Lett Math Phys (2008) 84:15–30
DOI 10.1007/s11005-008-0236-9

On Generalized Non-holonomic Systems

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Received: 29 August 2007 / Revised: 24 February 2008 / Accepted: 14 March 2008
Published online: 1 April 2008

Abstract. Generalized non-holonomic mechanical systems are analyzed from a geometric point of view. The existence and uniqueness of solutions, D'Alembert principle, Gauss principle of minimal constraint, the non-holonomic momentum and Gibbs–Appell equations for such systems are studied in an invariant Lagrangian framework.

Mathematics Subject Classification (2000). Primary 70F25, 70H45, 70G45.

Keywords. mechanics, generalized non-holonomic systems, geometric approach.

1. Introduction

Restricted mechanical systems have been traditionally studied assuming the vanishing of the work made by the constraining force on the virtual displacements. Nevertheless, in many interesting problems, for instance when the restriction is realized by the action of a servo mechanism [13], that assumption is not fulfilled: the constraining force yields a null work on vectors that non-necessarily coincide with the virtual displacements.

In order to encompass this kind of systems, Marle [13] developed, some years ago, the theory of *generalized non-holonomic systems* in a series of remarkable articles.

The analysis in [13], carried out in the Hamiltonian context, includes the proof of existence and uniqueness of solutions, the theory of reduction when symmetries are present, and a clever framework for dealing with forces in hamiltonian mechanics.

The aim of this letter is to consider some properties of these systems from the Lagrangian point of view, by using a simple geometrical setting.

An expression for the generator of the restricted dynamics in terms of the generator of the unrestricted one, furnishing a geometric proof of the existence and uniqueness of solutions, will be presented.

Furthermore, by means of suitable inner products defined on the tangent spaces of the configuration manifold, we will give a version of the Gauss principle of minimal constraint covering the generalized non-holonomic systems. As a particular case, we will recover the classical Chetaev's result about virtual displacements

[6,16]. Besides, the D'Alembert principle as stated in [19,20], the Gibbs–Appell equations [1,10] and the formula for non-holonomic momentum [2] will also be extended to the case of generalized non-holonomic systems.

We expect that the point of view adopted in this letter can contribute to enlighten some aspects of the remarkable Marle's contributions.

It is worth to note that the geometric setting presented below involves neither covariant derivatives, nor metrics on the phase space of velocities, nor jets.

The development of this letter is as follows:

In Section 2, a summary of the invariant setting introduced in [19,20] for the Lagrangian mechanics is presented.

In Section 3, we will give a geometric proof of the existence and uniqueness of solutions for generalized non-holonomic systems, and its expression in terms of the generator of the unrestricted system.

In Section 4, a version of the Gauss principle of minimal constraint encompassing the generalized non-holonomic systems is introduced.

In Section 5, two possible generalizations of the Gibbs–Appell equations as presented in [12] are analyzed.

The last section aims at extending the formula of the non-holonomic momentum [2] to the generalized case.

2. Lagrangian Mechanics in Invariant Form

In this section, we will make a summary of the framework introduced in [19,20] for the Lagrangian mechanics.

Given a differential manifold Q, we say that a vector $w \in T_{(q,v)}(TQ)$ is *vertical* if $\pi_*(w) = 0$, with $\pi : TQ \to Q$ the canonical projection $\pi(q, v) = q$. That is, w is vertical when it is tangent to the fiber.

For each $(q, v) \in TQ$, we will denote

$$\mathcal{V}_{(q,v)} := \{w \in T_{(q,v)}(TQ) \text{ s.t. } w \text{ is vertical}\}$$

and $\tau_{(q,v)} : T_q Q \to \mathcal{V}_{(q,v)}$ the canonical isomorphism given, in any coordinate system (q^i, v^i), by

$$\tau_{(q,v)}\left(u^i \frac{\partial}{\partial q^i}\right) = u^i \frac{\partial}{\partial v^i}.$$

When the point (q, v) is understood, we will write τ instead of $\tau_{(q,v)}$.

The space of tangent vector fields on a manifold $\mathcal{M}$ will be denoted by $\mathcal{X}(\mathcal{M})$.

The space of *vertical vector fields* on TQ, i.e. vector fields taking values in $\mathcal{V}_{(q,v)}$ at each $(q, v) \in TQ$, will be denoted by $\mathcal{V}(TQ)$.

For $\mathcal{T}(Q) := \{X : TQ \to TQ \text{ s.t. } X(q, v) \in T_q Q\}$, we will also denote by τ the isomorphism from $\mathcal{T}(Q)$ onto $\mathcal{V}(TQ)$ given by

$$[\tau(X)](q, v) = \tau[(X)(q, v)].$$

Forces are represented by *horizontal* differential 1-forms on TQ, that is 1-forms vanishing on vertical vectors. That is, if F is a horizontal 1-form, for every w tangent to TQ, $F(w)$ only depends on $\pi_* w$.

Let us denote

$$\mathcal{H}^1(TQ) := \{\text{horizontal 1-forms on } TQ\}. \tag{1}$$

The vector fields on TQ relevant for mechanics are the special ones [19,20]. A vector field $X \in \mathcal{X}(TQ)$ is *special* if, at each $(q, v) \in TQ$, $\pi_*(X(q, v)) = v$. In fact, only special vectors fields can be tangent to the lifting to TQ of curves in Q.

The space of special vector fields on TQ will be denoted by $\mathcal{S}(TQ)$.

Remark 1. It is clear that $\mathcal{S}(TQ)$ is an affine subspace of $\mathcal{X}(TQ)$ and $\mathcal{V}(TQ)$ is its associated vector subspace.

Let us now consider a mechanical system described by the smooth Lagrangian function

$$L(q, v) : TQ \to R,$$

where Q, its configuration space, is a n-dimensional manifold.

The equations of motion of such a system are given by the *Euler–Lagrange* equations. In any coordinate patch, they read

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i}, \quad i = 1, \ldots, n. \tag{2}$$

If an external force F^e acts on the system, the equations of motion become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i} + (F^e)_i, \quad i = 1, \ldots, n. \tag{3}$$

The symmetric matrix

$$M = (M_{ij})_{i,j=1,\ldots,n}, \tag{4}$$

with

$$M_{ij} := \frac{\partial^2 L}{\partial v^i \partial v^j}, \tag{5}$$

is assumed positive definite at every $(q, v) \in TQ$. Thus, for each $(q, v) \in TQ$, it gives rise to a inner product $\langle , \rangle_{M(q,v)}$ on $T_q Q$. Notice that, since L is C^∞, $\langle , \rangle_{M(q,v)}$ depends on (q, v)in a smooth way.

The *Lagrangian form* associated to L is the 2-form ω_L on TQ locally defined as

$$\omega_L := \left(\frac{\partial^2 L}{\partial q^i \partial v^j}\right) \mathrm{d}q^i \wedge \mathrm{d}q^j - \left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \mathrm{d}q^j \wedge \mathrm{d}v^i. \tag{6}$$

Remark 2. Note that $\omega_L = \mathrm{d}\left(\frac{\partial L}{\partial v^j}\mathrm{d}q^j\right)$. This definition of the Lagrangian form coincides with the one given in [20]. In [2] and [14] the definition is $\omega_L := -\mathrm{d}\left(\frac{\partial L}{\partial v^j}\mathrm{d}q^j\right)$

Since M is assumed positive definite at each point, ω_L turns out to be symplectic and then, it gives rise to an isomorphism

$$\alpha \dashrightarrow X_\alpha \tag{7}$$

from the space of 1-forms on TQ onto $\mathcal{X}(TQ)$, with X_α the unique vector field on TQ such that

$$\omega_L(X_\alpha, \cdot) = \alpha.$$

Remark 3. It is proved in [19] that

$$F \in \mathcal{H}^1(TQ) \Leftrightarrow X_F \in \mathcal{V}(TQ). \tag{8}$$

Straightforward computations show that, for $w^1 \in \mathcal{V}_{(q,v)}$,

$$\omega_L(w^1, w^2) = \langle \tau^{-1} w^1, \pi_* w^2 \rangle_{M(q,v)}, \quad \forall w^2 \in T_{(q,v)}(TQ). \tag{9}$$

The *energy function* associated to the Lagrangian L is

$$E_L(q,v) := \frac{\partial L}{\partial v}.v - L(q,v).$$

For simplicity, we will write

$$X_L := X_{-\mathrm{d}E_L}.$$

That is

$$\omega_L(X_L(q,v), w) = -\mathrm{d}E_L(w) \quad \forall w \in T_{(q,v)}TQ.$$

It is shown in [19,20] that X_L is the generator of the dynamics for the (unconstrained) system, i.e. $X_L \in \mathcal{S}(TQ)$ and a curve $q(t)$ on Q is a solution of the Euler-Lagrange equations (2) if and only if $(q(t), \dot{q}(t))$ is an integral curve of X_L on TQ.

It follows from Remark 3 and Equation (8) that $X_L + X_F \in \mathcal{S}(TQ)$ for any force F.

The *D'Alembert principle* in the sense of [19] or [20] (see also [2] or [14]) asserts that, if an external force F^e acts on the system, the dynamics is generated by the unique special vector field X_U satisfying

$$\omega_L(X_U, \cdot) + \mathrm{d}E_L = F^e. \tag{10}$$

That is, a curve $q(t)$ on Q is a solution of the Euler-Lagrange equations (3) if and only if $(q(t), \dot{q}(t))$ is an integral curve of

$$X_U := X_L + X_{F^e} \tag{11}$$

on TQ.

Now, let us consider the mechanical system restricted by the constraint

$$(q(t), \dot{q}(t)) \in \mathcal{C}, \tag{12}$$

with $\mathcal{C}$ a submanifold of TQ locally defined as the zeros of k smooth functions

$$\phi^l(q, v) = 0, \qquad l = 1, \dots, k, \tag{13}$$

such that their differential are independent at each $(q, v) \in \mathcal{C}$.

Notice that the set of generators of dynamics compatible with the restrictions is the set $\mathcal{S}(\mathcal{C})$ of special vector fields tangent to $\mathcal{C}$:

$$\mathcal{S}(\mathcal{C}) := \{X \in \mathcal{S}(TQ)/_{\mathcal{C}} \text{ s.t. } \mathrm{d}\phi^l(X) = 0 \text{ with } l = 1, \dots, k\}.$$

The constraint is *admissible* if, at each $(q, v) \in \mathcal{C}$,

$$\dim\left(\operatorname{span}\left\{\frac{\partial \phi^l}{\partial v^j} \mathrm{d}v^j(q, v)\right\}\right) = \dim\ (\operatorname{span}\ \{\mathrm{d}\phi^l(q, v)\}). \tag{14}$$

It is easy to show that, for admissible constraints, $\mathcal{S}(\mathcal{C})$ is non empty (see, for instance [19]).

In any coordinate patch, the equations of motion are now given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v^i} = \frac{\partial L}{\partial q^i} + (F^e + F^c)_i, \quad i = 1, \dots, n. \tag{15}$$

with F^c the force exerted by the physical realization of the constraint in order the restriction to be satisfied.

Consequently, the generator of the dynamics is the unique vector field X_R on $\mathcal{C}$ satisfying

$$\omega_L(X_L(q, v), w) + \mathrm{d}E_L(w) = (F^e + F^c)(w) \qquad \forall w \in T_{(q,v)}(\mathcal{C}).$$

It is equivalent to say

$$X_R := X_U + X_{F^c}. \tag{16}$$

Remark 4. We consider Lagrangians and constraints independent of time just for simplicity: the analysis for time-dependent ones is completely analogous.

The space of *virtual displacements* is, at each $(q,v)\in\mathcal{C}$ (see, for instance [18]),

$$\mathcal{D}_{(q,v)} := \{u\in T_qQ \text{ s.t. } \mathrm{d}\phi^l(q,v).\tau(u)=0,\ \ l=1,\dots,k\}. \tag{17}$$

If, at each $(q,v)\in\mathcal{C}$, the force F^c does a null work on the virtual displacements, the constraint is usually called *ideal*.

It is proved in [20] that, for ideal and admissible constraints, there exists a unique F^c such that $X_R = X_U + X_{F^c}\in\mathcal{S}(\mathcal{C})$. This X_R is the unique vector field on $\mathcal{C}$ generating the dynamics of the restricted system.

In this case, the equations of motion can be written as (*D'Alembert principle for non-holonomic systems*) [2,14,20]: X_R is the unique special vector field on $\mathcal{C}$ such that $X_R\in\mathcal{S}(\mathcal{C})$ and

$$\omega_L(X_R(q,v),w) + \ dE_L(w) = F^e(w), \tag{18}$$

$\forall\, w\in T_{(q,v)}(TQ)$ with $\pi_* w\in\mathcal{D}_{(q,v)}$.

3. Generalized Non-holonomic Systems

As quoted in Section 1, for the *generalized non-holonomic systems*, at each $(q,v)\in\mathcal{C}$ the constraining force F^c is known to vanish on a subspace of T_qQ, which can be different from $\mathcal{D}_{(q,v)}$.

We will call $\mathcal{W}_{(q,v)}$ that space and assume that it depends smoothly on (q,v).

Henceforth, all the maps between manifolds are assumed to be smooth.

Let us denote

$$\mathcal{W} := \{X:\mathcal{C}\to TQ \text{ s.t. } X(q,v)\in\mathcal{W}_{(q,v)}\}.$$

We will assume that $\mathcal{W}_{(q,v)}$ depends smoothly on (q,v) and

$$T_qQ = \mathcal{D}_{(q,v)}\oplus\mathcal{W}^{\perp}_{(q,v)}, \quad \forall\,(q,v)\in\mathcal{C}, \tag{19}$$

with $\mathcal{W}^{\perp}_{(q,v)}$ the $\langle,\rangle_{M(q,v)}$-orthogonal complement of $\mathcal{W}_{(q,v)}$ in T_qQ.

It is obvious that, in particular, this property holds in the classical non-holonomic case $\mathcal{W}_{(q,v)}=\mathcal{D}_{(q,v)}$. In fact, decompositions similar to (19) have already been considered in this case (see, for instance [7] or [12]).

The projections from T_qQ onto $\mathcal{D}_{(q,v)}$ and onto $\mathcal{W}^{\perp}_{(q,v)}$ associated to this direct sum will be denoted by $\Pi_{\mathcal{D}}$ and $\Pi_{\mathcal{W}^{\perp}}$, respectively.

Let us define

$$\begin{aligned}
&\mathcal{W}^0 := \{F\in\mathcal{H}^1(TQ)/_{\mathcal{C}} \quad \text{s.t., at each } (q,v)\in\mathcal{C},\ \ F/_{\mathcal{W}_{(q,v)}}\equiv 0\},\\
&\mathcal{W}^{\perp} := \{X:\mathcal{C}\to TQ \quad \text{s.t. } X(q,v)\in\mathcal{W}^{\perp}_{(q,v)},\ \ \forall(q,v)\in\mathcal{C}\},\\
&\mathcal{D} := \{X:\mathcal{C}\to TQ \quad \text{s.t. } X(q,v)\in\mathcal{D}_{(q,v)},\ \ \forall(q,v)\in\mathcal{C}\}.
\end{aligned}$$

In order the dynamics to satisfy $(q(t),\dot q(t))\in\mathcal{C}$ or, equivalently $X_R\in\mathcal{S}(\mathcal{C})$, the force $F^c\in\mathcal{W}^0$ must satisfy

$$X_R = X_U + X_{F^c}\in\mathcal{S}(\mathcal{C}). \tag{20}$$

Our proof of the existence and uniqueness of a force $F^c \in \mathcal{W}^0$ with this property will be based on the following two simple lemmas:

LEMMA 1. *For admissible constraints, the space $\mathcal{S}(\mathcal{C})$ is a non empty affine subspace of $\mathcal{X}(\mathcal{C})$. Its associated vector subspace is $\tau(\mathcal{D})$.*

Proof. As quoted above, $\mathcal{S}(\mathcal{C})$ is non-empty for admissible constraints.
On the other hand, if $X, Z \in \mathcal{S}(\mathcal{C})$, then $X - Z$ is vertical and

$$\mathrm{d}\phi^i(X - Z) \equiv 0.$$

So, according to definition (17), $\tau^{-1}(X - Z) \in \mathcal{D}$. □

LEMMA 2. *Let F belong to $\mathcal{H}^1(\mathcal{C})$. Then*

$$F \in \mathcal{W}^0 \Leftrightarrow \tau^{-1}(X_F) \in \mathcal{W}^\perp. \tag{21}$$

Proof. Due to Remark 3, X_F is vertical. Then, for each $w \in T_{(q,v)}TQ$ such that $\pi_*(w) = u$, we have from Equation (9)

$$F(q, v)(u) = \omega_L(X_F(q, v), w) = \langle \tau^{-1}(X_F(q, v)), u \rangle_{M(q,v)}.$$

□

The existence and uniqueness of $F^c \in \mathcal{W}^0$ such that $X_R = X_U + X_{F^c} \in \mathcal{S}(\mathcal{C})$ and its expression in terms of X_U is a direct consequence of the following geometric result:

PROPOSITION 1. *If condition (19) holds and the constraint is admissible, then*

$$\forall X \in \mathcal{S}(TQ)/_{\mathcal{C}}, \ \exists! F^X \in \mathcal{W}^0 \ \text{s.t.} \ X + X_{F^X} \in \mathcal{S}(\mathcal{C}).$$

If X_0 is any fixed element of $\mathcal{S}(\mathcal{C})$,

$$X_{F^X} = -\tau(\Pi_{\mathcal{W}^\perp}(\tau^{-1}(X - X_0))). \tag{22}$$

Proof. Let us choose any fixed vector field $X_0 \in \mathcal{S}(\mathcal{C})$. (Recall that $\mathcal{S}(\mathcal{C})$ is non empty for admissible constraints).
It is obvious that, for X_{F^X} as in Equation (22), $\tau^{-1}(X_{F^X}) \in \mathcal{W}^\perp$. So, the previous lemma implies $F^X \in \mathcal{W}^0$. (Note that we can take $\tau^{-1}(X - X_0)$ because $X - X_0$ is vertical.)
Now we have

$$\begin{aligned}(X + X_{F^X}) &= X_0 + (X - X_0) - \tau(\Pi_{\mathcal{W}^\perp}(\tau^{-1}(X - X_0))) \\ &= X_0 + \tau(\Pi_{\mathcal{D}}(\tau^{-1}(X - X_0)))\end{aligned}$$

Thus, from Lemma 1 we have $X + X_{F^X} \in \mathcal{S}(\mathcal{C})$.

On the other hand, if $\tilde{F} \in \mathcal{W}^0$ and $X + X_{\tilde{F}} \in \mathcal{S}(\mathcal{C})$ then, again from Lemma 1 we obtain $\tau^{-1}(X_{F^X} - X_{\tilde{F}}) \in \mathcal{D}$. But, from the previous lemma, since $F^X - \tilde{F} \in \mathcal{W}^0$, we have $\tau^{-1}(X_{F^X} - X_{\tilde{F}}) \in \mathcal{W}^\perp$. Condition (19) yields $(X_{F^X} - X_{\tilde{F}}) = 0$. So, $\tilde{F} = F^X$.

□

Notice that the characterization (22) of F^X is clearly independent of the choice of $X_0 \in \mathcal{S}(\mathcal{C})$.

Remark 5. On the hamiltonian side, via the Legendre transform, condition (19) becomes the condition under which the existence and uniqueness of solutions for generalized non-holonomic systems is shown in [13]: the constraining force must belong to the symplectic complement in $T_\xi(T^*Q)$, with respect to the canonical form on T^*Q, of $(\tilde{\pi}_*)^{-1}(\mathcal{D})$, where $\tilde{\pi} : T^*Q \to Q$ is the canonical projection.

Thus, the previous proposition can be seen as a geometric version, in the Lagrangian context, of that Marle's result.

It is worth to notice that we are only considering first order constraints (i.e when just $q(t)$ and $\dot{q}(t)$ are involved in them). Extensions of Marle's results to higher order constraints have been carried out in [3–5] and applied to pneumatic tyres and other models.

Remark 6. Lemma 2 and Proposition 1 enlighten the physical sense of condition (19): in order to determine the acceleration of the restricted system, we need n independent equations; $F^c \in \mathcal{W}^0$ implies $\tau^{-1}(X_{F^c}) \in \mathcal{W}^\perp$ and condition (19) guarantees that this amounts for $n-k$ equations, independent of the k equations derived from the constraint $X_U + X_{F^c} \in \mathcal{S}(\mathcal{C})$.

COROLLARY 1. *Under the assumptions of the previous proposition, X_R is the unique special vector field on $\mathcal{C}$ such that*

$$\tau^{-1}(X_R - X_U) \in \mathcal{W}^\perp. \tag{23}$$

Proof. It is a direct consequence of the previous proposition and Lemma 2. □

The previous result can be rewritten in the following way:

D'Alembert principle for generalized non-holonomic systems: For admissible constraints, if condition (19) holds and $F^c \in \mathcal{W}^0$, then the generator of the dynamics for the restricted system is the unique $X_R \in \mathcal{S}(\mathcal{C})$ such that, at each $(q, v) \in \mathcal{C}$, $\forall w \in T_{(q,v)}(TQ)$ s.t. $\pi_* w \in \mathcal{W}_{(q,v)}$,

$$\omega_L(X_R, w) + \mathrm{d}E_L(w) = F^e(w). \tag{24}$$

In fact, since $\omega_L(X_U, w) = F^e(w)$, $\forall w \in T_{(q,v)}(TQ)$, Equation (24) yields $\omega_L(X_R - X_U, w) = 0$, $\forall w \in T_{(q,v)}(TQ)$ s.t. $\pi_* w \in \mathcal{W}_{(q,v)}$. So, from Remark 3, $\tau^{-1}(X_R - X_U) \in \mathcal{W}^\perp$.

EXAMPLE. Two particles with masses m_1 and m_2, and the same electrical charge e, are jointed by a spring and move on a straight line l. At time t, their positions on l are given by $q^1(t)$ and $q^2(t)$ with $q^1 < q^2$.

Under the action of a time dependent electric field $\mathcal{E}(t)$, parallel to *l* and with (unknown) strength $E(t)$ constant on *l*, their velocities satisfy

$$\dot{q}^2 = A(q^1, q^2)\,\dot{q}^1, \tag{25}$$

with A a function such that

$$A(q^1, q^2) \neq \frac{m_1}{m_2}, \qquad \text{for } q^1 < q^2. \tag{26}$$

In order to apply the previous results for writing down the equations of motion of the system, we think about (25) as a constraint, and consider the force exerted by the electric field $\mathcal{E}(t)$ on the particles as the constraining force $F^c(t)$.

By doing so, we have:

$$Q = \{(q^1, q^2) \text{ s.t. } q^1 < q^2\}$$

and

$$\mathcal{C} = \{(q^1, q^2; u, A(q^1, q^2)u) \text{ s.t. } (q^1, q^2) \in Q\}.$$

At each $(q, v) = (q^1, q^2; v^1, v^2) \in \mathcal{C}$,

$$\mathcal{S}(\mathcal{C})_{(q,v)} = \{(v^1, v^2; w, A(q^1, q^2)w + b(q, v)\}, \tag{27}$$

with

$$b(q, v) = \left(\frac{\partial A}{\partial q^i} v^i\right) v^1 \tag{28}$$

and

$$\mathcal{D}_{(q,v)} = \{(u, A(q^1, q^2)u)\}. \tag{29}$$

The Lagrangian of the unrestricted system (i.e., disregarding the field $\mathcal{E}(t)$) is

$$L : Q \times R^2 \mapsto R \tag{30}$$

$$L(q^1, q^2; v^1, v^2) = \frac{1}{2}(m_1|v^1|^2 + m_2|v^2|^2) - U(q^1, q^2), \tag{31}$$

where

$$U(q^1, q^2) = \frac{1}{2}\kappa(q^2 - q^1 - a)^2 + \frac{k\,e^2}{q^2 - q^1} \tag{32}$$

whit k the constant of Coulomb, a the equilibrium length of the spring and κ its elasticity constant;

Since $E(t)$ is independent of q and the particles are identically charged, then

$$\mathcal{W}_{(q,v)} = \{(u, -u)\}. \tag{33}$$

Given that

$$M \equiv \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \tag{34}$$

the $\langle , \rangle_{M(q,v)}$-orthogonal complement of $\mathcal{W}_{(q,v)}$ is

$$\mathcal{W}^{\perp}_{(q,v)} = \left\{ \left(u, \frac{m_1}{m_2} u \right) \right\}. \tag{35}$$

It is easy to see that condition (19) is equivalent to assumption (26).

We now calculate X_{F^c}. A straightforward computation shows that the projections $\Pi_{\mathcal{D}}$ and $\Pi_{\mathcal{W}^{\perp}}$ are given by

$$\begin{aligned} \Pi_{\mathcal{D}}(u^1, u^2) &= \left(\frac{m_2 u^2 - m_1 u^1}{m_2 A - m_1},\ A \frac{m_2 u^2 - m_1 u^1}{m_2 A - m_1} \right) \\ \Pi_{\mathcal{W}^{\perp}}(u^1, u^2) &= \left(\frac{m_2 (A u^1 - u^2)}{m_2 A - m_1},\ \frac{m_1 (A u^1 - u^2)}{m_2 A - m_1} \right). \end{aligned} \tag{36}$$

According to Equation (27), we can take $X_0(q, v) = (v^1, v^2; 0, b(q, v))$ with b as in (28). If

$$X_U(q, v) = X_L(q, v) = (v^1, v^2; Y_U^1, Y_U^2),$$

Proposition 1 and Equation (36) yield

$$X_{F^c} = -\tau \left(\frac{m_2 (A Y_U^1 - (Y_U^2 - b))}{m_2 A - m_1},\ \frac{m_1 (A Y_U^1 - (Y_U^2 - b))}{m_2 A - m_1} \right). \tag{37}$$

The equations of motion of the system are given by $X_R = X_U + X_{F^c}$.

From Remark 3, we have that, if $X_{F^c} = X^i_{F^c} \dfrac{\partial}{\partial v^i}$, then $F^c = F^c_i \mathrm{d}q^i$, with $F^c_i = M_{ij} X^j_{F^c}$. So,

$$F^c_i = \frac{m_1 m_2 (A Y_U^1 - (Y_U^2 - b))}{m_2 A - m_1}.$$

Consequently,

$$E(t) = \frac{1}{e} \frac{m_1 m_2 (A Y_U^1 - (Y_U^2 - b))}{m_2 A - m_1}.$$

4. The Gauss Principle of Minimal Constraint

The classical *Gauss principle of minimal constraint* (see, for instance [11]) asserts that F^c vanishing on the virtual displacements is equivalent to characterize the generator X_R of the restricted dynamics as being the only vector field in $\mathcal{S}(\mathcal{C})$ such that, $\forall (q,v) \in \mathcal{C}$,

$$\|\tau^{-1}(X_R - X_U)(q,v)\|_{M(q,v)} = \inf_{X \in \mathcal{S}(\mathcal{C})} \|\tau^{-1}(X - X_U)(q,v)\|_{M(q,v)}, \tag{38}$$

with $\|.\|_{M(q,v)}$ the norm associated to $\langle .,.\rangle_{M(q,v)}$.

In order to establish a generalized Gauss principle of minimal constraints encompassing the generalized non-holonomic systems, we will consider for each $(q,v) \in \mathcal{C}$ a inner product $[.,.]_{(q,v)}$ on $T_q Q$, depending smoothly on (q,v), such that the subspaces $\mathcal{D}_{(q,v)}$ and $\mathcal{W}^{\perp}_{(q,v)}$ are $[.,.]_{(q,v)}$-orthogonal.

It is clear that, if condition (19) holds, then there exist inner products $[.,.]_{(q,v)}$ with these properties. For instance, we could take, for $u_1, u_2 \in T_q Q$,

$$[u_1, u_2]_{(q,v)} \;=\; \langle \Pi_{\mathcal{D}} u_1, \Pi_{\mathcal{D}} u_2\rangle_{M(q,v)} + \langle \Pi_{\mathcal{W}^{\perp}} u_1, \Pi_{\mathcal{W}^{\perp}} u_2\rangle_{M(q,v)}. \tag{39}$$

Obviously, $\Pi_{\mathcal{D}}$ and $\Pi_{\mathcal{W}^{\perp}}$ turn out to be orthogonal projections for $[.,.]_{(q,v)}$.

We will write $[.,.]$ instead of $[.,.]_{(q,v)}$ when no confusion can arise.

The norm on $T_q Q$ associated to $[.,.]_{(q,v)}$ will be denoted by $\|\cdot\|_{[,](q,v)}$.

We now introduce a generalized version of Gauss principle:

PROPOSITION 2. (Generalized Gauss principle of minimal constraint) *If condition* (19) *holds and the constraint is admissible, then X_R is the unique special vector field on $\mathcal{C}$ such that*

$$\|\tau^{-1}(X_R - X_U)(q,v)\|_{[,]} = \min_{X \in \mathcal{S}(\mathcal{C})} \|\tau^{-1}(X - X_U)(q,v)\|_{[,]}, \quad \forall (q,v) \in \mathcal{C}. \tag{40}$$

Proof. At each $(q,v) \in \mathcal{C}$,

$$\tau^{-1}(X - X_U)(q,v) = \tau^{-1}(X - X_R)(q,v) + \tau^{-1}(X_R - X_U)(q,v).$$

Now, since $X,\ X_R \in \mathcal{S}(\mathcal{C})$, then $\tau^{-1}(X - X_R)(q,v) \in \mathcal{D}_{(q,v)}$ (Lemma 1).

On the other hand, from Corollary 1 we have $\tau^{-1}(X_R - X_U)(q,v) \in \mathcal{W}^{\perp}_{(q,v)}$.

For $\mathcal{W}^{\perp}_{(q,v)}$ is the $[.,.]_{(q,v)}$-orthogonal complement of $\mathcal{D}_{(q,v)}$ in $T_q Q$,

$$\|\tau^{-1}(X - X_U)\|^2_{[,]} = \|\tau^{-1}(X - X_R)\|^2_{[,]} + \|\tau^{-1}(X_R - X_U)\|^2_{[,]}.$$

It is clear that the first term in the r.h.s in this equation vanishes only for $X = X_R$. $\square$

Remark 7. It is obvious that we can take $[.,.]_{(q,v)} = \langle , \rangle_{M(q,v)}$ if and only if $\mathcal{W}_{(q,v)} = \mathcal{D}_{(q,v)}$ at each $(q,v) \in \mathcal{C}$ and that we would obtain the classical Gauss principle of minimal constraint if we had $[.,.] = \langle , \rangle_{M(q,v)}$ in the equality (40). Then, from the previous proposition we can directly recover the well known Chetaev's result: for the (classical) Gauss principle of minimal constraint to hold, the constraining force has to make a null work along each $u \in \mathcal{D}_{(q,v)}$, $\forall (q,v) \in \mathcal{C}$, even if the restrictions are non-linear in the velocity variables.

EXAMPLE. In the example of the previous section, we can also determine X_R by means of the Gauss Principle:

If we take the inner product

$$[u,w]_{(q,v)} = \langle \Pi_{\mathcal{D}}\ u, \Pi_{\mathcal{D}}\ w\rangle_{M(q,v)} + \langle \Pi_{\mathcal{W}^\perp} u, \Pi_{\mathcal{W}^\perp} w\rangle_{M(q,v)},$$

then X_R is the unique vector field belonging to $\mathcal{S}(\mathcal{C})$ such that

$$\|\tau^{-1}(X_R - X_U)\|_{[,]} = \inf_{X \in \mathcal{S}(\mathcal{C})} \|\tau^{-1}(X - X_U)\|_{[,]}. \tag{41}$$

But we have seen that the minimum of

$$\|\tau^{-1}(X - X_U)\|^2_{[,]} = \|\Pi_{\mathcal{D}}(\tau^{-1}(X - X_U))\|^2_{M(q,v)} + \|\Pi_{\mathcal{W}^\perp}(\tau^{-1}(X - X_U))\|^2_{M(q,v)},$$

is reached when the first term in the r.h.s vanishes.

Thus, from (27) and (36), we have

$$X_R(q,v) = (v^1, v^2; Y_R(q,v), AY_R(q,v) + b(q,v))$$

with $Y_R = \dfrac{m_1 Y_U^1 + m_2(b - Y_U^2)}{m_1 - m_2 A}$.

5. Gibbs–Appell Equations

The Gibbs–Appell equations were introduced independently by Appell [1] and Gibbs [10] in the nineteenth century. A renewed attention has been payed to them in the last years (see, for instance [8,12] or [15]), in particular by their relation with Kane's method [9].

With our notations, the Gibbs–Appell function for classical non-holonomic systems [12] can be written as

$$\begin{aligned} G &: \mathcal{S}(TQ)/_{\mathcal{C}} \times \mathcal{C} \to R \\ G(X)(q,v) &= \frac{1}{2}\,\|\tau^{-1}(X - X_U)(q,v)\|^2_{M(q,v)}, \end{aligned} \tag{42}$$

where, as above, X_U is the generator of the unrestricted dynamics.

The vector field $X_R \in \mathcal{S}(\mathcal{C})$ generating the restricted dynamics is the solution of the Gibbs–Appell equation

$$D_X G(X_R)(\tau Y) = 0, \qquad \forall Y \in \mathcal{D}, \tag{43}$$

where

$$D_X G(X_R)(\tau Y)(q,v) := \frac{\mathrm{d}}{\mathrm{d}\epsilon}\ [G(X_R + \epsilon(\tau Y))(q,v)]_{\epsilon=0}.$$

For a generalized non-holonomic system with $F^c \in \mathcal{W}^0$, we can consider the same function G, or we can define another Gibbs–Appell function $G_{\mathcal{W}}$ besides the one defined by Equation (42):

$$G_{\mathcal{W}} : \mathcal{S}(TQ)/_{\mathcal{C}} \times \mathcal{C} \to R$$
$$G_{\mathcal{W}}(X)(q,v) = \frac{1}{2}\,\|\tau^{-1}(X - X_U)(q,v)\|_{[,]},$$

Both functions allow us for writing down equivalent Gibbs–Appell equations determining X_R, the generator of the restricted dynamics.

PROPOSITION 3. *Let us assume that condition* (19) *holds and that the constraint is admissible.*

The vector field X_R is the unique in $\mathcal{S}(\mathcal{C})$ satisfying

$$D_X G(X_R)(\tau Y) = 0, \quad \forall Y \in \mathcal{W}, \tag{44}$$

or the equivalent requirement

$$D_X G_{\mathcal{W}}(X_R)(\tau Z) = 0, \quad \forall Z \in \mathcal{D}. \tag{45}$$

Proof. At each $(q,v) \in \mathcal{C}$ and $\forall u \in T_q Q$,

$$D_X G(X_R)(\tau Y)(q,v) = \langle \tau^{-1}(X_R - X_U)(q,v), Y(q,v)\rangle_{M(q,v)}.$$

So, requirement (44) is equivalent to $\tau^{-1}(X_R - X_U)(q,v) \in \mathcal{W}^{\perp}_{(q,v)} \quad \forall (q,v) \in \mathcal{C}$.

Thus, as a consequence of Corollary 1, X_R is the generator of the restricted dynamics if and only if it belongs to $\mathcal{S}(\mathcal{C})$ and satisfies (44).

On the other hand,

$$D_X G_{\mathcal{W}}(X)(\tau Z)(q,v) = [\tau^{-1}(X - X_U)(q,v), Z(q,v)]_{(q,v)}$$

implies that condition (45) is equivalent to $\tau^{-1}((X_R - X_U)(q,v)) \in \mathcal{D}^{\perp[,]}_{(q,v)}$, the $[.,.]_{(q,v)}$-orthogonal complement of $\mathcal{D}_{(q,v)}$ in $T_q Q$.

But $[.,.]_{(q,v)}$ was chosen so as to have $\mathcal{W}^{\perp}_{(q,v)} = \mathcal{D}^{\perp[,]}_{(q,v)}$. Then, requirements (44) and (45) are equivalent. □

EXAMPLE. We now write down the Gibbs–Appell equations associated to G and $G_{\mathcal{W}}$ for the example considered in the previous sections.

If

$$X_U(q,v) = (v^1, v^2; Y_U^1(q,v), Y_U^2(q,v)) \quad \text{and}$$
$$X(q,v) = (v^1, v^2; Y(q,v), AY(q,v) + b(q,v)) \in \mathcal{S}(\mathcal{C}),$$

the extended Gibbs–Appell functions read

$$G(X)(q,v) = \frac{1}{2}\left\|(Y(q,v) - Y_U^1(q,v)\,,\ AY(q,v) + b(q,v) - Y_U^2(q,v)\right\|_{M(q,v)}^2$$

and

$$G_{\mathcal{W}}(X)(q,v) = \frac{1}{2}\left\|(Y(q,v) - Y_U^1(q,v)\,,\ AY(q,v) + b(q,v) - Y_U^2(q,v)\right\|_{[,]}^2.$$

Remark 8. The results presented hitherto could be rewritten in terms of any metric on TQ coinciding with $[\tau^{-1}, \tau^{-1}]_{(q,v)}$ when restricted to $\mathcal{V}_{(q,v)}$, $\forall (q,v) \in \mathcal{C}$.

If we can define $[.,.]_{(q,v)}$ only depending on q, for instance when $\mathcal{W} = \mathcal{D}$ and the restrictions are linear in v, these inner product give rise to a metric g on Q. In this case, we could take on TQ the Sasaki's extension g_S [17] of g.

6. The Generalized Non-holonomic Momentum Map

We now consider the action of a Lie group G on Q. The Lagrangian L is assumed to be invariant by the lifting of the action to TQ. We will see that an equation for the non-holonomic momentum map, similar to the one obtained in [2] for classical non-holonomic systems, can be deduced for the generalized case.

We will denote g the Lie algebra of G and, for each $(q,v) \in \mathcal{C}$,

$$g_{\mathcal{W}_{(q,v)}} := \{\xi \in g \quad \text{s.t.} \quad \xi_Q(q) \in \mathcal{W}_{(q,v)}\},$$

where $\xi_Q \in \mathcal{X}(Q)$ is the infinitesimal generator of the action on Q associated to ξ.

Let $g^{\mathcal{W}}$ be the bundle over $\mathcal{C}$ whose fiber at (q,v) is given by $g_{\mathcal{W}_{(q,v)}}$ and $[g^{\mathcal{W}}]^*$ its dual bundle.

The *non-holonomic momentum map* J^{nh} is the section of $[g^{\mathcal{W}}]^*$ defined by

$$J^{nh}(q,v)(\xi) := D_v L(\xi_Q(q)).$$

For classical non-holonomic constraints (i.e., $\mathcal{W} = \mathcal{D}$), it is shown in [2] that, if ξ is a section of the bundle $g^{\mathcal{D}}$ and $q(t)$ is a trajectory of the constrained system, the following *momentum equation* holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}[J^{nh}(\xi(q(t),\dot{q}(t)))] = D_v L\left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\ (\xi(q(t),\dot{q}(t)))\right]_Q\right) \tag{46}$$

For the generalized non-holonomic systems we have

PROPOSITION 4. *Let us assume that the Lagrangian L is invariant by the lifting to TQ of the action of the Lie group G on Q. Then, for every trajectory $(q(t),\dot{q}(t))$ of the restricted system and for any section ξ of the bundle $g^{\mathcal{W}}$, it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}[J^{nh}(\xi(q(t),\dot{q}(t)))] = D_v L\left(\left[\frac{\mathrm{d}}{\mathrm{d}t}\ \xi(q(t),\dot{q}(t))\right]_Q\right). \tag{47}$$

Proof. Just take $g_{\mathcal{W}_{(q(t),\dot{q}(t))}}$ instead of $g_{\mathcal{D}_{(q(t),\dot{q}(t))}}$ in the proof of Theorem 4.5 of [2]. □

The main applications of the non-holonomic momentum formula appear in the theory of reduction (see, for instance [2]). Although that theory has not been considered in this letter, the previous proposition was included just to present one more equation where the space $\mathcal{W}$ plays, in the generalized non-holonomic case, the role played by the space $\mathcal{D}$ in the classical non-holonomic systems.

Acknowledgements

We are indebted to Profs. R.E. Gamboa Saraví and R. Rodríguez for their useful suggestions about the example considered in this letter.

References

1. Appell, M.P.: Sur un forme nouvelle des équations de la Dynamique. Comptes Rendus Acad. Des Sciences **129**, 459 (1899)
2. Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., Murray, R.M.: Nohnolonomic mechanical systems with symmetry. Arch. Ration. Mech. Anal. **136**, 21–99 (1996)
3. Cendra, H., Ibort, A., de León, M., Martín de Diego, D.: A generalization of Chetaev's principle for a class of higher order nonholonomic constrainsts. J. Math. Phy. **45**, 2785 (2004)
4. Cendra, H., Grillo, S.: Lagrangian systems with higher order constraints. J. Math. Phys. **48**(5), 052904 (2007)
5. Cendra, H., Grillo, S.: Generalized nonholonomic mechanics, servomechanisms and related brackets. J. Math. Phys. **47**(2), 022902 (2006)
6. Cetaev, N.G.: On Gauss' principle. Izv Fiz-Mat **7**(3), 68–71 (1934)
7. de León, M., Martin de Diego, D.: On the geometry of non-holonomic Lagrangian systems. J. Math. Phys. **37**(7), 3389–3414 (1996)
8. Desloges, E.A.: The Gibbs–Appell equations of motion. Am. J. Phys. **56**, 9 (1988)
9. Desloges, E.A.: Relationship between Kane's equations and the Gibbs–Appell equations. J. Guid. Control Dyn. **10**(1), 120–122 (1987)
10. Gibbs, J.W.: On the fundamental formulae of dynamics. Am. J. Math **2**, 49 (1879)
11. Lanczos, C.: The variational principles of mechanics 3rd edn. University of Toronto Press, Toronto (1966)
12. Lewis, A.D.: The geometry of the Gibbs–Appell equation and Gauss principle of least constraint. Rep. Math. Phys. **38**(1), 11–28 (1996)
13. Marle, C.M.: Various approaches to conservative and non conservative non-holonomics systems. Rep. Math. Phys **42**, 211 (1998)
14. Marsden, J., Ratiu, T.: Symmetry and Mechanics, vols. I–II. Springer, Heidelberg (1999)
15. Mata, V., Provenzano, S., Cuadrado, J.L., Valero, F.: Inverse dynamic problem in robots using Gibbs–Appell equations. Robotica **20**(1), 59–67 (2002)
16. Neimark, J.I., Fufaev, N.A.: Dynamics of Non-holonomic systems. In: Translations of Mathematical Monographs, vol. 33. AMS, Providence (1972)
17. Sasaki, S.: On the differential geometry of tangent bundles of Riemannian manifolds. Tohoku Math. J. **10**, 338 (1958)

18. Solomin, J.E., Zuccalli, M.: A geometric approach to the extended D'Alembert principle of Udwadia-Kalaba-Hee-Chang. Q. Appl. Math. **63**, 269 (2005)
19. Vershik, A.M.: Classical and non-classical dynamics with constraints. In: Lectures Notes in Mathematics, vol. 1108, pp. 278–301. Springer, Heidelberg (1984)
20. Vershik, A.M., Fadeev, L.: Lagrangian mechanics in invariant form. Sel. Math. Sov. **1**, 339 (1981)