José Luis Castiglioni
Renato A. Lewin
Marta Sagastume

On a Definition of a Variety of Monadic ℓ-Groups

Abstract. In this paper we expand previous results obtained in [2] about the study of categorical equivalence between the category IRL_0 of integral residuated lattices with bottom, which generalize MV-algebras and a category whose objects are called **c**-differential residuated lattices. The equivalence is given by a functor $\mathsf{K}^\bullet$, motivated by an old construction due to J. Kalman, which was studied by Cignoli in [3] in the context of Heyting and Nelson algebras. These results are then specialized to the case of MV-algebras and the corresponding category $MV^\bullet$ of monadic MV-algebras induced by "Kalman's functor" $\mathsf{K}^\bullet$. Moreover, we extend the construction to ℓ-groups introducing the new category of *monadic ℓ-groups* together with a functor $\Gamma^\sharp$, that is "parallel" to the well known functor Γ between ℓ-groups and MV-algebras.

Keywords: Monadic ℓ-groups, Monadic MV algebras, Residuated Lattices.

1. Introduction

In [2] we proved that the category IRL_0 of integral residuated lattices with bottom is equivalent to DRL', a category whose objects are **c**-differential residuated lattices satisfying the condition ($\mathsf{CK}^\bullet$). The corresponding adjunction $\mathsf{C} \dashv \mathsf{K}^\bullet : IRL_0 \to DRL'$ extends the one between Heyting algebras and Nelson algebras obtained by Cignoli in [3].

Since we are building on previous work we recommend the reader to have the above mentioned paper [2] at hand while reading this paper.

After a preliminary section including previous definitions and results that will be used in the sequel, in Sect. 3 we prove that a c-differential residuated lattice satisfies the characterizing condition of DRL' if and only if we can define a unary operation κ that satisfies certain conditions. In particular, we prove that DRL' is a variety.

In Sect. 4, we find some properties of those algebras A of DRL' in which κA is closed under the involution. In this case, the operation κ turns out to be a quantifier, so we rename the variety (in the new signature) as monadic

Presented by **Constantine Tsinakis**; *Received* August 12, 2011

Studia Logica (2014) 102: 67–92
DOI: 10.1007/s11225-012-9464-1

DRL-algebras. We call MDRL the corresponding category. We prove that the category $iIRL_0$ of involutive integral residuated lattices with bottom is equivalent to MDRL.

In Sect. 5 we show that the category of MV-algebras, considered as a subcategory of $iIRL_0$, is equivalent to the category $\mathrm{MV}^\bullet$ (subvariety of monadic DRL-algebras), whose objects form a variety too.

In Sect. 6 we expand those results in the commutative case and put them in a larger context, in which the well known categorical relation between pointed ℓ-groups and MV-algebras is replicated in order to accommodate the new category $M\ell$-G, of monadic ℓ-groups,that is related to $\mathrm{MV}^\bullet$ in the same way as ℓ-groups are related to MV-algebras, that is, we define a functor $\Gamma^\sharp$ that sends each monadic ℓ-group to an interval and thus producing the following commutative diagram

$$\begin{tikzcd} LG \arrow[r, "\Gamma"] \arrow[d, "\mathsf{K}^u"'] & MV \arrow[d, "\mathsf{K}^\bullet"] \\ M\ell\text{-}G \arrow[r, "\Gamma^\sharp"'] & MV^\bullet \end{tikzcd}$$

Here $M\ell$-G is the new category and K^u is an unbounded version of Kalman's construction. In this way $\Gamma^\sharp$ is "parallel" to Γ.

The most important results of this paper are some categorical equivalences: from the category $iIRL_0$ to monadic differential residuated lattices (see Sect. 4), from the category of MV-algebras to monadic MV-algebras (Sect. 5) and from pointed ℓ-groups to monadic ℓ-groups (Sect. 6). The objects of "monadic" categories are algebras with a closure operator κ. This operator verifies conditions that allow us to consider it as some kind of quantifier.

In the groundbreaking work *"The algebra of Topology"*, McKinsey and Tarski [7] define the notion of closure algebra and establish a link between algebra and topology by proving, for example, that every closure algebra is isomorphic to a family of sets in a topological space. In *"On closed elements in closure algebras"*, the same authors [8] show a representation theorem for Heyting algebras: every Heyting algebra is isomorphic to the algebra of open sets of a closure algebra (see also [1], IX, 5, Theorem 9).

We can relate these results with some of those reported here. For example, in Sect. 4, we prove that if L is an object of the category $iIRL_0$ then L is isomorphic to the set of closed elements of $\mathsf{K}^\bullet(L)$ with its corresponding structure (see Theorem 11). Also, in Sect. 5 we prove an analogous result in the particular case of MV-algebras (see Corollary 15). Moreover, for pointed ℓ-groups, we show in 6.3 Lemma 19 that every object (G, g) is isomorphic to the closed elements of $\mathsf{K}^u(G, g)$ with its corresponding structure.

The ℓ-groups appear many times in algebraic logic research. According to Montagna and Tsinakis [9], *some of the most interesting algebras arising in algebraic logic are related to lattice-ordered groups*. In that paper, they show some results about ℓ-groups with a conucleus (interior operator with some "quantifier-like" conditions). For example, they prove that the category of cancellative commutative residuated lattices is equivalent to the category of abelian ℓ-groups with some conucleus. In a more general situation, they prove the categorical equivalence between residuated lattices and some category of involutive residuated lattices.

All these results can be related to the equivalences that appear in the previous diagram.

2. Preliminary Definitions and Results

All the residuated lattices considered in this paper are distributive and commutative, so we shall omit mentioning these two conditions in the sequel, assuming them as given. Recall that a residuated lattice is said to be *integral* if it is bounded above by the unit of the product.

Let IRL_0 be the category of integral residuated lattices with bottom. Let $\langle L, \wedge, \vee, \dot{}, \rightarrow, 0, 1\rangle$ be an object in this category. Define $\mathsf{K}^{\bullet}(L)$ in the following way,

$$\mathsf{K}^{\bullet}(L) = \{(x, y) \in L \times L^0,\ x \cdot y = 0\},$$

where L^0 is the order dual of L (see Sect. 7 of [2]).

An *involutive residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \sim, \rightarrow, e\rangle$ such that

1. $\langle L, \wedge, \vee, \cdot, \rightarrow, e\rangle$ is a residuated lattice,
2. $\sim$ is an involution of the lattice that is a dual automorphism; i.e., $\sim(\sim x) = x$ for all $x \in L$, and
3. $x \cdot y \leq z$ iff $x \leq\ \sim(y \cdot (\sim z))$.

Note that in any involutive residuated lattice we have that $\sim(y \cdot (\sim z)) = y \rightarrow z$.

An involutive residuated lattice is said to be *centered* if it has a distinguished element, called a *center*, that is, a fixed point for the involution.

A $\mathbf{c}$-*differential residuated lattice* is an integral involutive residuated lattice with bottom and center $\mathbf{c}$, satisfying the following "Leibniz condition"(Definition 7.2. of [2]):

$$\text{For any}\, x, y \in L, (x * y) \wedge \mathbf{c} = ((x \wedge \mathbf{c}) * y) \vee (x * (y \wedge \mathbf{c})).$$

We call such a $\mathbf{c} \in L$, a *Leibniz element*. We denote the category of $\mathbf{c}$-differential residuated lattices by DRL.

For $L \in IRL_0$, define the following operations in $\mathsf{K}^\bullet(L)$:

$$\begin{aligned}
(x, y) \vee (z, t) &:= (x \vee z, y \wedge t), \\
(x, y) \wedge (z, t) &:= (x \wedge z, y \vee t) \\
\sim (x, y) &:= (y, x) \\
(x, y) * (z, t) &:= (x \cdot z, (x \to t) \wedge (z \to y)) \\
(x, y) \to (z, t) &:= ((x \to z) \wedge (t \to y), x \cdot t)
\end{aligned}$$

Then, $\langle \mathsf{K}^\bullet(L), \vee, \wedge, \sim, *, \to, (0,1), (1,0), (0,0)\rangle$ is an object of DRL, where $\mathbf{c} = (0, 0)$ is the center. The assignment $L \mapsto \mathsf{K}^\bullet(L)$ extends to a functor $\mathsf{K}^\bullet : IRL_0 \to DRL$ (Lemma 7.1 of [2]).

For any $A \in DRL$, consider $\mathsf{C}(A) := \{a \in A \ : \ a \geq \mathbf{c}\}$. In $\mathsf{C}(A)$ define the product $x *_{\mathbf{c}} y = (x*y) \vee \mathbf{c}$, the bottom as the constant $\mathbf{c}$ and the other operations as those induced from A. Then we can prove that $\mathsf{C}(A)$ is an object of IRL_0 and that $A \longmapsto \mathsf{C}(A)$ defines another functor $\mathsf{C} : DRL \longrightarrow IRL_0$, which is left adjoint to $\mathsf{K}^\bullet$ (see Theorem 7.6. of [2]).

To obtain an equivalence we need to restrict the category DRL. We denote DRL' the full subcategory of DRL of those objects T that satisfy the following condition:

($\mathrm{C}\mathsf{K}^\bullet$) *For every pair of elements $x, y \in T$ such that $x, y \geq \mathbf{c}$ and $(x*y) \leq \mathbf{c}$ there exists $z \in T$ such that $z \vee \mathbf{c} = x$ and $\sim z \vee \mathbf{c} = y$.*

Then, we have the following result (Corollary 7.8. of [2]).

The adjunction $\mathsf{C} \dashv \mathsf{K}^\bullet : IRL_0 \to DRL$ restricts to an equivalence

$$\mathsf{C} \dashv \mathsf{K}^\bullet : IRL_0 \to DRL'.$$

This adjunction extends the one between Heyting algebras and Nelson algebras obtained by Cignoli in [3].

The following well known facts about integral involutive residuated lattices will be used throughout the paper without further mention.

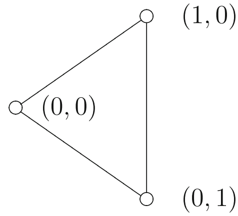

Figure 1. Image of [0,1] under $\mathsf{K}^\bullet$

$$\begin{aligned} x{*}\sim x &= 0 \\ x*y &\leq x\wedge y \\ x\to y &= \sim(x*\sim y) \end{aligned}$$

The condition ($\mathsf{CK}^\bullet$) was considered in order to restrict the category DRL to DRL' in a way that the DRL-morphism $t\mapsto(t\vee\mathbf{c},\sim t\vee\mathbf{c})$ from an object T into $\mathsf{K}^\bullet\mathsf{C}(T)$ results in an isomorphism.

To see the relationship between condition ($\mathsf{CK}^\bullet$) and the existence of the operator κ, let us take the example of the real interval $[0,1]$ endowed with its structure of MV-algebra.

The image of [0,1] under $\mathsf{K}^\bullet$ is the triangle in Figure 1. In the particular case depicted in this figure, let us write κ for the projection onto the segment joining $(0,1)$ with $(1,0)$, along the line joining the origin with $(1,0)$. Take $x=(a,0), y=(\neg a,0)$. Observe that $x,y\geq\mathbf{c}$ and $(x*y)\leq\mathbf{c}$. It is immediate that $z=\kappa(\sim x)$, satisfies ($\mathsf{CK}^\bullet$).

We prove that the assumption of ($\mathsf{CK}^\bullet$) is equivalent to the existence of an operation κ satisfying certain equations, as we will see in Theorem 1.

3. DRL' is a Variety

We begin with a theorem that improves on the characterization of the category DRL' defined in [2], helping us to prove that its objects form a variety.

THEOREM 1. *Let T be an algebra in DRL'. Then, there exists a map $\kappa: T\longrightarrow T$ such that*

$$\kappa x\vee\mathbf{c}=\mathbf{c}\to x, \tag{1}$$
$$\kappa x\wedge\mathbf{c}=x\wedge\mathbf{c}. \tag{2}$$

Conversely, if T is an object of DRL in which there exists an operator κ that satisfies Eqs. (1) and (2), then ($C\mathsf{K}^\bullet$) holds on T.

Proof. Let T be in DRL' and $x \in T$. Put $u = \mathbf{c} \to x, v = \sim x \vee \mathbf{c}$. It is easy to see that $u, v \geq \mathbf{c}$. Next observe that

$$u * v = ((\mathbf{c} \to x) * \sim x) \vee ((\mathbf{c} \to x) * \mathbf{c}),$$

but $\mathbf{c} * (\mathbf{c} \to x) * \sim x \leq x * \sim x = 0$, from where $(\mathbf{c} \to x) * \sim x \leq \mathbf{c} \to 0 = \mathbf{c}$.

On the other hand $(\mathbf{c} \to x) * \mathbf{c} \leq \mathbf{c}$ also, so $u * v \leq \mathbf{c}$.

We can apply condition $(C\mathsf{K}^\bullet)$ to u an v so there exists z such that $z \vee \mathbf{c} = u$ and $\sim z \vee \mathbf{c} = v$. We call $z = \kappa x$ and by distributivity, z is the unique that satisfies the Eqs. (1) and (2).

For the converse, assume $x, y \geq \mathbf{c}, x * y \leq \mathbf{c}$ and define $z = x \wedge \kappa(\sim y)$. Since $x * y \leq \mathbf{c}$ which, by (2), implies $x \leq y \to \mathbf{c} = \mathbf{c} \vee \kappa(\sim y)$, we have $z \vee \mathbf{c} = x \wedge (\kappa(\sim y) \vee \mathbf{c})$. So, $z \vee \mathbf{c} = x$.

Also, $z \wedge \mathbf{c} = x \wedge \kappa(\sim y) \wedge \mathbf{c} = x \wedge \sim y \wedge \mathbf{c} = \sim y$, so condition $(\mathsf{CK}^\bullet)$ holds. ■

Remark 1. A straightforward computation shows that for an element $x = (a, b) \in \mathsf{K}^\bullet(A), \kappa x = (\neg b, b)$, where $\neg b = b \to 0$.

In fact, assume that $\kappa(a, b)$ satisfies Eqs. (1) and (2). We have $\mathbf{c} \to (a, b) = (b \to 0, 0)$, so $\kappa(a, b) \vee \mathbf{c} = (b \to 0, 0)$. Also, $\kappa(a, b) \wedge \mathbf{c} = (0, b)$.

Corollary 2. *The map $x \longmapsto \kappa x$ is a closure operator. Moreover, $\kappa(x \wedge y) = \kappa x \wedge \kappa y$.*

Proof. From Eq. (1) of Theorem 1, we have $x \vee \mathbf{c} \leq \kappa x \vee \mathbf{c}$ (because $x \leq \mathbf{c} \to x$) and $x \wedge \mathbf{c} = \kappa x \wedge \mathbf{c}$, so, by distributivity, $x \leq \kappa x$.

Let us show $\kappa\kappa x = \kappa x$. We have that $\kappa\kappa x \wedge \mathbf{c} = \kappa x \wedge \mathbf{c} = x \wedge \mathbf{c}$ and $\sim (\mathbf{c} \to \kappa x) = \mathbf{c} * \sim \kappa x = \mathbf{c} * (\sim \kappa x \vee \mathbf{c}) = \mathbf{c} * (\sim x \vee \mathbf{c}) = \mathbf{c} * \sim x = \sim (\mathbf{c} \to x)$. So, $\mathbf{c} \to x = \mathbf{c} \to \kappa x = \kappa\kappa x \vee \mathbf{c}$.

In a similar way, we prove that $(\kappa x \wedge \kappa y) \vee \mathbf{c} = \mathbf{c} \to (x \wedge y)$ and $(\kappa x \wedge \kappa y) \wedge \mathbf{c} = (x \wedge y) \wedge \mathbf{c}$. ■

Remark 2. In what follows we also call DRL' (without distinction) the category whose objects have a unary operator κ in its signature, and verify the corresponding axioms.

Corollary 3. *The operation κ has the following properties:*

1. $\kappa x = 0$ *if and only if* $x = 0$,
2. $\kappa x = 1$ *if and only if* $x \geq \mathbf{c}$,
3. $\kappa x = \kappa(x \wedge \mathbf{c})$,
4. $x = (x \vee \mathbf{c}) \wedge \kappa x$,
5. $\sim x * \mathbf{c} = \sim \kappa x * \mathbf{c}$,

6. $\kappa x = \kappa y$ *if and only if* $x \wedge \mathbf{c} = y \wedge \mathbf{c}$*.*

7. $\sim \kappa x \leq \kappa \sim x$*.*

Corollary 4. *Let* T *be in* DRL' *and* $\kappa T = \{x \in T : x = \kappa x\}$*. Then*

1. *For every* $x \in T$ *there exist two elements* u *and* v *in* $\mathsf{C}(T)$ *such that* $x = u \wedge \kappa(\sim v)$*.*
2. *For every* $x \in T$ *there exist two elements* r *and* s *in* $\kappa T, \sim r \leq s$*, such that* $x = (\sim r \vee \mathbf{c}) \wedge s$*.*

Proof. *1.* This follows from *3* and *4* of the previous Corollary and the fact that $\kappa(x \wedge \mathbf{c}) = \kappa(\sim (\sim x \vee \mathbf{c}))$, by taking $u = x \vee \mathbf{c}, v = \sim x \vee \mathbf{c}$.

2. Define $r = \kappa(\sim x)$ and $s = \kappa x$. The claim follows from item *4* of Corollary 3 and the fact that $x \vee \mathbf{c} = \sim (\kappa(\sim x)) \vee \mathbf{c}$. We have $\sim r \leq s$ by item *7* Corollary 3. ■

Corollary 5. *Let* T *be in* DRL'*. Then,*

1. $\mathsf{C}(T)$ *generates* T *and*

2. $\kappa T \cup \{\mathbf{c}\}$ *generates* T*.*

4. Monadic Differential Residuated Lattices

In this section we study the particular case of an object T of DRL' such that κT is closed under the involution $\sim$.

The concept of quantifier, originally associated by P.R. Halmos (see [6]) to the classical propositional calculus and Boolean algebras, has been generalized in many contexts, mainly from an algebraic point of view. In the original approach of Halmos, a quantifier ∇ is a map satisfying the following conditions:

1. $\nabla(p \wedge \nabla q) = \nabla p \wedge \nabla q$
2. $\nabla 0 = 0$
3. $p \leq \nabla p$

From 1 and 2 we can deduce:

4. $\nabla(p \vee q) = \nabla p \vee \nabla q$
5. $\nabla\nabla p = \nabla p$
6. $\nabla 1 = 1$

Conditions 1, 3, 4 and 5 imply that ∇ is a closure operator. Condition 5 means that the image of ∇ is the set of fixed points of ∇. Conditions 1–4 imply that the image of ∇ is a subalgebra. This two results seem to be the essential ones, so, in a variety of algebras, any "reasonable" definition of quantifier must imply them. For example, in [4], for the definition of a Q-distributive lattice the author requests conditions 1–4 to hold. Also, see [10] where a quantifier for De Morgan algebras is defined by 1–4, plus the condition $\nabla \sim \nabla x = \sim \nabla x$, which means that the image of ∇x is closed under $\sim$. In [5] the authors define a quantifier in MV-algebras as a map such that E1–E6 hold, where E1, E2 are 3 and 4 above respectively, and

(E3) $\nabla\neg\nabla x = \neg\nabla x$,

(E4) $\nabla(\nabla x \oplus \nabla y) = \nabla x \oplus \nabla y$,

(E5) $\nabla(x \odot x) = \nabla x \odot \nabla x$

(E6) $\nabla(x \oplus x) = \nabla x \oplus \nabla x$

We will call *monadic differential residuated lattices* or, $MDRL$, the full subcategory of DRL' whose objects satisfy the following condition:

$$(\mathrm{Inv}^{\bullet}) \qquad \sim \kappa x = \kappa(\sim \kappa x).$$

Let $iIRL_0$ be the full category of IRL_0 whose objects satisfy

$$(\mathrm{Inv}) \quad \neg\neg a = a,$$

where $\neg a := a \to 0$. In the rest of this section we will prove that κ is a quantifier and the category MDRL is equivalent to $iIRL_0$.

LEMMA 6. *Let T be an object of DRL'. Then the following conditions are equivalent:*

$$\begin{array}{ll} (\mathit{Inv}^{\bullet}) & \sim \kappa x = \kappa(\sim \kappa x), \\ (\mathit{Inv}^{\bullet'}) & x \vee \mathbf{c} = \kappa(\sim x) \to \mathbf{c}, \\ (C) & \kappa T \textit{ is closed under the involution} \sim . \end{array}$$

PROOF. Let us prove the equivalence of (Inv$^{\bullet}$) and (Inv$^{\bullet'}$).

From Eqs. (1) and (2) of Theorem 1 and (Inv$^{\bullet}$) we deduce that $\kappa(\sim x) \to \mathbf{c} = \mathbf{c} \to\sim \kappa(\sim x) = \kappa(\sim \kappa(\sim x)) \vee \mathbf{c} = \sim \kappa(\sim x) \vee \mathbf{c}$. Also, from Eq. (2) in Theorem 1, $\kappa(\sim x) \wedge \mathbf{c} = \sim x \wedge \mathbf{c}$. Thus, $\sim \kappa(\sim x) \vee \mathbf{c} = x \vee \mathbf{c}$.

To prove the converse, it suffices to see that $\sim \kappa x$ satisfies Eqs. (1) and (2) of Theorem 1, that is to say, that $\sim \kappa x \vee \mathbf{c} = \mathbf{c} \to\sim \kappa x$ and $\sim \kappa x \wedge \mathbf{c} =\sim \kappa x \wedge \mathbf{c}$.

From (Inv$^\bullet$') we deduce that $\sim \kappa x \vee \mathbf{c} = \kappa \sim (\sim \kappa x) \to \mathbf{c} = \kappa\kappa x \to \mathbf{c} = \kappa x \to \mathbf{c} = \mathbf{c} \to \sim \kappa x$.

The equivalence with (C) is immediate. ∎

LEMMA 7. *If* **(Inv$^\bullet$)** *holds in T, then*

1. *for every $x \leq \mathbf{c}, \mathbf{c} * \kappa x = \mathbf{c} \wedge \kappa x = x$,*
2. *for every $x, y \leq \mathbf{c}, \kappa x * y = \kappa y * x$,*
3. *for every x, $\sim \kappa x = \kappa((\sim x) * \mathbf{c})$,*
4. *for every $x, y \leq \mathbf{c}, \kappa x * \kappa y = \kappa(\kappa x * y)$, so $\kappa x * \kappa y = \kappa(\kappa x * \kappa y)$,*
5. *for every $x, y \leq \mathbf{c}, \kappa x \to \kappa y = \sim \kappa \sim (x \to \kappa y)$,*
6. *$(x \to \mathbf{c}) \to \mathbf{c} = x \vee \mathbf{c}$,*
7. *for every $z \geq \mathbf{c}, z = \kappa(z * \mathbf{c}) \vee \mathbf{c}$,*
8. *for $u \geq \mathbf{c}, v \geq \mathbf{c}, u = v$ if and only if $u * \mathbf{c} = v * \mathbf{c}$.*

PROOF. In order to prove *1*, observe that

$$\mathbf{c} * \kappa x = \mathbf{c} * \sim (\sim \kappa x) = \sim (\mathbf{c} \to \sim \kappa x) = \sim (\sim \kappa x \vee \mathbf{c}) = \kappa x \wedge \mathbf{c} = x \wedge \mathbf{c} = x.$$

Item *2* follows from *1*, since $\kappa x * \kappa y * \mathbf{c} = \kappa x * y = \kappa y * x$.

From the definition of $\kappa, \mathbf{c} * \sim x = \sim \kappa x \wedge \mathbf{c}$, so *3* follows from Corollary 2 and Corollary 3,*2*.

To prove *4*, we observe that $\kappa x * \kappa y$ is the closure of $x * \kappa y$, that is, $(\kappa x * \kappa y) \vee \mathbf{c} = \mathbf{c} \to (x * \kappa y)$ and $(\kappa x * \kappa y) \wedge \mathbf{c} = (x * \kappa y) \wedge \mathbf{c}$. In fact, using Leibniz condition, $(\kappa x * \kappa y) \wedge \mathbf{c} = ((\kappa x \wedge \mathbf{c}) * \kappa y) \vee ((\kappa y \wedge \mathbf{c}) * \kappa x) = (x * \kappa y) \vee (y * \kappa x) = x * \kappa y = (x * \kappa y) \wedge \mathbf{c}$.

To prove $(\kappa x * \kappa y) \vee \mathbf{c} = \mathbf{c} \to (x * \kappa y)$ we use **(Inv$^\bullet$')** to obtain $(\kappa x * \kappa y) \vee \mathbf{c} = \kappa(\sim (\kappa x * \kappa y)) \to \mathbf{c} = \mathbf{c} \to \sim (\kappa(\sim (\kappa x * \kappa y)))$. From *3* we deduce that $\sim (\kappa(\sim (\kappa x * \kappa y))) = \kappa(\kappa x * \kappa y * \mathbf{c}) = \kappa(x * \kappa y)$. The last step is accomplished taking into account that, in general, $\mathbf{c} \to \kappa a = \kappa\kappa a \vee \mathbf{c} = \kappa a \vee \mathbf{c} = \mathbf{c} \to a$.

To prove *5*, we translate '$\to$' in terms of '$*$' in 4 to obtain

$$\sim (\kappa x \to \kappa y) = \kappa x * \sim \kappa y = \kappa x * \kappa \sim \kappa y = \kappa(x * \sim \kappa y) = \kappa \sim (x \to \kappa y).$$

Let us prove *6*. By definition of κ we have: $x \to \mathbf{c} = \mathbf{c} \to \sim x = \kappa(\sim x) \vee \mathbf{c}$. So: $(x \to \mathbf{c}) \to \mathbf{c} = (\kappa(\sim x) \vee \mathbf{c}) \to \mathbf{c} = \kappa(\sim x) \to \mathbf{c} = \kappa(\sim \kappa(\sim x)) \vee \mathbf{c}$. On the other hand, $x \vee \mathbf{c} = \sim (\sim x) \vee \mathbf{c} = \sim \kappa(\sim x) \vee \mathbf{c}$, which is the same, by (Inv$^\bullet$).

We now prove *7*. From *1*, $\sim z = \mathbf{c} \wedge \kappa \sim z$, that is, $z = \mathbf{c} \vee \sim \kappa \sim z$. Moreover, $z * \mathbf{c} = \sim (z \to \mathbf{c}) = \sim (\mathbf{c} \to \sim z) = \sim (\kappa \sim z \vee \mathbf{c})$, where last equality holds by Eq. (2) of Theorem 1. Then, $z * \mathbf{c} = \sim \kappa \sim z \wedge \mathbf{c}$, from where, $\kappa(z * \mathbf{c}) =\sim \kappa \sim z$ (by Corollary 2 and (2) of Corollary 3); so $z = \mathbf{c} \vee \kappa(z * \mathbf{c})$.

The last assertion follows from *7*. ∎

THEOREM 8. *Let A be an object of the category IRL_0 and T be an object of DRL'. Then,*

1. *If* (Inv) *holds in A, then* (Inv$^\bullet$) *holds in $\mathsf{K}^\bullet(A)$.*
2. *If* (Inv$^\bullet$) *holds in T, then* (Inv) *holds in $\mathsf{C}(T)$ and in $\kappa(T)$.*

PROOF. Using item *3* of Corollary 3, it suffices to take $x = (0, a)$, so $x \leq \mathbf{c}$, and prove that $\sim \kappa x = \kappa(\sim \kappa x)$. But $\sim \kappa x =\sim (\neg a, a) = (a, \neg a)$ and $\kappa(\sim \kappa x = (\neg\neg a, \neg a) = (a, \neg a)$.

For the second part, we have from item *6* of Lemma 7, that $z \geq \mathbf{c}$, implies that $(z \to \mathbf{c}) \to \mathbf{c} = z \vee \mathbf{c} = z$. Also, by (C) Lemma 6, $\sim$ is well defined in $\kappa(T)$, so (Inv) holds. ∎

COROLLARY 9. *Let A be an object of $iIRL_0$ considered as an object of IRL_0. Then, $\kappa\mathsf{K}^\bullet(A)$ is a subalgebra of the reduct $\langle \mathsf{K}^\bullet A, \vee, \wedge, *, \sim, 0, 1\rangle$. Moreover, $\kappa\mathsf{K}^\bullet(A) \cup \{\mathbf{c}\}$ generates $\mathsf{K}^\bullet A$.*

PROOF. By item *4* of Lemma 7, Corollary 2 and the previous Theorem, $\mathsf{K}^\bullet A$ is closed under the operations. Corollaries 4 and 5 prove the second assertion. ∎

THEOREM 10. *The assignment $T \longmapsto \kappa T$ extends to a functor $\mathcal{K} : MDRL \longrightarrow iIRL_0$.*

PROOF. By Theorem 8, $\mathcal{K}$ is well defined on objects. Given a morphism $f : T \longrightarrow U$ in $MDRL$, we define $\mathcal{K}(f)$ as the restriction of f to κT. ∎

THEOREM 11. *The functor $\mathcal{K}$ is left adjoint to $\mathsf{K}^\bullet$. Moreover, $\mathcal{K} \dashv \mathsf{K}^\bullet$ is an equivalence.*

PROOF. The proof is based on the fact that every object T in the category $MDRL$, verifies $\mathsf{C}(T) \approx \mathcal{K}(T)$.

We have already proved in Corollary 9 that $\mathcal{K}(T)$ is closed under $\vee, \wedge, *$ and $\sim$, and contains 0 and 1. We define $\alpha : \mathsf{C}(T) \longrightarrow \mathcal{K}(T)$ to be given by $\alpha(u) = \kappa(u * \mathbf{c})$ and prove that it is an isomorphism.

From item *7* of Lemma 7 we deduce the injectivity of α. To prove surjectivity, observe that, for $\kappa x \in \kappa T, \kappa x = \alpha(\kappa x \vee \mathbf{c})$. Indeed, we can assume $x \leq \mathbf{c}$, so by the same lemma, items *2* and *4*, $\alpha(\kappa x \vee \mathbf{c}) = \kappa(\kappa x * \mathbf{c}) = \kappa(x * \kappa\mathbf{c}) = \kappa x$.

Also, we see that $\alpha(\mathbf{c}) = 0$ and $\alpha(1) = 1$.

Let us prove that α preserves $*$. We have to show that, for u, v in $\mathsf{C}(T)$, $\kappa((u *_{\mathbf{c}} v) * \mathbf{c}) = \kappa(u * \mathbf{c}) * \kappa(v * \mathbf{c})$. In fact, again from Lemma 7,*2* and *4*, $(\kappa(u * \mathbf{c})) * (\kappa(v * \mathbf{c})) = \kappa((u * \mathbf{c}) * (\kappa(v * \mathbf{c}))) = \kappa(u * v * \mathbf{c})$, because $v = \kappa(v * \mathbf{c}) \vee \mathbf{c}$ implies $v * \mathbf{c} = \kappa(v * \mathbf{c}) * \mathbf{c}$.

Moreover, we can prove that $\alpha(\neg u) =\sim \alpha(u)$, where $\neg u = u \to \mathbf{c}$. Indeed, $\alpha(u \to \mathbf{c}) = \kappa((u \to \mathbf{c}) * \mathbf{c}) = \kappa(\sim (u * \mathbf{c}) * \mathbf{c})$. On the other hand, $\sim \alpha(u) =\sim \kappa(u * \mathbf{c})$, and the equality follows from, Lemma 7,*3*.

It is easy to see, from the fact that **(Inv)** holds in $\mathsf{C}(T)$ (see Theorem 8,*2*), that $u \to v = \neg(u *_{\mathbf{c}} \neg v)$.

Therefore, we can deduce $\alpha(u \to v) = \alpha(u) \to \alpha(v)$.

Finally we show that $\alpha(u \wedge v) = \alpha(u) \wedge \alpha(v)$, that is, $\kappa((u \wedge v) * \mathbf{c}) = \kappa(u * \mathbf{c}) \wedge \kappa(v * \mathbf{c})$. In fact, we have that $\neg(u \wedge v) = \neg u \vee \neg v$ so $\sim ((u \wedge v) * \mathbf{c}) =\sim (u * \mathbf{c}) \vee \sim (v * \mathbf{c})$. Then, $(u \wedge v) * \mathbf{c} = (u * \mathbf{c}) \wedge (v * \mathbf{c})$ and the result follows. From this we can easily deduce the preservation of $\vee$.

Therefore, for every A in $iIRL_0$ we have $A \approx \mathsf{C}\mathsf{K}^{\bullet}(A) \approx \kappa\mathsf{K}^{\bullet}(A)$. The isomorphism $A \approx \kappa\mathsf{K}^{\bullet}(A)$ is given by the assignment $a \longmapsto (a, \neg a)$.

Similarly, for every T in $MDRL, T \approx \mathsf{K}^{\bullet}\mathsf{C}(T) \approx \mathsf{K}^{\bullet}\kappa(T)$. The isomorphism $T \approx \mathsf{K}^{\bullet}\kappa(T)$ is given by the assignment $u \longmapsto (\kappa(u * \mathbf{c}), \kappa(\sim u * \mathbf{c}))$. ■

Corollary 12. *The operation κ verifies the following conditions.*

(E'1) $x \leq \kappa x$

(E'2) $\kappa(\kappa x \vee \kappa y) = \kappa x \vee \kappa y$

(E'3) $\kappa \sim \kappa x =\sim \kappa x$

(E'4) $\kappa(\kappa x + \kappa y) = \kappa x + \kappa y$, *where* $x + y := \sim (\sim x * \sim y)$,

(E'5) $\kappa(x * x) = \kappa x * \lambda x \leq \kappa x * \kappa x$, *where* $\lambda x :=\sim \kappa \sim x$,

(E'6) $\kappa(x + y) = \kappa x + \kappa y$

Proof. The first four items are immediate. We prove E'5).

First, we prove that $\kappa x * \lambda x = \kappa(\mathbf{c} * x * \kappa x)$. In fact, by *3* of Lemma 7, $\kappa x * \lambda x = \kappa x * \kappa(x * \mathbf{c})$ and by *4* of the same lemma, $\kappa x * \kappa(x * \mathbf{c}) = \kappa(\mathbf{c} * x * \kappa x)$.

Then we see that $\kappa(x * x) = \kappa(\mathbf{c} * x * \kappa x)$.

From *3* of Corollary 3 and Leibnitz condition, $\kappa(x * x) = \kappa((x * x) \wedge \mathbf{c}) = \kappa((x \wedge \mathbf{c}) * x)$.

By E'3), the definition of κ, *1* of Lemma 7 and item *3* of Corollary 3, $\kappa((x \wedge \mathbf{c}) * x) = \kappa((\kappa x \wedge \mathbf{c}) * x) = \kappa((\kappa(x \wedge \mathbf{c}) \wedge \mathbf{c}) * x) = \kappa((\kappa(x \wedge \mathbf{c}) * \mathbf{c}) * x)$, which is equal to $\kappa(\mathbf{c} * x * \kappa x)$.

In order to prove E'6), it suffices to see that $\lambda(\sim x* \sim y) = \sim \kappa x* \sim \kappa y$. We have $\lambda(\sim x* \sim y) =\sim \kappa(x+y) = \kappa(\sim (x+y) * \mathbf{c})$, by (3) of Lemma 7, and $\kappa(\sim (x+y) * \mathbf{c}) = \kappa(\sim x* \sim y * \mathbf{c})$.

On the other hand, by (3) and (4) of Lemma 7, $\sim \kappa x* \sim \kappa y = \kappa(\sim x * \mathbf{c})* \sim \kappa y = \kappa(\sim x * \mathbf{c}* \sim \kappa y)$. However, $\sim \kappa y * \mathbf{c} = \kappa(\sim y * \mathbf{c}) * \mathbf{c}$ and by (1) of Lemma 7 $\kappa(\sim y*\mathbf{c})*\mathbf{c} =\sim y*\mathbf{c}$. Then, $\sim \kappa x* \sim \kappa y = \kappa(\sim x* \sim y*\mathbf{c})$. ■

5. The Image of MV Under $\mathsf{K}^\bullet$

In this section we focus on the category of MV-algebras, a subcategory of $iIRL_0$, and the full subcategory of those objects of DRL' of the form $\mathsf{K}^\bullet(A)$, for some $A \in MV$. We will denote this category $MV^\bullet$ and call its objects monadic MV-algebras. This is justified because the operation κ has the properties of a quantifier as in [5] except for condition E'5), which in our case is slightly weaker than E5) and condition E'6), which in our case is slightly stronger than E5).

It is well known that an MV-algebra is term equivalent to an integral residuated lattice $\langle A, \vee, \wedge, \odot, \rightarrow, \neg, 0, 1\rangle$ that satisfies the following conditions:

$$\begin{array}{ll} \text{(Inv)} & \neg\neg x = x, \\ \text{(Lin)} & (x \rightarrow y) \vee (y \rightarrow x) = 1, \\ \text{(QHey)} & x \odot (x \rightarrow y) = x \wedge y. \end{array}$$

In the following Theorem we prove that these conditions are transmitted through $\mathsf{K}^\bullet$ to the dotted condition. We shall prove that the objects of $MV^\bullet$ are those residuated lattices which satisfy the identities of this theorem.

THEOREM 13. *Let A be an MV-algebra. Then $\mathsf{K}^\bullet A$ verifies*

$$\begin{array}{ll} (Inv^\bullet) & \sim \kappa x = \kappa(\sim \kappa x), \\ (Lin^\bullet) & (x \rightarrow y) \vee (y \rightarrow x) \geq \mathbf{c}, \\ (QHey^\bullet) & (x * \mathbf{c}) * (x \rightarrow (y \vee \mathbf{c})) = (x \wedge y) * \mathbf{c}. \end{array}$$

PROOF. Condition (Inv$^\bullet$) was proved in Theorem 8.

Condition (Lin$^\bullet$) is equivalent to $\kappa((x \rightarrow y) \vee (y \rightarrow x)) = 1$, by item *2* of 3. For $x = (u, v)$ and $y = (s, t)$ in $\mathsf{K}^\bullet A$ we have

$$\kappa((x \rightarrow y) \vee (y \rightarrow x)) = (\neg(u \odot t \wedge s \odot v), u \odot t \wedge s \odot v).$$

But $u \leq \neg v$ and $s \leq \neg t$, so $(u \odot t \wedge s \odot v) \leq (\neg v \odot t \wedge \neg t \odot v)$, and the last term is 0 in every MV-algebra, thus (Lin$^\bullet$) holds.

In order to prove (QHey$^\bullet$), we observe that $(u, v) * (0, 0) = (0, \neg u)$, and also $(u, v) \to (s, 0) = (u \to s, 0)$. So

$$(0, \neg u) * (u \to s, 0) = (0, (u \to s) \to \neg u).$$

But by (QHey), $(u \to s) \to (u \to 0) = (u \odot (u \to s)) \to 0 = (u \wedge s) \to 0$, so the first term in (QHey) is $(0, \neg(u \wedge s))$. On the other hand the second term verifies $(x \wedge y) * \mathbf{c} = (u \wedge s, v \vee t) * (0, 0) = (0, \neg(u \wedge s)$, thus proving that the condition holds. ■

THEOREM 14. *Let T be an object of DRL' in which (Inv$^\bullet$), (Lin$^\bullet$)and (QHey$^\bullet$) hold. Then κT is an MV-algebra.*

PROOF. Condition (Inv) follows from Theorem 8.

Assume (Lin$^\bullet$). So for $x, y \in T$,

$$(\kappa x \to \kappa y) \vee (\kappa y \to \kappa x)) \geq \mathbf{c}.$$

Without lost of generality, we can assume $x, y \leq \mathbf{c}$, because $\kappa u = \kappa(u \wedge \mathbf{c})$, for any u. From *5* of Lemma 7, $(\kappa x \to \kappa y) = \kappa(\sim \kappa \sim (x \to \kappa y))$ and similarly, $(\kappa y \to \kappa x) = \kappa(\sim \kappa \sim (y \to \kappa x))$, so $(\kappa x \to \kappa y) \vee (\kappa y \to \kappa x)) \in \kappa T$. But the only element in κT that is greater than $\mathbf{c}$ is 1, so (Lin) holds.

Let $x, y \geq \mathbf{c}$. Then (QHey$^\bullet$) becomes $x * (x \to y) * \mathbf{c} = (x \wedge y) * \mathbf{c}$. We would like to "cancel out" $\mathbf{c}$.

Observe that, since $\mathbf{c} * \mathbf{c} = 0$, $(x * (x \to y) \vee \mathbf{c}) * \mathbf{c} = x * (x \to y) * \mathbf{c}$. Thus,

$$\alpha(x * (x \to y) \vee \mathbf{c})) = \kappa((x * (x \to y) \vee \mathbf{c}) * \mathbf{c}) =$$
$$\kappa(x * (x \to y) * \mathbf{c}) = \kappa((x \wedge y) * \mathbf{c}) = \alpha(x \wedge y).$$

That is, $\alpha(x *_{\mathbf{c}} (x \to y)) = \alpha(x \wedge y)$, from where $x *_{\mathbf{c}} (x \to y) = x \wedge y$. So (QHey) holds in $\mathsf{C}(T)$, which implies that (QHey) holds in κT. ■

COROLLARY 15. *The categories MV and $MV^\bullet$ are equivalent.*

6. From Pointed ℓ-Groups to Monadic ℓ-Groups

Our purpose in this section is to find a category $M\ell$-G, of monadic ℓ-groups, that is related to the category MV$^\bullet$ in the same way as pointed ℓ-groups are related to MV-algebras, that is to say, we define a functor $\Gamma^\sharp$ that sends each monadic ℓ-group to one of its intervals. In this way $\Gamma^\sharp$ is "parallel" to Γ.

On the other hand, we define another functor K^u, "parallel" to $\mathsf{K}^\bullet$, between the category of pointed ℓ-groups and a new category $M\ell$-G of monadic ℓ-groups and we prove that the two categories are equivalent.

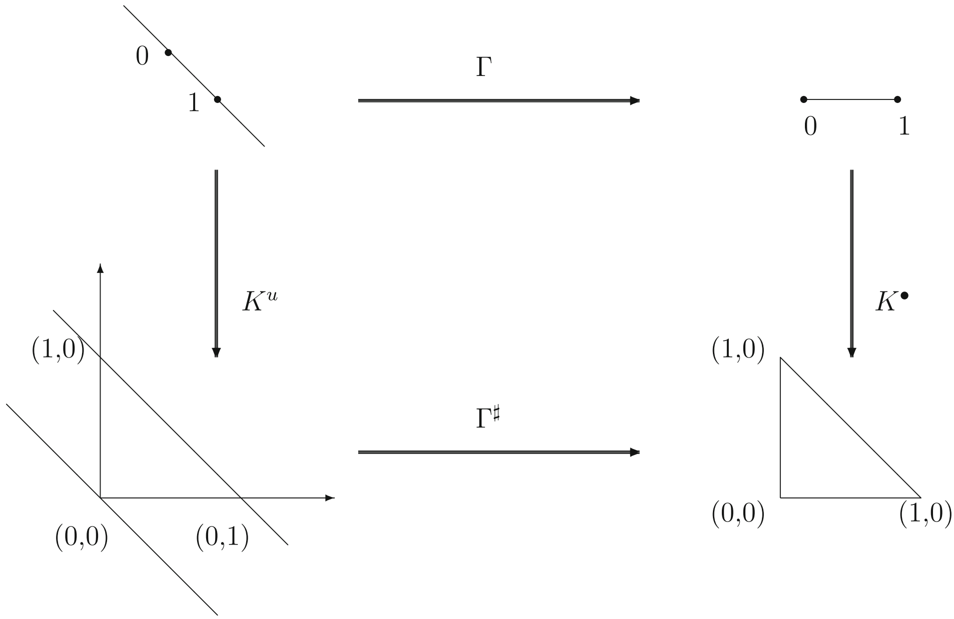

Figure 2. The commutative diagram

All these result in the following commutative diagram.

$$\begin{tikzcd} LG \arrow[r, "\Gamma"] \arrow[d, "\mathsf{K}^u"'] & MV \arrow[d, "\mathsf{K}^\bullet"] \\ M\ell\text{-}G \arrow[r, "\Gamma^\sharp"'] & MV^\bullet \end{tikzcd}$$

Figure 2 shows the example that motivates many results of this paper, where the four objects involved are: the ℓ-group of real numbers, the interval $[0,1]$ with its usual MV-structure, the monadic ℓ-group between the lines $x+y=0$ and $x+y=1$, provided with the structure defined in the sequel, and the triangle $(0,1),(0,0),(1,0)$, obtained from $[0,1]$ by $\mathsf{K}^\bullet$.

6.1. The Category $M\ell$-G

Definition 1. *A* monadic ℓ-group *is an algebra* $\langle U, \wedge, \vee, *, \rightarrow, \sim, \kappa, \mathbf{e}, \mathbf{c}\rangle$ *such that:*

$M\ell G$ 1. $\langle U, \wedge, \vee, *, \rightarrow, \sim, \mathbf{e}\rangle$ *is an involutive commutative residuated lattice,*

MℓG 2. $\mathbf{e} + (\sim \mathbf{e} * \sim \mathbf{e}) = \sim \mathbf{e}$, *(we write $x + y$ for $\sim(\sim x * \sim y)$),*

MℓG 3. $\sim \mathbf{c} = \mathbf{c}$,

MℓG 4. $\mathbf{e} + \mathbf{c} = \mathbf{c} + \mathbf{c}$,

MℓG 5. $\kappa \mathbf{c} = \mathbf{e}$,

MℓG 6. $\kappa\kappa u = \kappa u$,

MℓG 7. $\sim \kappa u = \kappa \sim \kappa u$,

MℓG 8. $\kappa(\kappa u * v) = \kappa u * \kappa v$,

MℓG 9. $\kappa u + \sim \kappa u = \mathbf{e}$,

MℓG 10. $\lambda u \vee (\kappa u * \mathbf{c}) = u$, *where* $\lambda u = \sim \kappa \sim u$,

MℓG 11. $\kappa(u \vee v) = \kappa u \vee \kappa v$,

MℓG 12. $\kappa(u \wedge v) = \kappa u \wedge \kappa v$,

MℓG 13. $\kappa u * (\kappa v \wedge \kappa w) = \kappa u * \kappa v \wedge \kappa u * \kappa w$.

The following consequences of the axioms will be used in the sequel.

LEMMA 16. *The following identities and inequalities hold in any monadic ℓ-group.*

1. $\sim \mathbf{e}$ *is an identity for addition, that is, for every* u, $\sim \mathbf{e} + u = u$,
2. $\kappa u + \sim (\mathbf{e} + \kappa u) = \sim \mathbf{e}$, *that is,* $\kappa u + -\kappa u = \sim \mathbf{e}$,
   where we write $-x = \sim (\mathbf{e} + x)$,
3. $\lambda u \leq u \leq \kappa u$,
4. $\kappa(\kappa u + v) = \kappa u + \kappa v$,
5. $\kappa u \leq \lambda u + \mathbf{e}$,
6. $\kappa(u + v) = \kappa u + \kappa v$,
7. $\lambda(u * v) = \lambda u * \lambda v$,
8. $(\mathbf{e} + \kappa u) * \sim \mathbf{e} = \kappa u$,

9. *If* $\sim \mathbf{e} \leq \kappa u + \kappa v \leq \mathbf{e}$, *then* $-\kappa v \leq \kappa u \leq \sim \kappa v$

10. $\kappa(v * \mathbf{c}) = \lambda v$,

11. $\kappa(u * v) = \kappa u * \lambda v \vee \kappa v * \lambda u$.

PROOF. *1.* Using the definition of $+$ and axiom MℓG*1*, we obtain

$$\sim \mathbf{e} + u = \sim (\mathbf{e} * \sim u) = \sim\sim u = u.$$

2. From axioms MℓG*7* and MℓG*8* we see that $\kappa(T)$ is closed under $\sim$ and $*$, so it is also closed under $+$. Also, $\mathbf{e} = \kappa \mathbf{c}$. Then, $\kappa(\mathbf{e} + \kappa u) = \mathbf{e} + \kappa u$, so we can apply axiom MℓG*9*, and get $(\mathbf{e} + \kappa u) + \sim (\mathbf{e} + \kappa u) = \mathbf{e}$. Adding $\sim \mathbf{e} * \sim \mathbf{e}$ and using axiom MℓG*2*, we get

$$\sim \mathbf{e} * \sim \mathbf{e} + (\mathbf{e} + \kappa u + \sim (\mathbf{e} + \kappa u)) = \sim \mathbf{e} * \sim \mathbf{e} + \mathbf{e} = \sim \mathbf{e}.$$

Since $*$ is associative, it is immediate that $+$ is associative too, so associating and using MℓG*2* again,

$$(\sim \mathbf{e} * \sim \mathbf{e} + \mathbf{e}) + (\kappa u + \sim (\mathbf{e} + \kappa u)) = \sim \mathbf{e} + (\kappa u + \sim (\mathbf{e} + \kappa u)) = \sim \mathbf{e},$$

so using item *1*, the identity holds.

3. From axiom MℓG*10* we get $\lambda u \leq u$. Taking "duals", we get the other inequality.

4. We have $\kappa(\kappa u + \kappa v) = \kappa u + \kappa v$, because $\kappa(T)$ is closed by $+$. Next, since $u + v \leq \kappa u + \kappa v$, we have

$$(*)\kappa(u + v) \leq \kappa u + \kappa v,$$

By items *1*, *2* and *3* we can see that $u \leq u + (v - \lambda v)$ so

$$\begin{array}{rcl} \kappa u & \leq & \kappa(u + v - \lambda v) \\ & \leq & \kappa(u + v) + \kappa(-\lambda v) \\ & = & \kappa(u + v) - \lambda v \\ \kappa u + \lambda v & \leq & \kappa(u + v) - \lambda v + \lambda v \\ & = & \kappa(u + v), \end{array}$$

where we used that $\kappa(-\lambda v) = -\lambda v$.

Since this is also true interchanging u and v,

$$(\kappa u + \lambda v) \vee (\lambda u + \kappa v) \leq \kappa(u + v).$$

This inequality and the inequality (*) imply that if either $u = \kappa u$ or $v = \kappa v$, then $\kappa u + \kappa v = \kappa(u + v)$.

5. From axiom MℓG*10* $\kappa u * \mathbf{c} \le u$, so using axioms M$\ell$G*8*, M$\ell$G*5* and previous item *4*,

$$\begin{aligned}\lambda u \ge \lambda(\kappa u * \mathbf{c}) &=\sim \kappa(\sim \kappa u + \mathbf{c}) =\sim (\sim \kappa u + \mathbf{e})\\ &= \kappa u * \sim \mathbf{e} = -\lambda \sim u\,,\end{aligned}$$

so by axiom MℓG*9*, $\lambda u + \mathbf{e} \ge -\lambda \sim u + (\kappa u + \lambda \sim u) = \kappa u$, the last assertion by item *2*.

6. We now use axiom MℓG*10* to obtain

$$\begin{aligned}&\sim u * \sim v = (\lambda \sim u \vee (\kappa \sim u * \mathbf{c})) * (\lambda \sim v \vee (\kappa \sim v * \mathbf{c})) =\\ &\qquad (\lambda \sim u * \lambda \sim v) \vee (\lambda \sim u * \kappa \sim v * \mathbf{c}) \vee (\lambda \sim v * \kappa \sim u * \mathbf{c})\\ &\qquad \vee(\kappa \sim u * \mathbf{c} * \kappa \sim v * \mathbf{c})).\end{aligned}$$

We apply λ to both sides of this equation and using axiom MℓG*12* and item *4* in dual form, we get

$$\begin{aligned}\lambda \sim (u + v) = (\lambda \sim u * \lambda \sim v) &\vee (\lambda \sim u * \kappa \sim v * \sim \mathbf{e})\\ \vee(\lambda \sim v * \kappa \sim u * \sim \mathbf{e}) &\vee (\kappa \sim u * \kappa \sim v * \lambda(\mathbf{c} * \mathbf{c}))\,,\end{aligned}$$

and thus

$$\begin{aligned}\kappa(u + v) = (\kappa u + \kappa v) &\wedge (\kappa u + \lambda v + \mathbf{e})\\ \wedge(\lambda u + \kappa v + \mathbf{e}) &\wedge (\lambda u + \lambda v + \sim \lambda(\mathbf{c} * \mathbf{c}))\,,\end{aligned}$$

Finally, using axioms MℓG*3*, MℓG*4*, MℓG*5* and item *4*,

$$\sim \lambda(\mathbf{c} * \mathbf{c}) = \kappa(\sim (\mathbf{c} * \mathbf{c})) = \kappa(\mathbf{c} + \mathbf{c}) = \kappa(\mathbf{e} + \mathbf{c}) = \mathbf{e} + \mathbf{e}\,,$$

and with this and item *5* of this lemma,

$$\begin{aligned}\kappa u + \kappa v &\le \kappa u + \lambda v + \mathbf{e}\\ \kappa u + \kappa v &\le \lambda u + \kappa v + \mathbf{e}\\ \kappa u + \kappa v &\le (\lambda u + \mathbf{e}) + (\lambda v + \mathbf{e}) = \lambda u + \lambda v + \sim \lambda(\mathbf{c} * \mathbf{c})\,,\end{aligned}$$

so $\kappa u + \kappa v \le \kappa(u + v)$, thus completing the proof.

7. This follows from *6.*

8. From item *2*, we get $\kappa u * (\sim \kappa u * (e + \kappa u)) = \kappa u * \mathbf{e} = \kappa u$, but by axiom M$\ell$G*9*, $\kappa u * \sim \kappa u =\sim \mathbf{e}$, so the property holds.

9. Adding $-\kappa v$ to each term in the hypothesis, $\sim \mathbf{e}\ -\kappa v \leq \kappa u + \kappa v - \kappa v \leq \mathbf{e} - \kappa v$, so $-\kappa v \leq \kappa u$.

On the other hand, $\mathbf{e} - \kappa v = \mathbf{e} + (\sim \mathbf{e} * \sim \kappa v) = \sim (\sim \mathbf{e} * (\mathbf{e} + \kappa v)) = \sim \kappa v$, by item *8*.

10. By axiom MℓG*4*, $\mathbf{c} + \mathbf{c} = \mathbf{e} + \mathbf{c}$, so by axiom M$\ell$G*3* $\mathbf{c} * \mathbf{c} = \sim \mathbf{e} * \mathbf{c}$, and by axioms M$\ell$G*8* and M$\ell$G*5*,

$$\kappa(\mathbf{c} * \mathbf{c}) = \kappa(\sim \mathbf{e} * \mathbf{c}) = \kappa \sim \mathbf{e} * \kappa\mathbf{c} = \sim \mathbf{e} * \mathbf{e} = \sim \mathbf{e}.$$

Using this and axioms MℓG*1*, MℓG*7*, MℓG*11* and MℓG*10*,

$$\begin{aligned}
v &= \lambda v \vee \kappa v * \mathbf{c} \\
v * \mathbf{c} &= \lambda v * \mathbf{c} \vee \kappa v * \mathbf{c} * \mathbf{c} \\
\kappa(v * \mathbf{c}) &= \kappa(\lambda v * \mathbf{c}) \vee \kappa(\kappa v * \mathbf{c} * \mathbf{c}) \\
&= (\lambda v * \kappa\mathbf{c}) \vee (\kappa v * \kappa(\mathbf{c} * \mathbf{c})) \\
&= (\lambda v * \mathbf{e}) \vee (\kappa v * \sim \mathbf{e}) \\
&= \lambda v \vee (\kappa v * \sim \mathbf{e})
\end{aligned}$$

But using item *5*, $\kappa v \leq \lambda v + \mathbf{e}$, and by item *8*, $\kappa v * \sim \mathbf{e} \leq (\lambda v + e) * \sim \mathbf{e} = \lambda v$, so $\kappa(v * \mathbf{c}) = \lambda v \vee \kappa u * \sim \mathbf{e} = \lambda v$.

11.

$$\begin{aligned}
u &= \lambda u \vee \kappa u * \mathbf{c} \\
u * v &= \lambda u * v \vee \kappa u * \mathbf{c} * v \\
\kappa(u * v) &= \kappa(\lambda u * v) \vee \kappa(\kappa u * \mathbf{c} * v) \\
&= (\lambda u * \kappa v) \vee (\kappa u * \kappa(\mathbf{c} * v)) \\
&= (\lambda u * \kappa v) \vee (\kappa u * \lambda v).
\end{aligned}$$

■

Remark 3. Item *1* of the previous lemma proves that there is an identity for addition restricted to all closed elements. Item *2* proves that for closed elements there is an additive inverse $-u = \sim \mathbf{e} * \sim u = \sim (e + u)$. Hence, for any given monadic ℓ-group U, the image κU with the obvious operations is a group.

Remark 4. It is worth noting that the last lemma together with the axioms amount to proving that the operation κ is a quantifier. See [5].

1. $x \leq \kappa x$

2. $\kappa(\kappa x \vee \kappa y) = \kappa x \vee \kappa y$

3. $\kappa \sim \kappa x = \sim \kappa x$

4. $\kappa(\kappa x + \kappa y) = \kappa x + \kappa y$

5. $\kappa(x * x) = \kappa x * \lambda x \leq \kappa x * \kappa x$

6. $\kappa(x + x) = \kappa x + \kappa x$

Observe that item *5* is slightly weaker than what is required in [5]. As a matter of fact, κ has several other stronger characteristics.

1. $\kappa(x \vee y) = \kappa x \vee \kappa y$

2. $\kappa(x \wedge y) = \kappa x \wedge \kappa y$

3. $\kappa(x + y) = \kappa x + \kappa y$

4. $\kappa(x * \kappa y) = \kappa x * \kappa y$

6.2. The Functor K^u

We define the functor K^u from the category of pointed ℓ-groups with a positive point g into the category $M\ell\text{-}G$ of monadic ℓ-groups as follows.

Definition 2. Let (G, g) be a pointed ℓ-group. We define the following operations on G:

$$\begin{aligned} x \cdot y &= x + y - g \\ x \to y &= g - x + y \\ \neg x &= g - x. \end{aligned}$$

Let $K^u(G) = \{(x, y) \in G \times G^{op},\ 0 \leq x + y \leq g\}$.

For any ℓ-group G and positive element $g \in G$ we define

$$\mathsf{K}^u(G, g) = \langle K^u(G), \wedge, \vee, *, \to, \sim, \kappa, \mathbf{e}, \mathbf{c} \rangle\,,$$

where the operations are

$$\begin{aligned} \mathbf{e} &= (g, 0) \\ \mathbf{c} &= (0, 0) \\ \sim (x, y) &= (y, x) \\ \kappa(x, y) &= (\neg y, y) \\ (x, y) * (z, t) &= (x \cdot z,\ (x \to t) \wedge (z \to y)\,) \\ (x, y) \vee (z, t) &= (x \vee z,\ y \wedge t\,) \\ (x, y) \wedge (z, t) &= (x \wedge z,\ y \vee t). \end{aligned}$$

We also define the auxiliary operations

$$\begin{array}{rcl} \lambda u & = & \sim \kappa \sim u \\ u \oplus v & = & \sim (\sim u * \sim v). \end{array}$$

LEMMA 17. *If G is an ℓ-group and g is a positive element of G, then $\mathsf{K}^u(G)$ is a monadic ℓ-group.*

PROOF. We first check that $\mathsf{K}^u(G)$ is a subCRL of $G \times G^{op}$. It is obvious that $\mathsf{K}^u(G)$ is closed under $\sim$.

If $(x, y), (z, t)$ are in $K^u(G)$, that is, if $0 \leq x + y \leq g$ and $0 \leq z + t \leq g$, then $(x, y) \vee (z, t) = (x \vee z, y \wedge t)$ is also in $K^u(G)$, because in an ℓ-group the sum distributes over $\wedge$ and $\vee$.

Also, the product $(x, y) * (z, t) = (x \cdot z, x \to t \wedge z \to y)$ is in $K^u(G)$ because $0 \leq x \cdot z + (x \to t \wedge z \to y) \leq g$. This follows from properties of ℓ-groups.

The identity $(g, 0)$ and the center $(0, 0)$ are in $K^u(G)$.

In order to check the validity in $\mathsf{K}^u(G)$ of axioms MℓG*2* to MℓG*9*, we use the definitions, the equality $(x, y)+(z, t) = (g-y+z\wedge g-t+x, y+t-g)$ and ℓ-group identities.

In order to check axiom MℓG*10*, for $u = (x, y)$, $\kappa u * \mathbf{c} = (g-y, y)*(0, 0) = (-y, y)$. Taking into account that for $(x, y) \in K^u(G), -y \leq x \leq g - y$, we have $\lambda u \vee (\kappa u * c) = (x, g - x) \vee (-y, y) = (x, y)$.

To check axioms MℓG*11* and MℓG*12*, we take into account that in G, $\neg r = g-r$ and thus we have that $\neg\neg r = r, \neg(r \vee s) = \neg r \wedge \neg s$ and $\neg(r \wedge s) = \neg r \vee \neg s$.

The validity of axiom MℓG*13* follows from the definitions. ■

6.3. The Functor $\boldsymbol{\kappa}$

Now we define the functor κ that maps monadic ℓ-groups into pointed ℓ-groups and prove that, together with K^u, it determines a categorical equivalence between these two categories.

LEMMA 18. *Let U be a monadic ℓ-group and $\kappa(U) = \{u \in U : \kappa u = u\}$. Define operations as follows:*

$$\begin{array}{rcl} u + v & = & \sim (\sim u * \sim v) \\ 0 & = & \sim \mathbf{e} \\ -u & = & \sim \mathbf{e} * \sim u. \end{array}$$

Then $\kappa_{\mathbf{e}}(U) = \langle \kappa(U), \vee, \wedge, +, -, 0 \rangle$ is a pointed ℓ-group.

PROOF. It is easy to see that the operations are well defined and we have already checked in Lemma 16 that $+$ is associative, commutative, $\sim e$ is an identity and that for elements such that $u = \kappa u$, $-u$ is an inverse.

From axioms MℓG*11* and MℓG*12* we have that $\kappa(U)$ is closed under $\wedge$ and $\vee$.

From axiom MℓG*13* we get that $+$ distributes over $\vee$. ■

LEMMA 19. *For every pointed ℓ-group $(G,g), \kappa\mathsf{K}^u(G,g)$ is isomorphic to (G,g).*

PROOF. We have $\kappa K^u(G) = \{(\neg y, y) : y \in G\}$, which is closed under the operations in $\mathsf{K}^u(G,g)$. Let the map φ be defined by $\varphi(r) = (r, \neg r)$ from G to $\kappa K^u(G)$.

This is a bijection that preserves the lattice operations and the involution by the properties of $\neg x = g - x$. Also, $\varphi(r+s) = \varphi r + \varphi s$ because $\neg(r \cdot s) = \neg r + \neg s$ in G. ■

To prove the equivalence between monadic ℓ-groups and pointed ℓ-groups it suffices to prove the isomorphism $\mathsf{K}^u\kappa(U) \approx U$.

We have $\mathsf{K}^u\kappa(U) = \{(a,b) :\ a = \kappa a, b = \kappa b, \sim \mathbf{e} \leq a + b \leq \mathbf{e}\}$.

THEOREM 20. *The object $\mathsf{K}^u\kappa(U)$ of the category $M\ell$-G is isomorphic to U.*

PROOF. Define the map ψ from U to $\mathsf{K}^u\kappa(U)$ by $\psi(u) = (\lambda u, \lambda \sim u)$.

From Lemma 16, item *7*, we have that

$$\begin{aligned}
\kappa \sim u &\leq \lambda \sim u + \mathbf{e} \\
\lambda u &\geq \sim \lambda \sim u * \sim \mathbf{e} \\
\lambda u &\geq -\lambda \sim u \\
\lambda u + \lambda \sim u &\geq \sim \mathbf{e},
\end{aligned}$$

On the other hand, by Lemma 16, *6* y *5*,

$$\begin{aligned}
\lambda u &\leq \kappa u \\
\lambda u + \mathbf{e} - \kappa u &\leq \mathbf{e} \\
\lambda u + (\mathbf{e} + \sim \mathbf{e} * \lambda \sim u) &\leq \mathbf{e} \\
\lambda u + \lambda \sim u &\leq \mathbf{e}.
\end{aligned}$$

The last inequality follows from dual of *8* of Lemma 16.

From these two conditions we get that ψ is well defined.

The injectivity of ψ follows from axiom MℓG*10*.

Let us prove its surjectivity. Let $(a,b) \in \mathsf{K}^u\kappa(U)$. Observe that $a = \kappa a$ and $b = \kappa b$ and $\sim \mathbf{e} \leq a + b \leq \mathbf{e}$. Define $u = a \vee (\sim b * \mathbf{c})$.

We have from axioms MℓG*8* and MℓG*5* that $\kappa(\sim b * \mathbf{c}) = \sim b * \kappa\mathbf{c} = \sim b * \mathbf{e} = \sim b$. So by axiom M$\ell$G*11* and *9* of Lemma 16 $\kappa u = \kappa a \vee \kappa(\sim b * \mathbf{c}) = a \vee \sim b = \sim b$, that is $\lambda \sim u = b$.

Also, by *7* of Lemma 16 $\lambda(\sim b * \mathbf{c}) = \lambda \sim b * \lambda\mathbf{c} = \sim b * \sim \mathbf{e} = -b$, so by *9* of Lemma 16 $\lambda u = \lambda a \vee \lambda(\sim b * \mathbf{c}) = a \vee -b = a$.

Thus $\psi(u) = (a, b)$.

The proof of the preservation under ψ of the lattice operations and involution is straightforward.

We check that $\psi(u * v) = \psi u \circledast \psi v$, where $\circledast$ is the product in $\mathsf{K}^u\mathcal{K}(U)$.

The equality of the first component follows from *2* of Lemma 16. For the second component we use axiom MℓG*8* and the fact that the operations $\rightarrow$ and $*$ are related by $\sim (r \rightarrow s) = r * \sim s$.

The element $\psi(\mathbf{c}) = (\sim \mathbf{e}, \sim \mathbf{e})$ is a center.

The identity element of $\mathsf{K}^u\mathcal{K}(U)$ is $\psi(\mathbf{e}) = (\mathbf{e}, \sim \mathbf{e})$, since

$$(a,b) \circledast (\mathbf{e}, \sim \mathbf{e}) = (a * \mathbf{e}, a \rightarrow \sim \mathbf{e} \wedge \mathbf{e} \rightarrow b) =$$
$$(a, \mathbf{e} \rightarrow (\sim a \wedge b)) = (a, \mathbf{e} \rightarrow b) = (a, b). \qquad \blacksquare$$

Corollary 21. *The category of monadic ℓ-groups is equivalent to the category of pointed ℓ-groups.*

6.4. Commutativity of the Diagram

We begin with a technical lemma that will be needed in our definitions.

Lemma 22. *Let L be an ℓ-group and g a positive element of G. In $L \times L^{op}$, define the operations of monadic ℓ-groups as in the definition of the functor K^u. The following properties hold for $u = (x, y)$ and $v = (z, t)$.*

1. $(\mathbf{c} \vee \kappa(u \vee \sim \mathbf{e})) \wedge \mathbf{e} = (\mathbf{c} \rightarrow (u \vee \sim \mathbf{e})) \wedge \mathbf{e}$
2. $(\kappa u \wedge \mathbf{c}) \vee \sim \mathbf{e} = (u \wedge \mathbf{c}) \vee \sim \mathbf{e}$
3. $\mathbf{c} \wedge (((u \wedge \mathbf{e}) * (v \wedge \mathbf{e})) \vee \sim \mathbf{e}) = (\mathbf{c} \wedge u) * (v \wedge \mathbf{e}) \vee (\mathbf{c} \wedge v) * (u \wedge \mathbf{e}) \vee \sim \mathbf{e}$

Proof. *1.* For $u = (x, y)$ we have

$$\begin{aligned}
u \vee \sim \mathbf{e} &= (x \vee 0, y \wedge g), \\
\kappa(u \vee \sim \mathbf{e}) &= (\neg(y \wedge g), y \wedge g), \\
\kappa(u \vee \sim \mathbf{e}) \vee \mathbf{c} &= (\neg(y \wedge g) \vee 0, y \wedge 0), \\
(\kappa(u \vee \sim \mathbf{e}) \vee \mathbf{c}) \wedge \mathbf{e} &= (\neg(y \wedge g) \wedge g, 0).
\end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbf{c} \to (u \vee \sim \mathbf{e}) &= ((0 \to (x \vee 0)) \wedge ((y \wedge g) \to 0), \;\; 0 \cdot (y \wedge g)), \\ (\mathbf{c} \to (u \vee \sim \mathbf{e})) \wedge \mathbf{e} &= ((0 \to (x \vee 0)) \wedge \neg (y \wedge g) \wedge g, \;\; 0 \cdot (y \wedge g) \vee 0). \end{aligned}$$

In L we have $0 \to (x \vee 0) \geq 0 \to 0 = g$ and $0 \cdot (y \wedge g) \leq 0 \cdot g = 0$, so $(\mathbf{c} \to (u \vee \sim \mathbf{e})) \wedge \mathbf{e} = (\neg(y \wedge g) \wedge g, 0)$ and the equality holds.

2. We have

$$\begin{aligned} (\kappa u \wedge \mathbf{c}) \vee \sim \mathbf{e} &= ((\neg y \wedge 0) \vee 0, \;\; (y \vee 0) \wedge g) \\ (u \wedge \mathbf{c}) \vee \sim \mathbf{e} &= ((x \wedge 0) \vee 0, \;\; (y \vee 0) \wedge g) \end{aligned}$$

so, the equality holds.

3. We have

$$\begin{aligned} &(u \wedge \mathbf{e}) * (v \wedge \mathbf{e}) \\ &\quad = (x \wedge g, y \vee 0) * (z \wedge g, t \vee 0) \\ &\quad = ((x \wedge g) \cdot (z \wedge g), \;\; (x \wedge g) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0)), \end{aligned}$$

So

$$\begin{aligned} &((u \wedge \mathbf{e}) * (v \wedge \mathbf{e})) \wedge \mathbf{c} = \\ &\qquad (((x \wedge g) \cdot (z \wedge g)) \wedge 0, \;((x \wedge g) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0)) \vee 0), \end{aligned}$$

Also,

$$\mathbf{c} \wedge (((u \wedge \mathbf{e}) * (v \wedge \mathbf{e})) \vee \sim \mathbf{e}) =$$

$$\begin{aligned} &= (0, \; (((x \wedge g) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0)) \vee 0) \wedge g) \\ &= (0, \;\; (((x \wedge g) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0) \wedge g)) \vee 0)) \\ &= (0, \;\; ((x \wedge g) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0) \wedge g)), \end{aligned}$$

this last equality because $0 \leq (x \wedge \mathbf{e}) \to (t \vee 0)$ and $0 \leq (z \wedge g) \to (y \vee 0)$. On the other hand,

$$\begin{aligned} (\mathbf{c} \wedge u) * (v \wedge \mathbf{e}) &= (0 \wedge x, y \vee 0) * (z \wedge g, t \vee 0) \\ &= ((x \wedge 0) \cdot (z \wedge g), \; (x \wedge 0) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0)), \end{aligned}$$

and in the same way,

$$(\mathbf{c} \wedge v) * (u \wedge \mathbf{e}) = ((z \wedge 0) \cdot (x \wedge g), \; (z \wedge 0) \to (y \vee 0) \wedge (x \wedge g) \to (t \vee 0)).$$

Taking into account that

$$\begin{aligned}(x \wedge 0) \to (t \vee 0) \wedge (x \wedge \mathbf{e}) \to (t \vee 0)) &= ((x \wedge 0) \vee (x \wedge \mathbf{e})) \to (t \vee 0)\\ &= (x \wedge \mathbf{e})) \to (t \vee 0)\end{aligned}$$

and in a similar way with the other two terms, we have that

$$\begin{gathered}(\mathbf{c} \wedge u) * (v \wedge \mathbf{e}) \vee (\mathbf{c} \wedge v) * (u \wedge \mathbf{e}) =\\ (((x \wedge 0) \cdot (z \wedge g) \vee (z \wedge 0) \cdot (x \wedge g),\ ((x \wedge g) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0)).\end{gathered}$$

Then

$$\begin{gathered}(\mathbf{c} \wedge u) * (v \wedge \mathbf{e}) \vee (\mathbf{c} \wedge v) * (u \wedge \mathbf{e}) \vee \sim \mathbf{e} =\\ (0,\ \ ((x \wedge g) \to (t \vee 0) \wedge (z \wedge g) \to (y \vee 0) \wedge g)).\end{gathered}$$ ■

Let U be an object of $M\ell\text{-}G$. Define $\Gamma^\sharp(U) = \{u \in U : \sim \mathbf{e} \leq u \leq \mathbf{e}\}$, and define the following operations on $\Gamma^\sharp(U)$:

$$\begin{aligned}u *_{\mathbf{e}} v &= (u * v) \vee \sim \mathbf{e}\,,\\ u \to_{\mathbf{e}} v &= (u \to v) \wedge \mathbf{e}.\end{aligned}$$

THEOREM 23. *The algebra* $\langle \Gamma^\sharp(U), \wedge, \vee, *_e, \to_e, \sim, \kappa, \sim \mathbf{e}, \mathbf{c}, \mathbf{e}\rangle$ *is an object of* $MV^\bullet$.

PROOF. It is straightforward to see that $\Gamma^\sharp(U)$ is an integral CRL with center $\mathbf{c}$ and that (Inv$^\bullet$) holds. From Lemma 22 we prove that $\mathbf{c}$ is a Leibniz element and that κ satisfies the Eqs. (1) and (2).

We now check that $\Gamma^\sharp U$ satisfies (Lin$^\bullet$) and (QHey$^\bullet$). Note that since U and $\mathsf{K}^u(\kappa(U))$ are isomorphic, every x in $\Gamma^\sharp(U)$ can be considered as a pair $x = (a, b)$ such that $\sim \mathbf{e} \leq a + b \leq \mathbf{e}$ and $\sim \mathbf{e} \leq a, b \leq \mathbf{e}$.

In order to prove that (Lin$^\bullet$), we have to prove that for $x = (a, b)$ and $y = (r, s)$, $(x \to_{\mathbf{e}} y) \vee (y \to_{\mathbf{e}} x) \geq \mathbf{c}$, that is,

$$(((a, b) \to (r, s)) \wedge (\mathbf{e}, \sim \mathbf{e})) \vee (((r, s) \to (a, b)) \wedge (\mathbf{e}, \sim \mathbf{e})) \geq (\sim \mathbf{e}, \sim \mathbf{e}),$$

or

$$(((a \to r \wedge s \to b) \vee (r \to a \wedge b \to s)) \wedge \mathbf{e}, (a \cdot s \wedge r \cdot b) \vee \sim \mathbf{e}) \geq (\sim \mathbf{e}, \sim \mathbf{e}).$$

But $\sim \mathbf{e} \leq r \leq a \to r, \sim \mathbf{e} \leq b \leq s \to b$, so $\sim \mathbf{e} \leq a \to r \wedge s \to b$.

In the same way, $\sim \mathbf{e} \leq r \to a \wedge b \to s$, so the first component is greater than $\sim \mathbf{e}$.

The second component of $(x \to_e y) \vee (y \to_e x)$ is $\sim \mathbf{e}$.

Indeed, $a \cdot s \wedge r \cdot b \leq \neg b \cdot s \wedge \neg s \cdot b$. This last expression is 0 in every MV-algebra, in particular, in the MV-algebra $[\sim \mathbf{e}, \mathbf{e}]$, interval of the ℓ-group $\kappa(U)$.

Let us prove (QHey$^\bullet$), that is,

$$x *_{\mathbf{e}} ((x \to_{\mathbf{e}} (y \vee c)) *_e \mathbf{c}) = (x \wedge y) *_e \mathbf{c}.$$

In the first place, observe that, for $z = (h, k) \in \Gamma^\sharp(U)$, $z *_{\mathbf{e}} \mathbf{c} = (\sim \mathbf{e}, \neg h)$. Then $x \to_{\mathbf{e}} (y \vee \mathbf{c}) = (a \to r \wedge \mathbf{e}, \sim \mathbf{e})$, so

$$(x \to_{\mathbf{e}} (y \vee \mathbf{c})) *_e \mathbf{c} = (\sim \mathbf{e}, \neg(a \to r \wedge \mathbf{e})) = (\sim \mathbf{e}, \neg(a \to r) \vee \sim \mathbf{e}).$$

Then, $x *_{\mathbf{e}} ((x \to_e (y \vee \mathbf{c})) *_e \mathbf{c}) = ((a, b) * (\sim \mathbf{e}, \neg(a \to r) \vee \sim \mathbf{e})) \vee (\sim \mathbf{e}, \mathbf{e})$.

The first component is $(a \cdot \sim \mathbf{e}) \vee \sim \mathbf{e} = \sim \mathbf{e}$ and the second is $(a \to (\neg(a \to r) \vee \sim \mathbf{e})) \wedge (\sim \mathbf{e} \to b) \wedge \mathbf{e}$.

But $a \to (\neg(a \to r) \vee \sim \mathbf{e}) = \neg(a \wedge r)$ (by ℓ-group properties) and $\neg(a \wedge r) \wedge (\sim \mathbf{e} \to b) \wedge \mathbf{e} = \neg(a \wedge r)$, so $x *_{\mathbf{e}} ((x \to_e (y \vee \mathbf{c})) *_e \mathbf{c}) = (\sim \mathbf{e}, \neg(a \wedge r))$.

On the other hand, $(x \wedge y) *_e \mathbf{c} = (a \wedge r, b \vee s) *_e \mathbf{c} = (\sim \mathbf{e}, \neg(a \wedge r))$ and we are done. ■

Theorem 24. *The following diagram commutes:*

$$\begin{tikzcd} LG \arrow[r, "\Gamma"] \arrow[d, "\mathsf{K}^u"'] & MV \arrow[d, "\mathsf{K}^\bullet"] \\ M\ell\text{-}G \arrow[r, "\Gamma^\sharp"'] & MV^\bullet \end{tikzcd}$$

Proof. Let G be an ℓ-group. We have proved that $\Gamma^\sharp(\mathsf{K}^u(G))$ and $\mathsf{K}^\bullet(\Gamma(G))$ are objects of $MV^\bullet$. Let us prove that they coincide.

As a matter of fact,

$(x, y) \in \Gamma^\sharp(\mathsf{K}^u(G))$

if and only if $\sim \mathbf{e} \le x + y \le \mathbf{e}$ and $(x, y) \in [(\sim \mathbf{e}, \mathbf{e}),\ (\mathbf{e}, \sim \mathbf{e})]$

if and only if $x \cdot y = \sim \mathbf{e}$ and $(x, y) \in [(\sim \mathbf{e}, \mathbf{e}),\ (\mathbf{e}, \sim \mathbf{e})]$

if and only if $(x, y) \in \mathsf{K}^\bullet(\Gamma(G))$. ■

Acknowledgments. We would like to acknowledge the anonymous referee for his careful reading of the paper. This work was partially supported by PIP 112-200801-02543-CONICET.

References

[1] Balbes, R., and P. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, 1974.

[2] Castiglioni, J. L., M. Menni, and M. Sagastume, On some categories of involutive centered residuated lattices, *Studia Logica* 90:93–124, 2008.

[3] Cignoli, R., The class of Kleene algebras satisfying an interpolation property and Nelson algebras, *Algebra Universalis* 23:262–292, 1986.

[4] Cignoli, R., Quantifiers on distributive lattices, *Discrete Mathematics* 96:183–197, 1991.

[5] Di Nola, A., and R. Grigolia, On Monadic MV-Algebras, *Annals of Pure and Applied Logic* 128:125–139, 2004.

[6] Halmos, P. R., *Algebraic Logic*, Chelsea Publishing Co., New York, 1962.

[7] McKinsey, J. C. C., and A. Tarski, The Algebra of Topology, *The Annals of Mathematics, Second Series* 45(1):141–191, 1944.

[8] McKinsey, J. C. C., and A. Tarski, On Closed Elements in Closure Algebras, *The Annals of Mathematics, Second Series* 47(1):122–162, 1946.

[9] Montagna, F., and C. Tsinakis, Ordered groups with a conucleus, *Journal of Pure and Applied Algebra* 214(1):71–88, 2010.

[10] Petrovich, A., Monadic De Morgan algebras, in C. Xavier, and M. Carlos (eds.), *Models, Algebras, and Proofs. Lecture Notes in Pure and Applied Mathematics*, 1999, vol. 203, pp. 315–333.

José Luis Castiglioni, Marta Sagastume
Departamento de Matemáticas
Universidad Nacional de La Plata
La Plata, Argentina
jlc@mate.unlp.edu.ar

Marta Sagastume
martasagas@gmail.com

Renato A. Lewin
Facultad de Matemáticas
Pontificia Universidad Católica de Chile
Santiago, Chile
rlewin@mat.puc.cl