge_p D\psi$ is the p-fold exterior product of the differential mapping $D\psi$ (considered as a linear operator). Here, $\mathcal{L}_\varphi^{(0)} = \mathcal{L}_\varphi$.

The Fredholm determinant is related with the *Ruelle zeta function* [20], which is defined as

$$\varsigma(z, q) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(q)\right).$$

This series converges in $\{z : |z| < \exp(-T(q))\}$. The Fredholm determinant is used to show that the Ruelle zeta function may have a meromorphic extension to the whole complex plane.

For example, if $d = 1$ [12], then

$$\varsigma(z, q) = \frac{\det(1 - z\mathcal{L}_q^{(1)})}{\det(1 - z\mathcal{L}_q^{(0)})}. \tag{8}$$

Therefore, z-poles of $\varsigma(z, q)$ are found among z-zeros of $\det(1 - z\mathcal{L}_q^{(0)})$, i.e., nonzero eigenvalues of $\mathcal{L}_q^{(0)} \equiv \mathcal{L}_q$. The zeta-function has a pole localized in $\exp(T(q))$.

Next, we will prove, as we commented in the Introduction, the absence of phase transitions, i.e., the analyticity of the free energy function $T(q)$, by showing that any operator $\mathcal{L}_q$ has an isolated eigenvalue. Then, since $\exp(T(q))$ is an isolated singularity of the mapping ς, the leading eigenvalue of $\mathcal{L}_q$ is isolated. □

To complete the analysis, we present a description of the spectrum of transfer operators under consideration as in Proposition 3 for proving the nuclearity, the operators of the form $\mathcal{L} = \phi C_\psi$, where C_ψ is the composition operator. For $\psi \in \mathcal{A}_\infty(D)$, this composition operator has a discrete spectrum [12]. Let $\psi \in \mathcal{A}_\infty(D)$. We have the equation for eigenvalues:

$$\mathcal{L}\chi(z) = \phi(z)\chi(\psi(z)) = E\chi(z).$$

Clearly, if $\chi(\overline{z}) \neq 0$, then an eigenvalue of $\mathcal{L}$ is $E = \phi(\overline{z})$, where $\overline{z}$ is a fixed point of ψ. If $\chi(\overline{z}) = 0$, then, differentiating with respect to z, we obtain the following form of the above equation:

$$D\phi(\overline{z}) \times \chi(\overline{z}) + \phi(\overline{z}) \times D\chi(\overline{z})D\psi(\overline{z}) = ED\psi(\overline{z}).$$

Thus, if $D\phi(\overline{z}) \neq 0$, then $E = \phi(\overline{z})D\psi(\overline{z})$. Now the eigenvalues of $\mathcal{L}$ (recall that it is discrete) form the set

$$E_n = \{\phi(\overline{z})(D\psi(\overline{z}))^n\}.$$

Recall that, by the Earle–Hamilton theorem, $\|D\psi(\overline{z})\| < 1$ and, therefore, 0 is the only point of accumulation.

Note that

$$\mathrm{Tr}(\mathcal{L}) = \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \phi(\overline{z})(D\psi(\overline{z}))^n = \frac{\phi(\overline{z})}{\det(1 - D\psi(\overline{z}))},$$

which is the Mayer trace formula.

3.1. Expanding analytic mappings with finite Markov partitions. We consider an interesting particular case, which was treated in [10] to calculate Hausdorff dimensions of some sets. A mapping $f : X \to X$, $X \subset \mathbb{R}^d$, is *expanding* with respect to a finite Markov partition $\mathcal{P} = \{W_1, W_2, \dots, W_k\}$ if

(i) f restricted to any W_j is injective;
(ii) $\|D_x(f^n)\| \geq \delta > 1$ for some $n \in \mathbb{N}$ and for every $x \in X$.

Recall that the set $\{W_1, W_2, \dots, W_k\}$ for (X, f) is a Markov partition of X if $W_i \cap W_i = \emptyset$, $i \neq j$, $f(W_j) = \bigcup_{l=0}^{k} W_{j_l}$, and $W_i = \overline{\operatorname{int}(W_i)}$. The transition rules are established as $A_{i,j=}1$ if $f(W_j) \cap W_i \neq \emptyset$. Note that, for the conditions for expanding mappings, there exist inverse branches ψ_j. If these branches are analytic, the mapping is called an *expanding analytic mapping*. The contraction property for the branches indeed holds for this kind of mappings.

For the potential interaction, $\varphi(x) := -\log \|Df(x)\|$, where f is an expanding analytic mapping; the free energy has the form

$$T(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in P_n(f)} \prod_{i=0}^{n-1} \|Df^i(x)\|^{-q}. \tag{9}$$

The transfer operators can be expressed as

$$\mathcal{L}_q(\varkappa)(z) = \sum_{i\in\Omega} \sum_{j:A_{i,j=1}} \|D\psi_{i,j}(z)\|^{-q} \chi(\psi_{i,j}(z)), \tag{10}$$

where $\psi_{i,j} : U_j \to U_i$ are the branches $\widehat{f} \circ \psi_{i,j} = \mathrm{id}\,|_{U_j}$ and $\widehat{f}$ is the holomorphic extension of f to complex neighborhoods U_j of W_j.

Thus, for a finite Markov partition, we have the validity of the results about the nuclearity of the transfer operator for any q, absence of phase transitions, etc.

3.2. A case of an expanding mapping with infinite partition. Important cases of analytic expanding mappings are obtained as follows. Let H^2 denote the hyperbolic plane in its disk model. Let Γ be a Klein group on H^2, i.e., a group which acts discontinuously on H^2. Recall that ξ is a limit point of Γ if and only if there exists a point $w \in H^2$ such that the Γ-orbit $\Gamma(w) = \{\gamma w : \gamma \in \Gamma\}$ accumulates at ξ. The set Λ is called the *limit set of* Γ. Since Γ acts discontinuously, $\Lambda \subset \partial H^2$. This action generates functions $f : \partial H^2 \to \partial H^2$, called *boundary hyperbolic mappings*. They are introduced by Series, and the details of the construction of them can be found in [21, 22]. We used them in connection with multifractal analysis [14] and dimension theory [15].

We have the following result.

Theorem (see [21, 22])**.** *There exist a one-sided finite type subshift Σ and a mapping $p : \Sigma \to \Lambda$ continuous and bijective, except possibly for a countable set of points such that $p \circ \tau = f \circ p$ ($\tau : \Sigma \to \Sigma$ is the Bernoulli shift).*

Therefore, if ∂H^2 is partitioned in finite arcs, then the absence of phase transitions is ensured. Now it is interesting to investigate a similar case, where the partition is infinite.

We consider the Gauss mapping $f : I \to I$ given by $f(\xi) = \xi^{-1} - [\xi^{-1}]$ ($I = [0, 1]$, $[a]$ is the integer part of a). It is a boundary hyperbolic mapping originated from the action of the modular group $\mathrm{SL}_2(\mathbb{Z})$ (for details, see [15, 21, 22]).

Let us consider the Markov partition $\mathcal{P} = \left\{ I_n = \left[\frac{1}{n+1}, \frac{1}{n}\right) \right\}_{n\in\mathbb{N}}$. We have $f\,|_{I_n}(\xi) = \frac{1}{\xi} - n$ and, therefore, $f\,|_{I_n}$ is analytic if $\xi \neq 0$ and $|(f^2)'| \geq 4$. The branches are $\psi_n(z) = \dfrac{1}{z+n}$.

Remark. The reason for which there is no finite Markov partition in this case can be explained by the above theorem. In the proof of this result, it is shown that ∂H^2 is into a set partitioned at most countable set of arcs $\{I_j\}$ such that $f(I_j) = \bigcup_{j_l=0}^{r} I_{j_l}$, i.e., $\mathcal{P} = \{I_j\}$ is a Markov partition for (Λ, f). The set $\mathcal{P}$ is infinite if and only if Γ contains parabolic elements (i.e., hyperbolic isometries with fixed points in ∂H^2, see [3]).

We assign to any $\xi \in \Lambda$ (the limit set of Γ) its expansion into a continued fraction:

$$\xi = \cfrac{1}{m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 \cdots}}},$$

and, therefore, we may identify ξ with the string $(m_0 m_1 \cdots)$; we denote this as $\xi \leftrightarrow (m_0 m_1 \cdots)$. Thus, we have $f^n(m_0, m_1, \ldots) = (m_{n+1} m_{n+2} \cdots)$, and, therefore, $\xi \in P_n(f)$ if and only if the continued fraction $(m_0, m_1, \ldots)$ associated with ξ is such that $m_{i+n} = m_i$ for each n. Now, for $\xi \in P_n(f)$, we can write $\xi \leftrightarrow [m_0, m_1, \ldots, m_n]$.

Thus, the partition function is defined as follows:

$$Z_n(q) = \sum_{m_0, m_1, \ldots, m_{n-1}} \prod_{j=0}^{n-1} \exp(q\varphi([m_j, m_{j+1}, \ldots, m_n, m_0, \ldots, m_{1+j-1}])). \tag{11}$$

Theorem 4. *Consider the dynamical system (I, f), where $I = [0, 1]$ and f is the Gauss mapping. It ensures absence of a phase transition for $q > \frac{1}{2}$.*

Proof. A condition that the above sums converge is that $|\varphi(\xi)| \sim |\xi|^{2q}$ as $\xi \to 0$ for some $q > 1$. The transfer operators act on $\mathcal{A}_\infty(D)$, where

$$D = \left\{ z \in \mathbb{C} : |z - 1| < \frac{3}{2} \right\}$$

and the mappings $\varphi \circ\ \psi_n$ must be holomorphic in the disk D [11]. The mapping $\varphi(\xi) = -\log|f^{'}(\xi)|$ satisfies the above conditions.

Now the transfer operators are given by [11]

$$\mathcal{L}_q(\varkappa)(z) = \sum_{n=1}^{\infty} \left(\frac{1}{z+n}\right)^{2q} \chi\left(\frac{1}{z+n}\right). \tag{12}$$

By the above-mentioned convergence reasons, the nuclearity of the transfer operators is ensured for $q > \frac{1}{2}$, and the Fredholm determinant $\det(1 - z\mathcal{L}_q)$ for the corresponding transfer operator is entire in z and analytic for $q > \frac{1}{2}$. □

3.3. The critical exponent of the group and dimension.

Definition. The critical exponent of a group Γ acting on H^2 is the number

$$\delta = \lim_{N \to \infty} \frac{1}{N} \log \operatorname{card}\{\gamma \in \Gamma : d_h(x, \gamma y) < N\},$$

where d_h is the hyperbolic metric on the hyperbolic disk and $x, y \in H^2$.

Simple hyperbolic geometry arguments imply that this limit is finite and, moreover, it does not depend on x or y. This is proved by considering the Poincaré series [17]

$$\eta_{x,y}(s) = \sum_{\gamma \in \Gamma} \exp(-s d_h(x, \gamma y)),$$

for which

$$\exp(-s d_h(x, y)) \eta_{y,y}(s) \le \eta_{x,y}(s) \le \exp(s d_h(x, y)) \eta_{y,y}(s)$$

(using triangle inequality). This shows that the critical exponent depends only on Γ. Under certain conditions, for example, if the group is geometrically finite, we have $\delta = \dim_H \Lambda$ [16], where as above, Λ denotes the limit set of the Γ-action on H^2 and $\dim_H$ is the Hausdorff dimension.

The potential $\varphi = -\delta \log |f'|$ has an equilibrium state μ_δ, precisely, the Patterson–Sullivan measure [18], which is concentrated on Λ, and, which is more important, it is the unique Gibbs state. An explicit proof of this fact, mentioned in [21], is presented in [15]. Therefore, the case $q = \delta$ can be analyzed by a special technique in order to establish the absence of phase transitions.

We complete this section by a brief comment about zeros of the free energy. Recall that the set of z-zeros of the Fredholm determinant $\det(1 - z\mathcal{L}_q)$ is equal to the set of nonzero eigenvalues of $\mathcal{L}_q$. Also, recall that the spectral radius of $\mathcal{L}_q$ is $\exp(T(q))$. Hence $\mathcal{L}_q$ has 1 as an eigenvalue if and only if $T(q) = 0$. This condition is equivalent to $\det(1 - \mathcal{L}_q) = 0$. Therefore, to find the maximum zero for the free energy $T(q)$, we should find the values of q for which $\det(1 - \mathcal{L}_q)$ vanishes.

For the case of boundary hyperbolic mappings, by the Bowen equation and Lemma 2, the largest zero of the free energy is given by $\dim_H \Lambda$. In some cases, for example, if the group is geometrically finite, $\delta = \dim_H \Lambda$.

In [10], an algorithm to compute the largest zero of the free energy for dynamics derived from expanding analytic mappings was designed. In this case, the largest zero of the corresponding free energy agrees with the Hausdorff dimension of the so-called limit set of the iterative scheme. These calculations can be extended to our more general systems by using the expansion of the Fredholm determinant from the Grothendieck theory and estimate from the Hadamard matrix algebra. We omit details since it is not an aim of this article.

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