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Algebraic Cycles as Residues of Meromorphic Forms

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According to a classical result of Weil [15], a divisor α of a smooth n-dimensional projective variety X is homologous to zero if and only if it is the residue of a closed meromorphic 1-form on X.

Griffiths proved recently [9, pp. 3–8] that a 0-cycle α of X is homologous to zero if and only if it is the Grothendieck residue of a meromorphic n-form $\tilde{\omega}$ on X having poles in the union of a family of complex hypersurfaces $Y_1, \ldots, Y_n$ of X, such that $\bigcap Y_i$ is 0-dimensional and contains the support of α.

We show in this paper (Theorem 3.7) that, in fact, any q-dimensional algebraic cycle α of X, $0 \leqq q \leqq n$, is the *analytic residue* of a semimeromorphic $(n-q)$-form $\tilde{\omega}$ on X, having poles in the union of a family $\mathscr{F} = \{Y_1, \ldots, Y_{n-q}\}$ of hypersurfaces in X such that $\bigcap \mathscr{F}$ contains the support of α. The form $\tilde{\omega}$ is not closed, in general, but its differential verifies

(1) $$d\tilde{\omega} = \sum_{i=1}^{n-q} \tilde{\omega}_i,$$

where $\tilde{\omega}_i$ is semimeromorphic with poles in $\bigcup (Y_j : 1 \leqq j \leqq n-q, j \neq i)$.

If α is homologous to zero in X, the form $\tilde{\omega}$ can be chosen meromorphic. Moreover, α is algebraically equivalent to zero if and only if one can choose $\mathscr{F}$ such that

$$d\tilde{\omega} = \sum_{i=1}^{n-q-1} \tilde{\omega}_i.$$

To describe the analytic residue of $\tilde{\omega}$, allow in principle $\dim \cap \mathscr{F} \geqq q$. The $2q$-dimensional residue current $R_{\mathscr{F}}[\tilde{\omega}]$ [5, 6], has support in a pure q-dimensional complex variety $V_e(\mathscr{F}) \subset \bigcap \mathscr{F}$ canonically associated to $\mathscr{F}$. By a result of Poly [14], the current $R_{\mathscr{F}}[\tilde{\omega}]$ determines, when it is closed, a unique (Borel-Moore) homology class $c \in H_{2q}(V_e(\mathscr{F}); \mathbb{C})$ and, consequently, an algebraic q-cycle with coefficients in $\mathbb{C}$ and support in $V_e(\mathscr{F})$. We call this cycle the *analytic residue* of $\tilde{\omega}$, and denote it by $[R_{\mathscr{F}}[\tilde{\omega}]]$.

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If $q=0$ and $\tilde{\omega}$ is meromorphic, or if $q=n-1$ and $\tilde{\omega}$ is closed meromorphic, $[R_{\mathscr{F}}[\tilde{\omega}]]$ can be identified with Grothendieck's residue cycle or with the divisor of $\tilde{\omega}$, respectively; one refinds in this way the quoted results of Griffiths and Weil.

These statements are equally true on Stein manifolds, replacing algebraic equivalence to zero of a q-cycle α by its homological equivalence to zero in some $(q+1)$-dimensional subvariety of X that contains the support of α. This makes no difference in the projective case, by a result of Bloch and Ogus [2, 7.3].

Our proofs are based in a chain level construction, already sketched in [11], of the Poincaré isomorphism $H^m_Y(X;\mathbb{C})\to H_{2n-m}(Y;\mathbb{C})$ for a complete intersection Y in X, using meromorphic forms, currents and the residue-principal value operators of [5, 6].

The same method gives the following additional result (3.8): if the q-cycle α is homologous to zero in X, it is homologous to zero in some $(2q+1)$-dimensional complex subvariety of X.

Most of the results described also hold when X is a complex space, either holomorphically complete or projective, provided $q\geqq$ dimension of the singular set of X. We sketch in (3.9) the extension of the proofs to these cases.

Finally, we compare in (3.10) the geometric residue defined by Atiyah and Hodge [1] with our notion of analytic residue.

We would like to mention that Dolbeault [17] has results on the representation of divisors as residues of 1-forms, in the compact non-Kähler case, which are not included by our methods.

1. The Residual Complex of a Family of Hypersurfaces

X will always denote a paracompact complex manifold of dimension n.

(1.1) Let $\mathscr{F}=\{Y_0,\dots,Y_s\}$ be an ordered family of $s+1$ complex hypersurfaces in X, $s\geqq 0$.

We shall make use of the following notations:

$$\mathscr{F}(j)=\{Y_i\in\mathscr{F}: i\neq j\},\qquad 0\leqq j\leqq s.$$

$\Delta_s=\langle 0,\dots,s\rangle$ is the standard s-simplex and Δ^t is the family of t-simplexes $T=\langle i_0,\dots,i_t\rangle\subset\Delta_s$, $t\leqq s$.

$Y_T=\bigcup(Y_i: i\in T)$ and $Y[T]=\bigcap(Y_i: i\in T)$, for $T\in\Delta^t$. For each r, $0\leqq r\leqq s$, $Y^r=\bigcap(Y_i: 0\leqq i\leqq r)$ and $\mathscr{U}^r=\{X-Y_i,\ 0\leqq i\leqq r\}$, which is an open covering of $X-Y^r$.

For any hypersurface Y in X, $\Omega^q(*Y)$ and $\mathscr{E}^q(*Y)$ denote the sheaves of meromorphic and semimeromorphic q-differential forms on X with poles on Y [12, 2.1].

$\check{\mathscr{C}}^{t,q}=\check{\mathscr{C}}^{t,q}(\mathscr{E}*)=\check{\mathscr{C}}^{t,q}(\mathscr{U}^s,\mathscr{E}*)$, for $0\leqq t\leqq s$ and $0\leqq q\leqq 2n$, is the sheaf of alternated t-cochains of Čech with values in $\{\mathscr{E}^q(*Y_T): T\in\Delta^t\}$; on an open set $W\subset X$ its sections are:

$$\Gamma(W,\check{\mathscr{C}}^{t,q})=\prod_{T\in\Delta^t}\Gamma(W,\mathscr{E}^q(*Y_T));$$

a section $\lambda\in\Gamma(W,\check{\mathscr{C}}^{t,q})$ is thus represented as

$$\lambda=(\lambda_T\in\Gamma(W,\mathscr{E}^q(*Y_T)): T\in\Delta^t).$$

$\check{\mathscr{C}}^{\cdot,\cdot}=\sum(\check{\mathscr{C}}^{t,q}:0\leqq t\leqq s, 0\leqq q\leqq 2n)$ is the double complex with De Rham differential $d:\check{\mathscr{C}}^{\cdot,q}\to\check{\mathscr{C}}^{\cdot,q+1}$ and Čech differential $\delta:\check{\mathscr{C}}^{t,\cdot}\to\check{\mathscr{C}}^{t+1,\cdot}$. If $\lambda=(\lambda_T)\in\check{\mathscr{C}}^{t,q}$, one has $d\lambda=(d\lambda_T)$ and, given $T=(i_0,\dots,i_{t+1})\in\Delta^{t+1}$,

$$(\delta\lambda)_T=\sum_k(-1)^k\lambda_{T(k)},$$

where $T(k)=(i_0,\dots,\hat{i}_k,\dots,i_{t+1})$.

$\check{\mathscr{C}}^m=\sum(\check{\mathscr{C}}^{t,m-t}:0\leqq t\leqq s)$, and $\check{\mathscr{C}}^\cdot=\sum_m\check{\mathscr{C}}^m$ is the total complex with differential $D=\delta+(-1)^t d$ on $\check{\mathscr{C}}^{t,\cdot}$.

$\check{\mathscr{C}}^{\cdot,\cdot}(\Omega*)$ and $\check{\mathscr{C}}^\cdot(\Omega*)$ are defined similarly, using meromorphic instead of semimeromorphic differential forms.

There are obvious inclusions $\check{\mathscr{C}}^\cdot(\Omega*)\to\check{\mathscr{C}}^\cdot(\mathscr{E}*)$, $\Omega^\cdot_X\to\check{\mathscr{C}}^\cdot(\Omega*)$, and $\mathscr{E}^\cdot_X\to\check{C}^\cdot(\mathscr{E}*)$, all compatible with boundaries. For instance, the second inclusion associates to the germ $\omega\in\Omega^\cdot_X$ the 0-cochain $(\omega_i=\omega, i\in\Delta^0)$. For each r, $0\leqq r\leqq s$, we denote by $\check{\mathscr{C}}^\cdot_{Y^r}$ the total complex of sheaves constructed as above with the family $\{Y_0,\dots,Y_r\}$; we may write $\check{\mathscr{C}}^\cdot_{Y^r}(\mathscr{E}*)$ or $\check{\mathscr{C}}^\cdot_{Y^r}(\Omega*)$, according to the case.

There are differential homomorphisms

$$\Pi_r:\check{\mathscr{C}}^\cdot_{Y^s}\to\check{\mathscr{C}}^\cdot_{Y^r},\qquad 0\leqq r<s,$$

obtained by restriction of cochains in $\check{\mathscr{C}}^{t,q}(\mathscr{U}^s)$ to the t-simplexes $T\subset\Delta_r$. Denote $\check{\mathscr{C}}_{Y^r/Y^s}=\ker(\Pi_r)$, and define the quotient sheaves $\check{Q}_{Y^r}=\check{\mathscr{C}}_{Y^r}(\mathscr{E}*)/\mathscr{E}^\cdot_X$. We have a commutative diagram with exact lines and columns:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{E}^\cdot_X \arrow[r] \arrow[u] & \check{\mathscr{C}}^\cdot_{Y^r} \arrow[r] \arrow[u] & \check{Q}^\cdot_{Y^r} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathscr{E}^\cdot_X \arrow[r] & \check{\mathscr{C}}^\cdot_{Y^s} \arrow[r] \arrow[u, "\pi_r"] & \check{Q}^\cdot_{Y^s} \arrow[r] & 0 \\
 & & \check{\mathscr{C}}^\cdot_{Y^r/Y^s} \arrow[u] \arrow[r, equal] & \check{\mathscr{C}}^\cdot_{Y^r/Y^s} \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}\tag{1.2}$$

We omit the proof of the following theorem, which has been sketched in [11], and of Theorem (1.5), which is similar and can be recovered from Remark (1.10).

(1.3) **Theorem.** *Integration induces isomorphisms I, $\check{I}$, and $\check{I}_s$*

$$\begin{tikzcd}
\dots \arrow[r] \arrow[d, phantom, "(a)"] & H^m\Gamma(X,\mathscr{E}^\cdot_X) \arrow[r] \arrow[d, "I"] \arrow[dr, phantom, "(c)"] & H^m\Gamma(X,\check{\mathscr{C}}^\cdot_{Y^s}) \arrow[r] \arrow[d, "\check{I}"] \arrow[dr, phantom, "(c)"] & H^m\Gamma(X,\check{Q}^\cdot_{Y^s}) \arrow[r] \arrow[d, "\check{I}_s"] & \dots \arrow[d, phantom, "(a)"] \\
\dots \arrow[r] & \mathrm{H}^m(X,\mathbb{C}) \arrow[r] & H^m(X-Y^s,\mathbb{C}) \arrow[r] & H^{m+1}_{Y^s}(X,\mathbb{C}) \arrow[r] & \dots
\end{tikzcd}\tag{1.4}$$

from the cohomology sequence associated to the sequence

$$0 \to \mathscr{E}^{\cdot}_X \to \check{\mathscr{C}}^{\cdot}_{Y^s} \to \check{Q}^{\cdot}_{Y^s} \to 0$$

to the local cohomology sequence of the couple (X, Y^s). *The* (c)-*squares commute and the* (a)-*squares anticommute.*

(1.5) **Theorem.** *Integration induces isomorphisms* $\check{I}_{r,s}$

$$(1.6)\qquad \begin{array}{ccccccc} \dots \longrightarrow & H^m\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{E}*)) & \longrightarrow & H^m\Gamma(X, \check{Q}^{\cdot}_{Y^s}) & \longrightarrow & H^m\Gamma(X, \check{Q}^{\cdot}_r) & \longrightarrow \dots \\ \text{(a)} & \simeq\downarrow \check{I}_{r,s} & \text{(c)} & \simeq\downarrow \check{I}_s & \text{(c)} & \simeq\downarrow \check{I}_r & \text{(a)} \\ \dots \longrightarrow & H^m_{Y^r-Y^s}(X, \mathbb{C}) & \longrightarrow & H^{m+1}_{Y^s}(X, \mathbb{C}) & \longrightarrow & H^{m+1}_{Y^r}(X, \mathbb{C}) & \longrightarrow \dots \end{array}$$

from the cohomology sequence associated to the sequence

$$0 \to \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{E}*) \to \check{Q}^{\cdot}_{Y^s} \to \check{Q}^{\cdot}_{Y^r} \to 0$$

to the local cohomology sequence of the couple (Y^r, Y^s). *The* (c)-*squares commute and the* (a)-*squares anticommute.*

(1.7) **Corollary.** *Suppose that* X *is Stein or that* X *is smooth projective and each open* $X - Y_i$ *is affine,* $0 \leqq i \leqq s$. *The canonical homomorphisms*

$$H^m\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\Omega*)) \to H^m\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{E}*)),$$
$$H^m\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^s}(\Omega*)) \to H^m\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^s}(\mathscr{E}*)),$$

$m \geqq 0$, *are then isomorphisms.*

Proof. One can deduce from Grothendieck's theorem [10] that

$$(1.8)\qquad H^{\cdot}(\Omega_X(*Y_i)) \xrightarrow{\simeq} H^{\cdot}(\mathscr{E}_X(*Y_i)) \xrightarrow{\simeq} R_{j*}(\mathbb{C}_{X-Y_i})$$

for all $0 \leqq i \leqq s$, where $j: X - Y_i \to X$ denotes the inclusion. This implies that, in the two cases of this corollary, one has isomorphisms

$$(1.9)\qquad H^{\cdot}\Gamma(X, \Omega_X(*Y_i)) \xrightarrow{\simeq} H^{\cdot}\Gamma(X, \mathscr{E}_X(*Y_i)) \xrightarrow{\simeq} H^{\cdot}(X - Y_i, \mathbb{C}).$$

But then the homomorphism $E_1^{\cdot\cdot}(\Omega*) \to E_1^{\cdot\cdot}(\mathscr{E}*)$ between the first spectral sequences of the double complexes $\Gamma(X, \check{\mathscr{C}}^{\cdot\cdot}_{Y^r/Y^s}(\Omega*))$ and $\Gamma(X, \check{\mathscr{C}}^{\cdot\cdot}_{Y^r/Y^s}(\mathscr{E}*))$ is an isomorphism, from where the first assertion of the corollary follows. The proof of the second assertion is similar.

(1.10) *Remark.* Let X be a complex space and $\mathscr{F} = \{Y_0, \dots, Y_s\}$ a family such that $Y_i \supset sX =$ singular set of X, $0 \leqq i \leqq s$. Suppose that X is holomorphically complete, or that X is projective and each $X - Y_i$ is affine, for $0 \leqq i \leqq s$. Then one still has isomorphisms

$$\check{I}_{r,s}: H^m\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\Omega*)) \to H^m_{Y^r-Y^s}(X, \mathbb{C})$$
$$\check{I}: H^m\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^s}(\Omega*)) \to H^m(X - Y^s, \mathbb{C})$$

for all $m \geqq 0$. To prove it in the first case, for instance, consider the standard double Čech complex $\check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{S}^{\cdot})$ with values in the sheaf $\mathscr{S}^{\cdot}$ of differentiable cochains on $X_* = X - sX$, the associated total complex $\check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{S})$ and the homomorphism

$$\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\Omega *)) \to \Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{S}))$$

given by integration.

Grothendieck's theorem (1.8) is still true in the present case, thanks to the condition $sX \subset Y_i$, so that we still have (1.9) and, as in the proof of (1.7):

$$H^m \Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\Omega *)) \xrightarrow{\simeq} H^m \Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{S})) \simeq H^m_{Y^r - Y^s}(X, \mathbb{C}),$$

where the last isomorphism can be obtained by standard methods.

2. Poincaré Duality

The purpose of this section is to give a chain level construction of the Poincaré isomorphism between the bottom sequence in (1.6) and the exact sequence of Borel-Moore homology with closed support and coefficients in $\mathbb{C}$:

$$\ldots \to H_{2n-m}(Y^r - Y^s) \to H_{2n-m-1}(Y^s) \to H_{2n-m-1}(Y^r) \to \ldots .$$

(2.1) *Residual Currents.* Denote by $'\mathscr{D}_{\cdot,X}$ the sheaf complex of currents on X with differentials $b \cdot T(\alpha) = (-1)^{p+1} T(d\alpha)$, for each $T \in \Gamma(W, \mathscr{D}'_{p,X})$ and $\alpha \in \Gamma_c(W, \mathscr{E}^p)$, W open set in X.

Let $\mathscr{F} = \{Y_0, \ldots, Y_t\}$, $0 \leqq t \leqq n-1$, be an ordered family of complex hypersurfaces in X. We do not impose for the moment any restriction on the dimension of the variety $\bigcap \mathscr{F}$. The *t-residue-principal value* operator

$$RP_{\mathscr{F}}: \begin{cases} \Gamma(X, \mathscr{E}^{2n-p-t}(*\bigcup \mathscr{F})) \to \Gamma(X, '\mathscr{D}_{p,X}, \\ \tilde{\omega} \qquad\qquad\qquad\qquad \to RP_{\mathscr{F}}[\tilde{\omega}] \end{cases}$$

has the following local definition [5, 6]: suppose that the hypersurfaces Y_j are given by equations $\phi_j \in \mathscr{O}(W)$, $0 \leqq j \leqq t$, on an open set $W \subset X$: then

$$RP_{\mathscr{F}}[\tilde{\omega}](\alpha) = \lim_{\delta \to 0} \int_{D_\delta(\phi)} \tilde{\omega} \wedge \alpha.$$

$\alpha \in \Gamma_c(W, \mathscr{E}^p_X)$, where $D_\delta(\phi)$ is the tube $\{|\phi_0| = \delta_0, \ldots, |\phi_{t-1}| = \delta_{t-1}, |\phi_t| > \delta_t\}$ with an adequate orientation, and where $\delta = (\delta_0, \ldots, \delta_t) \in \mathbb{R}^{t+1}_+$ tends to zero in a convenient way.

The $(t+1)$-*residue*

$$R_{\mathscr{F}}: \begin{cases} \Gamma(X, \mathscr{E}^{2n-p-t-1}_X) \to \Gamma(X, '\mathscr{D}_{p,X}) \\ \tilde{\omega} \qquad\qquad\qquad \to R_{\mathscr{F}}[\tilde{\omega}] \end{cases}$$

is defined as

$$R_{\mathscr{F}}[\tilde{\omega}](\alpha) = \lim_{\delta \to 0} \int_{T_\delta(\phi)} \tilde{\omega} \wedge \alpha,$$

$\alpha \in \Gamma_c(W, \mathscr{E}^p_X)$, where $T_\delta(\phi) = \{|\phi_0| = \delta_0, \ldots, |\phi_t| = \delta_t\}$ is oriented similarly.

The following properties of the RP operators will be of importance for us:

(2.2) The support of $R_{\mathscr{F}}[\tilde{\omega}]$ is contained in a complex variety $V_e(\mathscr{F})\subset\bigcap\mathscr{F}$, the *essential intersection of* $\mathscr{F}$, which has pure codimension $t+1$ or is empty. The support of $RP_{\mathscr{F}}[\tilde{\omega}]$ is contained in a complex variety $\tilde{V}_e(\mathscr{F})\subset\bigcap(Y_j:0\leqq j\leqq t)$ which has pure codimension t or is empty.

(2.3) $b\cdot RP_{\mathscr{F}}[\tilde{\omega}]+(-1)^{t+1}RP_{\mathscr{F}}[d\tilde{\omega}]=R_{\mathscr{F}}[\tilde{\omega}]$

and

$$b\cdot R_{\mathscr{F}}[\tilde{\omega}]=(-1)^{t+1}R_{\mathscr{F}}[d\tilde{\omega}],$$

$\tilde{\omega}\in\Gamma(X,\mathscr{E}^{\cdot}(*\mathscr{F}))$.

We will say that $\tilde{\omega}\in\Gamma(X,\mathscr{E}^{\cdot}(*\mathscr{F}))$ is regular on $Y_j\in\mathscr{F}$ if $\tilde{\omega}\in\Gamma(X,\mathscr{E}^{\cdot}(*\mathscr{F}(j)))$, where $\mathscr{F}(j)=\{Y_i\in\mathscr{F}:i\neq j\}$.

Suppose now that $\mathscr{F}$ *satisfies the complete intersection conditions:*

(2.4) $\dim_{\mathbb{C}}\cap\mathscr{F}=n-t-1$ and $\dim_{\mathbb{C}}\cap\mathscr{F}(t)=n-t$.

We can state in this case the following additional properties [6, 1.7.7(2), 1.7.7(3)]:

(2.5) If $\tilde{\omega}$ is regular on the surface Y_j, $0\leqq j<t$, one has

$$RP_{\mathscr{F}}[\tilde{\omega}]=R_{\mathscr{F}}[\tilde{\omega}]=0.$$

(2.6) If $\tilde{\omega}$ is regular on Y_t, one has

$$RP_{\mathscr{F}}[\tilde{\omega}]=R_{\mathscr{F}(t)}[\tilde{\omega}]\quad\text{and}\quad R_{\mathscr{F}}[\tilde{\omega}]=0.$$

(2.7) $R_{\mathscr{F}}$ and $RP_{\mathscr{F}}$ are alternating functions of the order of $\mathscr{F}$ and $\mathscr{F}(t)$, respectively.

(2.8) $V_e(\mathscr{F})=\bigcap\mathscr{F}$ and $\tilde{V}_e(\mathscr{F})=\bigcap\mathscr{F}(t)$.

We do not know if properties (2.5) and (2.6) are true when (2.4) is not satisfied; property (2.7) certainly fails in this case. For this reason, we are not able to prove Lemma (2.11) below without imposing condition (2.10)

(2.9) *Construction of Poincaré Duality.* From this moment on we consider a family $\mathscr{F}=\{Y_0,\dots,Y_s\}$, $0\leqq s\leqq n-1$, of hypersurfaces in X such that

(2.10) $\dim_{\mathbb{C}}Y_0\cap\dots\cap Y_t\leqq n-t-1,\quad 0\leqq t\leqq s.$

For each r, $0\leqq r\leqq s$, we define homomorphisms

$$\check{V}_r:\begin{cases}\check{C}^{2n-p}_{Y^r}(\mathscr{E}*)=\Gamma(X,\check{\mathscr{C}}^{2n-p}_{Y^r}(\mathscr{E}*))\to\Gamma(X,{}'\mathscr{D}_{p,X})\\ \lambda=\sum\limits_{t=0}^{r}\lambda^t\to\check{V}_r[\lambda]=\sum\limits_{t=0}^{r}(-1)^{\sigma(t)}RP_{\mathscr{F}_t}[\lambda^t_{\langle 0,\dots,t\rangle}],\end{cases}$$

where $\sigma(t)=t(t+1)/2$, $\lambda^t\in\Gamma(X,\check{\mathscr{C}}^{t,2n-p-t}(\mathscr{U},\mathscr{E}*))$ and $\mathscr{F}_t=\{Y_0,\dots,Y_t\}$. We observe that only the value of λ^t on the particular simplex $\langle 0,\dots,t\rangle$ gives a contribution to $\check{V}_r$. It is clear also that $\check{V}_r$ can be defined in the sheaf level.

(2.11) **Lemma.**

$$b\cdot \check{V}_r[\lambda]-\check{V}_r[D\lambda]=(-1)^{\sigma(r)}R_{\mathscr{F}_r}[\lambda^r_{\langle 0,\dots,r\rangle}]\,.$$

Proof. Let $\lambda=\sum_{t=0}^{r}\lambda^t\in\Gamma(X,\check{\mathscr{C}}^{2n-p}(\mathscr{E}*))$. By (2.5)

$$(2.12)\quad b\cdot \check{V}_r[\lambda]=\sum_{t=0}^{r}(-1)^{\sigma(t)}b\cdot RP_{\mathscr{F}_t}[\lambda^t_{\langle 0,\dots,t\rangle}]$$

$$=\sum_{t=0}^{r}(-1)^{t+\sigma(t)}RP_{\mathscr{F}_t}[d\lambda^t_{\langle 0,\dots,t\rangle}]$$

$$+\sum_{t=0}^{r}(-1)^{\sigma(t)}R_{\mathscr{F}_t}[\lambda^t_{\langle 0,\dots,t\rangle}]\,.$$

On the other hand $D\lambda=\sum_{t=0}^{r}(-1)^t d\lambda^t+\sum_{t=1}^{r}\delta(\lambda^{t-1})$, so that

$$\check{V}_r D[\lambda]=\sum_{t=0}^{r}(-1)^{t+\sigma(t)}RP_{\mathscr{F}_t}[d\lambda^t]$$

$$+\sum_{t=1}^{r}(-1)^{\sigma(t)}RP_{\mathscr{F}_t}[\delta(\lambda^{t-1})_{\langle 0,\dots,t\rangle}]\,.$$

The general term of the last summation is, by (2.5):

$$(-1)^{\sigma(t)}RP_{\mathscr{F}_t}\left[\sum_{j=0}^{t}(-1)^j\lambda^{t-1}_{\langle 0,\dots,\hat{j},\dots,t\rangle}\right]$$

$$=(-1)^{\sigma(t)}RP_{\mathscr{F}_t}[(-1)^t\lambda^{t-1}_{\langle 0,\dots,t-1\rangle}]\,;$$

according to (2.6), this last expression is equal to:

$$(-1)^{t+\sigma(t)}R_{\mathscr{F}_{t-1}}[\lambda^{t-1}_{\langle 0,\dots,t-1\rangle}]$$

so that we finally obtain

$$(2.13)\quad \check{V}_r D[\lambda]=\sum_{t=0}^{r}(-1)^{t+\sigma(t)}RP_{\mathscr{F}_t}[d\lambda^t_{\langle 0,\dots,t\rangle}]$$

$$+\sum_{t=0}^{r-1}(-1)^{t+1+\sigma(t+1)}R_{\mathscr{F}_t}[\lambda^t_{\langle 0,\dots,t\rangle}]\,.$$

Since $t+1+\sigma(t+1)=\sigma(t)(\mathrm{mod}\cdot 2)$, we deduce from (2.12) and (2.13) that

$$b\cdot \check{V}_r[\lambda]-\check{V}_r D[\lambda]=(-1)^{\sigma(r)}R_{\mathscr{F}_r}[\lambda^r_{\langle 0,\dots,t\rangle}]\,,$$

as wanted.

(2.14) Given any closed set $F\subset X$, denote $'\mathscr{D}_{\cdot,F^\infty}$ the sheaf of currents on X with supports contained in F, with the differential $b\cdot$ induced by that of $'\mathscr{D}_{\cdot,X}$. We quote here for further reference the following result of Poly [14] (cf. also Darchen [7]):

(2.15) **Theorem.** *Let F be a semianalytic set in the paracompact real analytic manifold X. There are canonical isomorphisms*

$$H_m(F,\mathbb{C}) \to H_m\Gamma(X, '\mathscr{D}_{\cdot,F^\infty}), \quad m \geqq 0,$$

which can be obtained by associating to each semianalytic chain α in F (cf. [3]) its integration current $I[\alpha]$ on X.

(2.16) For each r, $0 \leqq r \leqq s$, we have the inclusions of complexes

$$'\mathscr{D}_{\cdot,(Y^s)^\infty} \subset {'\mathscr{D}_{\cdot,(Y^r)^\infty}} \subset {'\mathscr{D}_{\cdot,X}};$$

we denote by

$$'\mathscr{D}_{\cdot,(Y^r/Y^s)^\infty} = {'\mathscr{D}_{\cdot,(Y^r)^\infty}}/{'\mathscr{D}_{\cdot,(Y^s)^\infty}},$$

and

$$'\mathscr{D}_{\cdot,(X/Y^s)^\infty} = {'\mathscr{D}_{\cdot,X}}/{'\mathscr{D}_{\cdot,(Y^s)^\infty}}$$

the quotient complexes.

By (2.2), (2.8), and (2.11), $\check{V}_r : \check{\mathscr{C}}^{2n-\cdot}_{Y^r} \to {'\mathscr{D}_{\cdot,X}}$ induces differential homomorphisms

$$(2.17) \quad \check{V}_r : \check{\mathscr{C}}^{2n-\cdot}_{Y^r} \to {'\mathscr{D}_{\cdot,(X/Y^r)^\infty}}, \quad 0 \leqq r \leqq s.$$

The restriction of $\check{V}_r$ to $\check{\mathscr{C}}^{2n-\cdot}_{Y^r/Y^s}(\mathscr{E}*)$ (1.2) gives currents in $'\mathscr{D}_{\cdot,(Y^r)^\infty}$ by (2.2) and (2.8). By (2.7) again, $\check{V}_r$ induces differential homomorphisms

$$(2.18) \quad \check{V}_{r,s} : \check{\mathscr{C}}^{2n-\cdot}_{Y^r/Y^s}(\mathscr{E}*) \to {'\mathscr{D}_{\cdot,(Y^r/Y^s)^\infty}}, \quad 0 \leqq r \leqq s.$$

Consider now the sheaf homomorphisms

$$R_r : \check{\mathscr{C}}^{2n-1-\cdot}_{Y^r} \to {'\mathscr{D}_{\cdot,(Y^r)^\infty}}$$

defined for any open set $W \subset X$ by

$$R_r[\lambda] = (-1)^{\sigma(r)} R_{\mathscr{F},r}[\lambda^r_{\langle 0,\dots,r\rangle}],$$
$$\lambda = \sum_{t=0}^{r} \lambda^t \in \Gamma(W, \check{\mathscr{C}}^{2n-1-\cdot}_{Y^r}).$$

It is clear that R_r is zero on the subsheaf $\mathscr{E}^{\cdot}_X$ of $\check{\mathscr{C}}^{\cdot}_{Y^r}$, so that we obtain homomorphisms

$$(2.19) \quad R_r : \check{Q}^{2n-1-\cdot}_{Y^r} = \check{\mathscr{C}}^{2n-1-\cdot}(\mathscr{E}*)/\mathscr{E}^{2n-1}_X \to {'\mathscr{D}_{\cdot,(Y^r)^\infty}}$$

which, up to signs, are compatible with boundaries, by (2.3) and (2.5). The homomorphisms $\check{V}_{r,s}$, R_r, and R_s, $0 \leqq r \leqq s$, give mappings from the terms of the upper sequence in (1.4) into the terms of the long exact sequence of homology associated to the exact sequence

$$0 \to {'\mathscr{D}_{\cdot,(Y^s)^\infty}} \to {'\mathscr{D}_{\cdot,(Y^r)^\infty}} \to {'\mathscr{D}_{\cdot,(Y^r/Y^s)^\infty}} \to 0.$$

(2.20) **Theorem.** *The diagram so obtained:*

$$(2.21) \quad \begin{array}{ccccccc}
\dots \to & H^{2n-q-1}\Gamma(X, \check{\mathscr{C}}^{\cdot}_{Y^r/Y^s}(\mathscr{E}*)) & \to & H^{2n-q-1}\Gamma(X, \hat{\mathscr{Q}}^{\cdot}_{Y^s}) & \to & H^{2n-q-1}\Gamma(X, \hat{\mathscr{Q}}^{\cdot}_{Y^r}) & \to \dots \\
& \downarrow{\scriptstyle \check{V}_{r,s}} & \text{(b)} & \downarrow{\scriptstyle R_s} & \text{(a)} & \downarrow{\scriptstyle R_r} & \text{(c)} \\
\dots \to & H_{q+1}\Gamma(X, '\mathscr{D}_{\cdot,(Y^r/Y^s)^\infty}) & \to & H_q\Gamma(X, '\mathscr{D}_{\cdot,(Y^s)^\infty}) & \to & H_q\Gamma(X, '\mathscr{D}_{\cdot,(Y^r)^\infty}) & \to \dots
\end{array}$$

is commutative, and isomorphic up to signs to the Poincaré duality diagram (with coefficients in $\mathbb{C}$*):*

$$
\begin{array}{ccccccccc}
\ldots\to & H^{2n-q-1}_{Y^r-Y^s}(X) & \xrightarrow{\delta} & H^{2n-q}_{Y^s}(X) & \to & H^{2n-q}_{Y^r}(X) & \to & H^{2n-q}_{Y^r-Y^s}(X) & \to\ldots \\
 & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & \\
\ldots\to & H_{q+1}(Y^r-Y^s) & \xrightarrow{\partial} & H_q(Y^s) & \to & H_q(Y^r) & \to & H_q(Y^r-Y^s) & \to\ldots
\end{array}
\tag{2.22}
$$

In particular, $\check{V}_{r,s}$, R_s, *and* R_r *in* (2.21) *are all isomorphisms.*

Proof. Commutativity of (a):

Let $\lambda=\sum\limits_{t=0}^{s}\lambda^t\in Z\Gamma(X,\check{Q}^{2n-q-1}_{Y^s})$; by (2.3)

$$
\begin{aligned}
R_s[\lambda]&=(-1)^{\sigma(s)}R_{\langle 0,\ldots,s\rangle}[\lambda^s_{\langle 0,\ldots,s\rangle}]\\
&=(-1)^{\sigma(s)}b\cdot R_{\langle 0,\ldots,s-1\rangle}P_s[\lambda^s_{\langle 0,\ldots,s\rangle}]\\
&\quad+(-1)^{\sigma(s)+s+1}R_{\langle 0,\ldots,s-1\rangle}P_s[d\lambda^s_{\langle 0,\ldots,s\rangle}].
\end{aligned}
$$

Since λ is a cycle, we have

$$
(-1)^s d\lambda^s=-\delta(\lambda^{s-1})=-\sum_{j=0}^{s}(-1)^j\lambda^{s-1}_{\langle 0,\ldots,\hat{j},\ldots,s\rangle},
$$

so that by (2.5) and (2.6)

$$
\begin{aligned}
&(-1)^{\sigma(s)+s+1}R_{\langle 0,\ldots,s-1\rangle}P_s[d\lambda^s_{\langle 0,\ldots,s\rangle}]\\
&\quad=(-1)^{\sigma(s)}R_{\langle 0,\ldots,s-1\rangle}P_s[(-1)^s\lambda^{s-1}_{\langle 0,\ldots,s-1\rangle}]\\
&\quad=(-1)^{\sigma(s-1)}R_{\langle 0,\ldots,s-1\rangle}[\lambda^{s-1}_{\langle 0,\ldots,s-1\rangle}],
\end{aligned}
$$

and

$$
\begin{aligned}
R_s[\lambda]&=(-1)^{\sigma(s-1)}R_{\langle 0,\ldots,s-1\rangle}[\lambda^{s-1}_{\langle 0,\ldots,s-1\rangle}]\\
&\quad+(-1)^{\sigma(s)}b\cdot R_{\langle 0,\ldots,s-1\rangle}P_s[\lambda^s_{\langle 0,\ldots,s\rangle}].
\end{aligned}
$$

Iterating the method one obtains

$$
R_s[\lambda]=(-1)^{\sigma(r)}R_{\langle 0,\ldots,r\rangle}[\lambda^r_{\langle 0,\ldots,r\rangle}]+b\cdot T.
$$

$$
T=\sum_{j=r+1}^{s}(-1)^{\sigma(j)}R_{\langle 0,\ldots,j-1\rangle}P_j[\lambda^j_{\langle 0,\ldots,j\rangle}]\in\Gamma(X,{}'\mathcal{D}_{q-1,(Y^r)^\infty})
$$

so that $R_s[\lambda]-R_r[\Pi_r\lambda]\in B\Gamma(X,{}'\mathcal{D}_{q,(Y^r)^\infty})$, which implies the commutativity of (a).

To show commutativity of (c), denote by D_r and D_s the boundaries of $\check{Q}_{Y^r}$ and $\check{Q}_{Y^s}$, respectively. If a cycle $\tilde{\lambda}\in Z\Gamma(X,\check{Q}^{2n-q-1}_{Y^r})$ is represented by a cochain $\lambda=\sum\limits_{t=0}^{r}\lambda^t$, then

$$
D_s\tilde{\lambda}=D_r\tilde{\lambda}+\delta\lambda^r=\delta\lambda^r\in Z\Gamma(X,\check{\mathscr{C}}^{2n-q}_{Y^r})
$$

and

$$\begin{aligned}\check{V}_{r,s}[D_s\tilde{\lambda}] &= \check{V}_{r,s}[\delta\lambda^r]\\ &=(-1)^{\sigma(r+1)}R_{\langle 0,\dots,r\rangle}P_{r+1}[(\delta\lambda^r)_{\langle 0,\dots,r+1\rangle}]\\ &=(-1)^{\sigma(r+1)+r+1}R_{\langle 0,\dots,r\rangle}[\lambda^r_{\langle 0,\dots,r\rangle}],\end{aligned}$$

by (2.5) and (2.6), so that one obtains $V_{r,s}[D_s\tilde{\lambda}]=R_r[\tilde{\lambda}]$; as wanted. The commutativity of (b) is obvious, since a cycle $\lambda=\sum_{t=0}^{s}\lambda^t\in Z\Gamma(X,\check{\mathscr{C}}^{2n-q-1}_{Y^r/Y^s})$ verifies $b\cdot\check{V}[\lambda]=R_s[\lambda]$, by (2.11). This proves the first assertion of (2.20).

As for the isomorphism between diagrams (2.21) and (2.22), the top sequences have already been identified in (1.3). To identify the bottom sequences, we represent each homology class in $H_q(Y^s)$ by a semianalytic cycle α of Y^s, and associate to this class the integration current on α [3]. We proceed in the same way with Y^r. Similarly, classes in $H_q(Y^r-Y^s)$ are represented by semianalytic chains in Y^r whose boundaries lie in Y^s; we also associate to such chains their integration currents. One can see that this method defines a mapping between the bottom sequences of (2.21) and (2.22). By the result of Poly (2.16) this mapping is an isomorphism.

We will accept the fact that the interior squares between diagrams (2.21) and (2.22) are, up to signs, commutative. The proof is similar to that given in [12, No. 5] for the case of one hypersurface in X. This implies that (2.21) and (2.22) are isomorphic.

(2.23) We want to compare the cohomology sequences associated to the sequences

$$0\to\mathscr{E}^{\cdot}_X\to\check{\mathscr{C}}^{\cdot}_{Y^s}\to\hat{\mathscr{Q}}^{\cdot}_{Y^s}\to 0$$

and

$$0\to{}'\mathscr{D}_{\cdot,(Y^s)^\infty}\to{}'\mathscr{D}_{\cdot,X}\to{}'\mathscr{D}_{\cdot,(X/Y^s)^\infty}\to 0.$$

Denote $V:\mathscr{E}^{2n-\cdot}_X\to{}'\mathscr{D}_{\cdot,X}$ the mapping $\omega\to V[\omega]$, where $V[\omega]$ is the current $\alpha\to\int\omega\wedge\alpha$, for $\omega\in\Gamma(X,\mathscr{E}^{2n-p}_X)$ and $\alpha\in\mathscr{D}_p(W)$, W open in X.

We omit the proof of the following theorem, which is similar to that of (2.20) [11]:

(2.24) **Theorem.** *In the following diagram:*

$$\begin{array}{ccccccc}\dots\to H^{2n-p}\Gamma(X,\mathscr{E}^{\cdot}_X) & \to & H^{2n-p}\Gamma(X,\check{\mathscr{C}}^{\cdot}_{Y^s}) & \to & H^{2n-p}\Gamma(X,\check{Q}^{\cdot}_{Y^s}) & \to\dots\\ \simeq\downarrow V & \text{(c)} & \simeq\downarrow V_s & \text{(c)} & \simeq\downarrow & \text{(a)}\\ \dots\to H_p\Gamma(X,{}'\mathscr{D}_{\cdot,X}) & \to & H_p\Gamma(X,{}'\mathscr{D}_{\cdot,(X/Y^s)^\infty}) & \to & H_{p-1}\Gamma(X,{}'\mathscr{D}_{\cdot,(Y^s)^\infty}) & \to\dots\end{array}$$

the (c)*-squares commute and the* (a)*-squares anticommute. Moreover, this diagram is isomorphic, up to signs, to the Poincaré duality diagram*

$$\begin{array}{ccccccc}\dots\to H^{2n-p}(X) & \to & H^{2n-p}(X-Y^s) & \xrightarrow{\delta} & H^{2n-p+1}_{Y^s}(X) & \to\dots\\ \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & \\ \dots\to H_p(X) & \to & H_p(X-Y^s) & \to & H_{p-1}(Y^s) & \to\dots\ .\end{array}$$

3. Cycles as Residues

(3.1) *The Analytic Residue.* Let X be a paracompact complex manifold of dimension n. Choose a family $\mathscr{F}=\{Y_1,\ldots,Y_{n-q}\}$ of hypersurfaces in X, without restriction on $\dim\cap\mathscr{F}$, and a form $\tilde{\omega}\in\Gamma(X,\mathscr{E}_X^{n-q}(*\bigcup\mathscr{F}))$. If $R_{\mathscr{F}}[\tilde{\omega}]$ *is closed*, this current defines a class (2.15)

$$\{R_{\mathscr{F}}[\tilde{\omega}]\}\in H_{2q}\Gamma(X,{}'\mathscr{D}_{\cdot,V_e(\mathscr{F})\infty})\simeq H_{2q}(V_e(\mathscr{F});\mathbb{C}).$$

According to Borel and Haefliger [4], the top dimensional homology class $\{R_{\mathscr{F}}[\tilde{\omega}]\}$ of the q-variety $V_e(\mathscr{F})$ determines an analytic q-cycle $[R_{\mathscr{F}}[\tilde{\omega}]]$ with support in $V_e(\mathscr{F})$, which we call *the analytic residue of* $\tilde{\omega}$.

(3.2) **Proposition.** *In these conditions, suppose that $\tilde{\omega}$ is closed. Then $R_{\mathscr{F}}[\tilde{\omega}]$ is closed and the analytic q-cycle $[R_{\mathscr{F}}[\tilde{\omega}]]$ is homologous to zero in the $(q+1)$-variety $\tilde{V}_e(\mathscr{F})$.*

Proof. According to (2.3) the condition $d\tilde{\omega}=0$ implies first that $R_{\mathscr{F}}[\tilde{\omega}]$ is closed, and secondly that $b\cdot RP_{\mathscr{F}}[\tilde{\omega}]=R_{\mathscr{F}}[\tilde{\omega}]$. By (2.20), this implies that $[R_{\mathscr{F}}[\tilde{\omega}]]$ bounds in $\tilde{V}_e(\mathscr{F})=$ support of $RP_{\mathscr{F}}[\tilde{\omega}]$.

(3.3) *Remark.* Suppose in particular that $q=0$ and $\tilde{\omega}$ is meromorphic, so that $d\tilde{\omega}=0$. Then $[R_{\mathscr{F}}[\tilde{\omega}]]\simeq 0$ in X, a result first proved by Griffiths [9] with the condition $\dim\cap\mathscr{F}=0$.

(3.4) **Proposition.** *Suppose that $\mathscr{F}=\{Y_1,\ldots,Y_{n-q}\}$ verifies*

$$\dim_{\mathbb{C}}\cap\mathscr{F}=q \quad \textit{and} \quad \dim_{\mathbb{C}}\cap\mathscr{F}(n-q)=q+1,$$

and choose $\tilde{\omega}\in\Gamma(X,\mathscr{E}_X^{n-q}(*\bigcup\mathscr{F}))$. Then

a) *denote $\mathscr{F}(i)=\{Y_j\in\mathscr{F}:j\neq i\}$. If*

$$d\tilde{\omega}=\sum(\tilde{\omega}(i):1\leqq i\leqq n-q),$$

where $\tilde{\omega}(i)\in\Gamma(X,\mathscr{E}_X^{n-q+1}(*\bigcup\mathscr{F}(i)))$, one has $b\cdot R_{\mathscr{F}}[\tilde{\omega}]=0$, so that the analytic residue $[R_{\mathscr{F}}[\tilde{\omega}]]$ can be defined.

b) *If $d\tilde{\omega}=\sum(\tilde{\omega}(i):1\leqq i<n-q)$, $[R_{\mathscr{F}}[\tilde{\omega}]]$ is homologous to zero in $\bigcap\mathscr{F}(n-q)$.*

The proof follows immediately from (2.3), (2.5), and (2.6).

(3.5) **Corollary.** *If X is smooth projective and $\mathscr{F}$ verifies*

$$\dim_{\mathbb{C}}\cap\mathscr{F}=q \quad \textit{and} \quad \dim_{\mathbb{C}}\cap\mathscr{F}(n-q)=q+1,$$

the analytic residues forms $\tilde{\omega}\in\Gamma(X,\mathscr{E}_X^{n-q}(*\bigcup\mathscr{F}))$ satisfying (3.4b) *are algebraically equivalent to zero.*

Proof. According to Bloch and Ogus [2, 7.3] the q-cycles of X algebraically equivalent to zero are exactly those homologous to zero in some $(q+1)$-dimensional subvariety of X.

Our purpose is now to obtain a converse to (3.4). We omit the proof of the following property, which can be obtained from standard results in the projective case [16, I.6] or can be deduced easily, in the Stein case, from arguments in [8].

(3.6) **Lemma.** *Let X be a complex manifold of dimension n, and let $F\subset Y$ be subvarieties of X of pure dimension q and q', respectively, with $q<q'$. Then*

(a) *If X is Stein, there exists a family $\mathscr{F}=\{Y_1,\dots,Y_{n-q}\}$, of hypersurfaces in X such that*

(a_1) $\dim\cap(Y_j:1\leqq j\leqq i)=n-i,\ 1\leqq i\leqq n-q$;

(a_2) $F\subset Y^{n-q}=\bigcap(Y_j:1\leqq j\leqq n-q)$; *and*

(a_3) $Y\subset Y^{n-q'}=\bigcap(Y_j:1\leqq j\leqq n-q')$.

(b) *If X is projective, the family $\mathscr{F}$ can be chosen such that* (a_1), (a_2), *and* (a_3) *hold, together with:*

(b_1) *the sets $X-Y_i$ are affine, for $1\leqq i\leqq n-q$.*

(3.7) **Theorem.** *Suppose that X is Stein or smooth projective. Let α be an analytic q-cycle in X. There exists then a family $\mathscr{F}=\{Y_1,\dots,Y_{n-q}\}$ of $n-q$ hypersurfaces in X and a form $\tilde\omega\in\Gamma(X,\mathscr{E}_X^{n-q}(*\mathscr{F}))$ such that*

(a) $\dim_{\mathbb{C}}Y_1\cap\dots\cap Y_i=n-i,\ 1\leqq i\leqq n-q$;

(b) *the support of α is contained in $\bigcap\mathscr{F}$;*

(c) $d\tilde\omega=\sum(\tilde\omega(i):1\leqq i\leqq n-q)$, *where* $\tilde\omega(i)\in\Gamma(X,\mathscr{E}^{n-q+1}(*\bigcup\mathscr{F}(i))$;

(d) *α is the analytic residue of $\tilde\omega$.*

Moreover:

(e) *If α is homologous to zero in X, $\tilde\omega$ can be chosen meromorphic: $\tilde\omega\in\Gamma(X,\Omega^{n-q}(*\bigcup\mathscr{F}))$.*

(f) *If α is homologous to zero in some $(q+1)$-dimensional variety of X, $\tilde\omega$ can be chosen in addition such that*

$$d\tilde\omega=\sum(\tilde\omega(i):1\leqq i<n-q),$$

where $\tilde\omega(i)\in\Gamma(X,\Omega^{n-q+1}(*\bigcup\mathscr{F}(i))$.

Proof. The q-cycle α can be represented by a couple $[F,c]$, where F is a pure q-dimensional variety and $c\in H_{2q}(F;\mathbb{C})$.

Let $\mathscr{F}=\{Y_1,\dots,Y_{n-q}\}$ be a family of hypersurfaces in X that satisfies properties (a_1), (a_2), and (b_1) of (3.6) with respect to F, according to the case. If α is homologous to zero in some $(q+1)$-dimensional variety Y of X, we demand (a_3) to be also satisfied with respect to Y.

We now apply Theorem (2.20), renumbering $\mathscr{F}=\{Y_0,\dots,Y_s\}$, $s=n-q-1$, and setting $r=s-1$. The mapping $i_*:H_{2q}(F;\mathbb{C})\to H_{2q}(Y^s;\mathbb{C})$ is injective, since $\dim F=\dim Y^s=q$. The cycle α can then be represented by the couple $[Y^s,c']$, $c'=i_*(c)$. By (2.20), there is a cycle

$$\lambda=\sum_{t=0}^{s}\lambda^{t,2n-2q-1-t}\in\Gamma(X,\check{Q}_{Y^s}^{2n-2q-1})$$

such that $R_s[\lambda]=R_{\mathscr{F}}[(-1)^{\sigma(s)}\lambda^s_{\langle 0,\dots,s\rangle}]$ is homologous in $\Gamma(X,{}'\mathscr{D}_{\cdot,(Y^s)\infty})$ to the integration current $I[Y^s,c']$. If we set $\tilde\omega=(-1)^{\sigma(s)}\lambda^s_{\langle 0,\dots,s\rangle}$, the assertions (a)–(d) above are verified.

Suppose now that α is homologous to zero in a $(q+1)$-variety Y. By construction, $Y\subset Y^r$ and the image of c' in $H_{2q}(Y^r;\mathbb{C})$ is zero, which implies $c'=\partial c''$, $c''\in H_{2q+1}(Y^r-Y^s;\mathbb{C})$.

By (2.20) and (1.7), there exists a cycle $\mu=\sum\mu^{t,2(n-q)-t-1}$ in $\Gamma(X,\check{C}^{2(n-q)-1}_{Y^r/Y^s}(*\Omega))$ such that $\check{V}_{r,s}(\mu)$ and the integration current $I[c'']$ are homologous in

$\Gamma(X, '\mathscr{D}_{2q+1,(Y^r/Y^s)\infty})$. Their boundaries

$$b\cdot \check{V}_{r,s}[\mu]=(-1)^{\sigma(s)}R_{\mathscr{F}}[\mu^s_{\langle 0,\ldots,s\rangle}]$$

and

$$b\cdot I[c'']=I[\partial c'']=I[c']$$

cobound then in $\Gamma(X, '\mathscr{D}_{2q,(Y^s)\infty})$, so that $\alpha=[Y^s, c']$ is the analytic residue of $\tilde{\omega}=(-1)^{\sigma(s)}\mu^s_{\langle 0,\ldots,s\rangle}$. Finally

$$\begin{aligned} d\mu^s_{\langle 0,\ldots,s\rangle} &= (-1)^s(\delta\mu^{s-1})_{\langle 0,\ldots,s\rangle} \\ &= (-1)^s \sum_{j=0}^{s} (-1)^j \mu^{s-1}_{\langle 0,\ldots,\hat{j},\ldots,s\rangle}, \end{aligned}$$

where $\mu^{s-1}_{\langle 0,\ldots,s-1,\hat{s}\rangle}=0$ since the restriction of μ to $\check{\mathscr{C}}_{Y^r}$ is zero. This proves property (f) above.

Property (e) can be obtained in a similar way. If α is homologous to zero in X, Theorem (2.22) and Corollary (1.7) assure the existence of a cycle $\lambda\in\Gamma(X, \check{\mathscr{C}}^{2(n-q)-1}_{Y^s}(*\Omega))$, $\lambda=\sum(\lambda^t:0\leqq t\leqq s)$, such that $b\cdot \check{V}_s[\lambda]$ and the integration current $I[\alpha]$ are homologous in $\Gamma(X, '\mathscr{D}_{2q,(Y^s)\infty})$.

Since $b\cdot \check{V}_s[\lambda]=(-1)^{\sigma(s)}R_{\mathscr{F}}[\lambda^s_{\langle 0,\ldots,s\rangle}]$, the result follows taking $\tilde{\omega}=(-1)^{\sigma(s)}\lambda^s_{\langle 0,\ldots,s\rangle}$

Observe moreover that $\lambda^t=0$ if $2(n-q)-t-1>n$, and

$$\check{V}[\lambda]=\sum_{t=n-2q-1}^{2(n-q)-1}(-1)^{\sigma(t)}R_{\langle 0,\ldots,t-1\rangle}P_t[\lambda^t_{\langle 0,\ldots,t\rangle}]\,.$$

This current has support in the $(2q+1)$-variety $T=Y_0\cap\ldots\cap Y_{n-2q-2}$, and $b\cdot\check{V}[\lambda]$ is homologous to $I[\alpha]$ in $\Gamma(X, '\mathscr{D}_{2q,(Y^s)\infty})$. One deduce from these two facts and Theorem (2.20) that α bounds in T. Finally, given any q'-dimensional variety T' in X that contains the support of α, with $q\leqq q'\leqq 2q+1$, one can always suppose that $T\supset T'$, by (3.6). We have proved then the following:

(3.8) **Corollary.** *Suppose that X is Stein or smooth projective and consider an analytic q-cycle α homologous to zero in X. Given any q'-dimensional variety T' of X that contains the support of α, $q\leqq q'\leqq 2q+1$, there exist a $(2q+1)$-variety T such that $T\supset T'$ and α is homologous to zero in T.*

(3.9) *The Singular Case.* We want to make a few remarks about the way our results can be extended to the case where X is a complex space either holomorphically complete or projective. We refer to [6, 12] for the definitions of meromorphic forms on spaces.

(a) The operators $RP_{\mathscr{F}}$ and $R_{\mathscr{F}}$ still exist in this case, for any locally principal family $\mathscr{F}$, and they verify properties (2.2)–(2.8) [6]. Poly's theorem in no longer available now, but may be replaced by the canonical splitting

$$H\Gamma(X, '\mathscr{D}_{\cdot,Y\infty})\simeq H\cdot(Y;\mathbb{C})\oplus A$$

described in [12, Sect. 4] which is valid for any semianalytic set $Y\subset X$. Consequently, one can still define, as in (3.1), as in (3.1), the analytic cycle associated to a meromorphic form $\tilde{\omega}$ on X with closed residue $R_{\mathscr{F}}[\tilde{\omega}]$, by projecting the class of $R_{\mathscr{F}}[\tilde{\omega}]$ into $H\cdot(Y;\mathbb{C})$, where $Y=V_e(\mathscr{F})$ contains the support of $R_{\mathscr{F}}[\tilde{\omega}]$.

Propositions (3.2) and (3.4) are now true without any change in the proofs.

(b) In Lemma (3.6) we add the hypothesis $q \geqq \dim sX$, where sX is the singular set of X. The family $\mathscr{F}=\{Y_1, \ldots, Y_{n-q}\}$ can then be chosen such that all conditions of the lemma are verified, and besides

$$Y_j \supset sX, \quad 1 \leqq j \leqq n-q.$$

According (1.10), we still have isomorphisms

$$H^m \Gamma(X, \mathscr{C}^{\cdot}_{Y^r/Y^s}(\Omega *)) \simeq H^m_{Y^r-Y^s}(X;\mathbb{C}) \simeq H_{2n-m}(Y^r-Y^s;\mathbb{C}),$$

where we use the notations of No. 1 with respect to the renumbered family $\mathscr{F}=\{Y_0, \ldots, Y_{n-q-1}\}$; observe that $Y^s \supset sX$.

If we consider an analytic q-cycle α of X *homologous to zero in some* $(q+1)$-*dimensional subspace* Y *of* X, and $q \geqq \dim sX$, we can always suppose that the support of α is Y^s and that $\alpha \simeq 0$ in Y^r, $r=s-1$. The proof of (3.7f) applies without change, and (3.7a–d) will be verified. One obtains (3.7e) and (3.8) in the same way.

In particular, these results can be applied taking Y as the ambient space of α, obtaining the following statement: *There exist a closed meromorphic form* $\tilde{\omega} \in \Gamma(Y, \Omega^1_Y(*A))$, *where the hypersurface* A *contains* sX *and the support of* α, *such that* $R_A[\tilde{\omega}]=\alpha$.

We don't know if the statement of Theorem (3.7) about cycles not homologous to zero are still true, when X is singular.

(3.10) *Forms of the Second Kind.* It may be useful to compare the analytic residue defined in (3.1) with the geometric residues and related definitions of Atiyah and Hodge [1].

Let X be a smooth projective variety and $\tilde{\omega} \in \Gamma(X, \Omega^{2n-p}(*\tilde{Y}))$ be a *closed form* with poles on a hypersurface $\tilde{Y}$; $\tilde{\omega}$ defines a class $[\tilde{\omega}] \in H^{2n-p}(X-\tilde{Y}; C)$. The *geometric residues* of $\tilde{\omega}$ are the periods of $[\tilde{\omega}]$ on the cycles in $X-\tilde{Y}$ that bound in X. The form $\tilde{\omega}$ is of the *second kind* if there exists a hypersurface $W \supset \tilde{Y}$ such that the geometric residues of $\tilde{\omega}$ on $X-W$ are all zero (cf. [1, p. 84, Definition C]). This is equivalent to demand that the restriction of $[\tilde{\omega}]$ to $X-W$ be in the image of $H^{2n-p}(X;\mathbb{C}) \to H^{2n-p}(X-W;\mathbb{C})$.

If we consider Theorem (2.22), with $s=0$, $Y^0=W$, we see that $\tilde{\omega}$ is of the second kind if and only if $R_W[\tilde{\omega}]$ bounds in $\Gamma(X, '\mathscr{D}_{\cdot, W^\infty})$, where $R_W[\tilde{\omega}]$ is the residue current associated to the family $\mathscr{F}=\{W\}$. If in particular $2n-p=1$, $\tilde{\omega}$ is of the second kind if and only if the analytic residue of $\tilde{\omega}$ is zero.

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