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TREATMENT OF GAMOW STATES, USING TEMPERED ULTRADISTRIBUTIONS[1]

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In the present manuscript, and with the use of tempered ultradistributions, we extend analitically the pseudonorm of Gamow states as defined originally by T. Berggren. We define this pseudonorm for all states determined by the zeros of the Jost function for any short range potential. As a particular example we study the s-states corresponding to the square-well potential.

Key words: tempered ultradistributions, Gamow states.

1. INTRODUCTION

The appearance of long-living quasi-bond states in target-projectile scattering, and the explanation of decaying nuclei through α emission, are some of the many situations in nuclear physics where the usual concepts of quantum mechanics have to be improved in order to incorporate these non-stationary states to the familiar set of bound and continuum ones. These non-stationary (or unstable) states are called in general resonant states, and in particular they are given the name of Gamow states when only the decaying (or growing) aspect of the problem is taken into account.

In the common formulation of quantum mechanics, these states are associated with the complex energy poles of the S scattering ma-

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trix [4]. An equivalent point of view is to define resonant states as solutions of the time-independent Schrödinger equation with purely out-going waves at large distances [4]. In this respect, if we call $E = E_R \pm i\Gamma/2$ ($\Gamma > 0$) one of these complex energy solutions, then we can describe a growing $(+)$ or a decaying state $(-)$ with mean energy E_R and growing (decaying) constant Γ. The main obstacle to handle matrix elements involving resonant states is their exponentialy divergent behaviour at large distances, which makes it impossible to normalize them in an infinite volume within the conventional framework of quantum mechanics.

Several attempts have been performed to handle adequately resonant states. Many of them usually destroy the self-adjoint character of the Hamiltonian operator. The first succesful work that incorporates resonant wave functions in a general basis containig also bound and continuum states has been made by Tore Bergreen [5]. Using a regularization method first suggested by Zel'dovich, Bergreen has shown that at least for finite range potentials it is possible to define an ortogonality criteria among bound and resonant states, and also a pseudonorm can be evaluated using the general analysis of Newton [4].

A proper inclusion of resonant states within the general framework of quantum mechanics can be done through the rigged Hilbert Space (RHS) or Gelfand's Triplet (GT) formulation [3]. Resonant states are described, within the RHS, by generalized complex energy solutions of a self-adjoint Hamiltonian. The structure of the RHS guarantees that any matrix element involving resonant states is a well defined quantity, provided the topology in the GT has been properly choosen to handle the exponential growing of Gamow States at large distances. Equivalently, the GT formalism allows an analytical extension of physical states in the whole complex energy plane.

The literature concerning the aplication of RHS to resonant states is extensive [8-11]. We can mention, for instance, the works of Bohm [8], Gadella [9], and also Ref. 10, where resonant states are introduced using a RHS of entire Hardy-class functions defined in a half complex energy-plane. This allows to extend analyticaly the concept of a resonant state as an antilinear complex functional over the intersection of Schwartz test functions with Hardy class.

A more general theory of resonant states follows if the RHS is built up on tempered ultradistributions [12], i.e., in this case resonant states arise as continuous linear functionals over rapidly decreasing entire analytical test functions. This can be obtained by using the Dirac's formula, which allows a more direct determination of these states. Another advantage of using tempered ultradistributions is that only the physical spectrum appears in the definition of complex-energy states [12].

In this work we want to extend the application of tempered ultradistributions to the evaluation of a complex pseudonorm for resonant states in the sense of Bergreen. With this pseudonorm we generalize the Bergreen's result [5], and hence it can be considered as the proper analytical extension of a pseudoscalar product among resonant states.

This can be considered as a first step towards the evaluation of more general calculations involving matrix elements of physical operators including Gamow states. Thus for instance, for nuclei in the drip line, the transition emission of particles into the continuum is governed by multipole matrix elements between a bound state and a Gamow state. As another example the study of nuclei far from β-stability lines, which is a very active research field, could be also handled within this formalism.

Tempered ultradistributions and Gelfand triplet are introduced in the next section. In Sec. 3, we define resonant states starting from the Schrödinger equation for a spherical symmetric potential, and then we focus our atention on the calculus of the pseudonorm of a complex-energy state. We apply in Sec. 4 the results of the previous section to the case of a square well potential. We give a resume in Sec. 5.

2. THE TEMPERED ULTRADISTRIBUTIONS

We define the space H of test functions $\phi(x)$ such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any p and q by means of the set of countably norms [1]:

$$\|\hat{\phi}\|_p = \sup_{0\le q\le p,\ x} e^{p|x|}\left|D^q\hat{\phi}(x)\right|, \quad p = 0,1,2,\ldots. \tag{2.1}$$

The dual of H, Λ_∞, consists of the distributions of exponential type T [1]:

$$T = D^p\left[e^{p|x|}f(x)\right], \quad p = 0,1,2\ldots, \tag{2.2}$$

where $f(x)$ is bounded continuous. The triplet

$$(H, \mathcal{H}, \Lambda_\infty) \tag{2.3}$$

is a rigged Hilbert space or Guelfand's triplet [3]. In (2.3), $\mathcal{H}$ is the Hilbert space of square integrable functions.

The space $h = \mathcal{F}\{H\}$ ($\mathcal{F}$= Fourier transform) consist of entire analytic rapidly decreasing test functions given by the countable set of norms:

$$\|\phi\|_{pn} = \sup_{|Im(z)|\le n} (1+|z|)^p|\phi(z)|. \tag{2.4}$$

The dual of h is the space $\mathcal{U}$ of tempered ultradistributions [1]. The triplet$(h, \mathcal{H}, \mathcal{U})$ is also a rigged Hilbert space.

The space $\mathcal{U}$ can be characterized as follow [1]. Let be $\mathcal{A}_\omega$ the space of all functions $F(z)$ such that:

(i) $F(z)$ is analytic in $\{z \in \mathcal{C} : |Im(z)| > p\}$.
(ii) $F(z)/z^p$ is bounded continuous in $\{z \in \mathcal{C} : |Im(z)| \geq p\}$, where p depends of $F(z)$; here $p = 0, 1, 2, \ldots$.

Let Π be the set of all z-dependent polinomials $P(z)$, $z \in \mathcal{C}$. Then $\mathcal{U}$ is the quotient space

$$\mathcal{U} = \frac{\mathcal{A}_\omega}{\Pi}. \tag{2.5}$$

Due to these properties any ultradistribution can be represented as a linear functional where $F(z) \in \mathcal{U}$ is the indicatrix of this functional [1]:

$$F(\phi) =< F(z), \phi(z) >= \oint_\Gamma F(z)\phi(z)dz, \tag{2.6}$$

where the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $Im(z) > \rho$, $\rho > p$ and back from ∞ to $-\infty$ for $Im(z) < -\rho$, $-\rho < -p$ (Γ lies outside a horizontal band of width $2p$ that contain all the singularities of $F(z)$).

As a well-known example in the physical literature we choose for the linear functional F the Dirac delta distribution. In this case the analogous formula to (2.6) is

$$\delta(\phi) =< \delta(z), \phi(z) >=< -\frac{1}{2\pi i z}, \phi(z) >= \oint_\Gamma -\frac{1}{2\pi i z}\phi(z)dz = \phi(0). \tag{2.7}$$

To evaluate (2.7) we have used the usual Cauchy's formula, where, in this case, the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $Im(z) > 0$, and back from ∞ to $-\infty$ for $Im(z) < 0$.

In a similar way to this example we can treat all the tempered distributions (the usual distributions used in the physical literature), since, as has been proved in Ref. 2, the Schwartz distributions are a particular case of tempered ultradistributions.

In the rigged Hilbert spaces $(\Phi, \mathcal{H}, \Phi^*)$ is valid the following very important property:

Every symmetric operator A acting on Φ, that admit a self-adjoint prolongation operating on $\mathcal{H}$, has in Φ^* a complete set of generalized eigenvectors (or proper distributions) that correspond to real eigenvalues [3].

This property is then valid in $(H, \mathcal{H}, \Lambda_\infty)$ and in $(h, \mathcal{H}, \mathcal{U})$.

3. THE PSEUDONORM OF EIGENSTATES OF SHORT RANGE POTENTIALS

In this paragraph we describe the main properties of the solutions of the Schrödinger equation for a central short range potential [4]. According to this reference the above mentioned equation has two types of solutions: regular ($\phi_l(k,r)$) and irregular ($f_l(k,r)$).

Both solutions are related by

$$\phi_l(k,r) = \frac{1}{2} i k^{-l-1} \left[f_l(-k) f_l(k,r) - (-1)^l f_l(k) f_l(-k,r) \right] . \quad (3.1)$$

In (3.1), $f_l(k)$ is the Jost function, defined by

$$f_l(k) = k^l \mathcal{W} \left[f_l(k,r), \phi_l(k,r) \right] , \quad (3.2)$$

where $\mathcal{W}[f, \phi]$ is the Wronskian of the two solutions. The zeros of the Jost function $f_l(k)$ are the bound ($Re(k) = 0, Im(k) \leq 0$), virtual ($Re(k) = 0, Im(k) > 0$) and resonant states ($Re(k) \neq 0, Im(k) > 0$) [4,5]. With these definitions we are now in position to calculate the pseudonorm of the above states. According to Ref. 4, a zero of the Jost function, $k = k_0$, satisfies:

$$\dot{f}_l(k_0) = k_0^l \mathcal{W} \left[\dot{f}_l(k_0,r), \phi_l(k_0,r) \right] + k_0^l \mathcal{W} \left[f_l(k_0,r), \dot{\phi}_l(k_0,r) \right] . \quad (3.3)$$

Using Eq. (3.1) at $k = k_0$, we have the equality

$$f_l(k_0,r) = C(k_0) \phi_l(k_0,r), \quad C(k_0) = \frac{-2ik_0^{l+1}}{f_l(-k_0)}; \quad (3.4)$$

and, following the procedure of Ref. 4, get

$$\begin{aligned} \dot{f}_l(k_0) = {k_0}^l \lim_{\beta \to \infty} \Bigg\{ & \mathcal{W} \left[\dot{f}_l(k_0,\beta), \phi_l(k_0,\beta) \right] \\ & - 2k_0 C(k_0) \int_0^\beta {\phi_l}^2(k_0,r) \, dr \Bigg\}. \end{aligned} \quad (3.5)$$

We want to show now that in (3.5): (i) the integral can be defined as a ultradistribution in the variable k_0, and (ii) in the limit

$\beta \to \infty$, as an ultradistribution in k_0, the Wronskian $\mathcal{W}$ vanishes. With this purpose and following Ref. 4 we note that $k^l f_l(k_0,r)=h_l(k_0,r)$ is an entire analytic function of the variable k_0 and therefore $k^{l+1} f_l(k_0,r)$ is also too. It is easy to conclude that $k^{l+1} f_l(k_0,r) = g_l(k_0,r)$ is an entire analytic function of k_0. Moreover, in Ref. 4 it has been shown $h_l(0,r) = C\phi_l(0,r)$. As a consequence we have $h_l(0,0) = g_l(0,0) = 0$ since ϕ_l has the property $\phi_l(0,0) = 0$.

We can write now (3.5) in terms of $g_l(k,r)$ as

$$\dot{f}_l(k_0) = \lim_{\beta\to\infty} \left\{ \frac{\mathcal{W}\left[g_l(k_0,\beta), \phi_l(k_0,\beta)\right]}{k_0} - 2k_0^{l+1} C(k_0) \int_0^\beta {\phi_l}^2(k_0,r)\, dr \right\}. \tag{3.6}$$

But

$$\lim_{\beta\to\infty} \oint_\Gamma \frac{\mathcal{W}\left[g_l(k_0,\beta), \phi_l(k_0,\beta)\right]}{k_0} \phi(k_0)\, dk_0 = \lim_{\beta\to\infty} \mathcal{W}\left[g_l(0,\beta), \phi_l(0,\beta)\right] \phi(0), \tag{3.7}$$

where $\phi(k_0) \in h$ is an entire analytic test function and the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $Im(k_0) > \rho$, $\rho > 0$ and back from ∞ to $-\infty$ for $Im(k_0) < -\rho$, $-\rho < 0$.

Taking into account that f_l satisfies

$$\frac{d}{dr}\mathcal{W}\left[\dot{f}_l(k,r), f_l(k,r)\right] = 2k f_l^2(k,r), \tag{3.8}$$

it is easy to show that

$$\frac{d}{dr}\mathcal{W}\left[g_l(k,r), h_l(k,r)\right] = 2k^2 h_l^2(k,r), \tag{3.9}$$

and thus

$$\frac{d}{dr}\mathcal{W}\left[g_l(0,r), h_l(0,r)\right] = 0. \tag{3.10}$$

Equation (3.10) implies that

$$\mathcal{W}\left[g_l(0,r), h_l(0,r)\right] = \text{constant}. \tag{3.11}$$

From $h_l(0,0) = g_l(0,0) = 0$, we get

$$\mathcal{W}\left[g_l(0,r), h_l(0,r)\right] = 0. \tag{3.12}$$

This implies that

$$\mathcal{W}\left[g_l(0,r), \phi_l(0,r)\right] = 0, \tag{3.13}$$

and thus we have

$$\lim_{\beta\to\infty} \oint_{\Gamma} \frac{\mathcal{W}\left[g_l(k_0,\beta), \phi_l(k_0,\beta)\right]}{k_0} \phi(k_0)\, dk_0 = 0 \tag{3.14}$$

and, as a consequence,

$$\lim_{\beta\to\infty} \frac{\mathcal{W}\left[g_l(k_0,\beta), \phi_l(k_0,\beta)\right]}{k_0} = P(k_0), \tag{3.15}$$

where $P(k_0)$ is an arbitrary polynomial in the variable k_0.

Now we have the freedom of select $P(k_0) \equiv 0$, which allows us to rewrite (3.5) in the form

$$\lim_{\beta\to\infty} \int_0^{\beta} {\phi_l}^2(k_0,r)\, dr = \frac{\dot{f}_l(k_0) f_l(-k_0)}{4ik_0^{2l+2}}, \tag{3.16}$$

where the limit is taken in the sense of ultradistributions. By definition, the pseudonormalized state is

$$\psi_l(k_0,r) = \left[\frac{4ik_0^{2l+2}}{\dot{f}_l(k_0) f_l(k_0)}\right]^{1/2} \phi_l(k_0,r), \tag{3.17}$$

which can be thought of as a tempered ultradistribution in the variable k_0.

4. THE SQUARE-WELL POTENTIAL

We start from the Schrödinger equation for the radial component $\mathcal{R}_l(r)$ (Refs. 6 and 7):

$$\mathcal{R}_l''(r) + \frac{2}{r}\mathcal{R}_l'(r) + \left[q^2 - \frac{l(l+1)}{r^2}\right]\mathcal{R}_l(r) = 0, \tag{4.1}$$

with the primes denoting differentiation with respect to r; herein

$$q^2 := \frac{2m}{\hbar^2}\left[E - \mathcal{V}(r)\right] = k^2 - \frac{2m}{\hbar^2}\mathcal{V}(r), \tag{4.2}$$

where

$$\mathcal{V}(r) = \begin{cases} 0, & \text{for } r > a, \\ -\mathcal{V}_0, & \text{for } r \leq a, \end{cases} \tag{4.3}$$

and

$$k^2 = \frac{2mE}{\hbar^2}. \tag{4.4}$$

The regular solution is:

$$\phi_l(k,r) = \begin{cases} q^{-l} r\, j_l(qr), & \text{for } r < a, \\ r\left[A_l\, j_l(kr) + B_l\, n_l(kr)\right], & \text{for } r > a, \end{cases} \tag{4.5}$$

where j_l and n_l are,respectively, the spherical Bessel and Neumann functions. The constants A_l and B_l are

$$\begin{aligned} A_l &:= ka^2 q^{-l}\left[k\, j_l(qa)\, n_l'(ka) - q\, j_l'(qa)\, n_l(ka)\right], \\ B_l &:= ka^2 q^{-l}\left[q\, j_l(ka)\, j_l'(qa) - k\, j_l'(ka)\, j_l(qa)\right]. \end{aligned} \tag{4.6}$$

The irregular solution $f_l(k,r)$ is given by

$$f_l(k,r) = \begin{cases} r\left[C_l\, j_l(qr) + D_l\, n_l(qr)\right], & \text{for } r < a, \\ -ikr\, h_l^-(kr), & \text{for } r > a, \end{cases} \tag{4.7}$$

where $h_l^- = j_l - in_l$ is the spherical Hankel function and the constants C_l and D_l are given by

$$\begin{aligned} C_l &= -ikqa^2\left[q\, h_l^-(ka)\, n_l'(qa) - k\, h_l^{-\prime}(ka)\, n_l(qa)\right] \\ D_l &= ikqa^2\left[q\, h_l^-(ka)\, j_l'(qa) - k\, h_l^{-\prime}(ka)\, j_l(qa)\right]. \end{aligned} \tag{4.8}$$

Using Eqs. (3.4), (4.5) and (4.7) we can evaluate the corresponding Jost function $f_l(k)$:

$$f_l(k) = \left(\frac{k}{q}\right)^l ika^2\left[k\, j_l(qa)\, h_l^{-\prime}(ka) - q\, j_l'(qa)\, h_l^-(ka)\right]. \tag{4.9}$$

We wish to calculate Eq. (3.20) for this example in the case $l = 0$. With this purpose we need the expressions of $f_0(-k_0)$ and $\dot{f}_0(k_0)$ which, taking into account that $f_0(k_0) = 0$, are given by

$$\begin{aligned} f_0(-k_0) &= -\frac{2ik_0}{q_0} e^{ik_0 a} \sin q_0 a \\ \dot{f}_0(k_0) &= i\frac{q_0^2 - k_0^2}{q_0^3} (1 + ik_0 a)\, e^{-ik_0 a} \sin q_0 a, \end{aligned} \tag{4.10}$$

where

$$q_0^2 := k_0^2 + \frac{2m}{\hbar^2} \mathcal{V}_0.$$

Finally, Eq. (3.20) takes the form

$$\int_0^\infty \phi_0^2(k_0, r) dr = \frac{1 + ik_0 a}{2ik_0} \, \frac{q_0^2 - k_0^2}{q_0^4} \, \sin^2 q_0 a. \tag{4.11}$$

It should be noted that when k_0 corresponds to a bound state the integral (4.11) is real and positive. When k_0 corresponds to a virtual state or a resonant state ($Rek_0 \neq 0$, $Imk_0 > 0$) the integral (4.11) is in general a complex number. It is not surprising, since (4.11) is an analytical extension in the sense of ultradistributions of the usual Lebesgue integral.

5. DISCUSSION

It has been shown in this work that the RHS formulation based on tempered ultradistributions allows to perform a general treatment of complex-energy states, incorporating in a natural way bound and continuum states as well as resonant and virtual states together. Thus resonant states are treated on an equal foot with respect to the usual real energy states, i.e., they have been incorporated within a more general framework of quantum mechanics. In this work we have applied this formulation to the specific evaluation of the complex pseudonorm, showing that the results come out in a more transparent way, since they are free from regularization schemes.

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