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On the asymptotic exactness of Bank-Weiser's estimator

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Summary. In this paper we analyze an error estimator introduced by Bank and Weiser. We prove that this estimator is asymptotically exact in the energy norm for regular solutions and parallel meshes. By considering a simple example we show that this is not true for general meshes.

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1 Introduction

Several a posteriori error estimators have been introduced for the approximation by finite element methods of second order elliptic problems. Many of them, the so-called residual type estimators, are constructed by exploiting in some way the error equation [1–3, 6, 13]. In particular, some estimators are obtained by solving local problems for the error.

In this paper we analyze an estimator introduced by Bank and Weiser [6] which belongs to the last class. This estimator has been used in the code PLTMG [5] and has been extended to the Stokes problem [7, 8].

Although this estimator has been introduced in a general context we consider it in the particular case of linear triangular elements. We prove that this estimator is asymptotically exact in the energy norm for parallel meshes and regular solutions (i.e., in this case, the ratio between the estimator η and the energy norm of the error $\|e\|_E$ converges to one when the meshsize goes to zero). The proof is based on known local superconvergence results.

On the other hand, by analyzing a simple example, we show that the asymptotic exactness is not true for general meshes, even for very regular solutions. However, it is known [6] that under mild regularity assumptions on the solution, there exist

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positive constants c and C such that

$$c\eta \leqq \|e\|_E \leqq C\eta ; \tag{1.1}$$

these constants depend on the regularity of the mesh. Therefore, the behavior of this estimator is similar to that of other estimator defined using the jumps of the normal derivatives of the approximate solution properly weighted (see [9]).

2 Asymptotic exactness

We consider as a model problem the Laplace equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega , \\ u = 0, & \text{on } \Gamma_{\mathrm{d}} , \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g, & \text{in } \Gamma_{\mathrm{n}} ; \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygon with boundary $\partial\Omega = \Gamma_{\mathrm{d}} \cup \Gamma_{\mathrm{n}}$ and $\boldsymbol{n}$ is the outer normal vector to $\partial\Omega$.

Given a regular family of triangulations $\{\mathscr{T}_h\}$ of Ω, let u_h be the piecewise linear finite element approximation of u corresponding to $\mathscr{T}_h$. Integration by parts in each element shows that the error

$$e := u - u_h$$

satisfies the equation

$$\int_\Omega \nabla e \cdot \nabla v = \int_\Omega f v + \int_{\Gamma_{\mathrm{n}}} g v - \sum_{T \in \mathscr{T}_h} \int_{\partial T} \frac{\partial u_h}{\partial \boldsymbol{n}_T} v, \quad \forall v \in H^1_{\Gamma_{\mathrm{d}}}(\Omega) , \tag{2.2}$$

where $H^1_{\Gamma_{\mathrm{d}}}(\Omega) := \{v \in H^1(\Omega) \colon v|_{\Gamma_{\mathrm{d}}} = 0\}$ and $\boldsymbol{n}_T$ denotes the outer normal to ∂T. (Here and thereafter we use standard notation for Sobolev spaces, norms and seminorms.)

Let E_T be the set of edges of T and for each interior edge l let us choose an arbitrary normal direction $\boldsymbol{n}$ and denote the two triangles sharing this edge T_{in} and T_{out}, where the normal $\boldsymbol{n}$ is outwards T_{in}. Let

$$\left[\!\left[\frac{\partial u_h}{\partial \boldsymbol{n}} \right]\!\right]_l := \nabla(u_h|_{T_{\mathrm{out}}}) \cdot \boldsymbol{n} - \nabla(u_h|_{T_{\mathrm{in}}}) \cdot \boldsymbol{n}$$

denote the jump of $\dfrac{\partial u_h}{\partial \boldsymbol{n}}$ across the edge l; this value is independent of the choice of $\boldsymbol{n}$. Equation (2.2) can be written as

$$\int_\Omega \nabla e \cdot \nabla v = \sum_T \left(\int_T f v + \frac{1}{2} \sum_{l \in E_T} \int_l J_l v \right), \quad \forall v \in H^1_{\Gamma_{\mathrm{d}}}(\Omega) , \tag{2.3}$$

where

(2.4)
$$J_l := \begin{cases} \left[\!\left[\dfrac{\partial u_h}{\partial \boldsymbol{n}}\right]\!\right]_l, & \text{if } l \not\subset \partial\Omega , \\ 2\left(g - \left.\dfrac{\partial u_h}{\partial \boldsymbol{n}}\right|_l\right), & \text{if } l \subset \Gamma_{\mathrm{n}} , \\ 0, & \text{if } l \subset \Gamma_{\mathrm{d}} . \end{cases}$$

To recall the definition of the estimator of Bank and Weiser, consider for $T \in \mathcal{T}_h$ the space

$$\mathcal{P}_2^0(T) := \{v \in \mathcal{P}_2(T) : v(P) = 0, \forall P \text{ vertex of } T\}$$

(where $\mathcal{P}_2(T)$ denotes the space of polynomials of degree not greater than 2 restricted to T). Let $\varepsilon_T \in \mathcal{P}_2^0(T)$ be the solution of the local problem

$$\int_T \nabla\varepsilon_T \cdot \nabla v = \int_T fv + \frac{1}{2} \sum_{l \in E_T} \int_l J_l v, \quad \forall v \in \mathcal{P}_2^0(T) ;$$

then the local error estimator is defined by

$$\eta_T := \| \nabla \varepsilon_T \|_{0,T} .$$

We shall prove that this estimator is asymptotically exact in the energy norm on those portions of the mesh which are parallel (i.e., such that the union of two neighboring triangles is a parallelogram) and where the solution is regular.

Let $\Omega_0 \Subset \Omega_1$ be subdomains of Ω and assume that the meshes are parallel in Ω_1. It is known [14] that for this kind of meshes the following superconvergence result holds:

(2.5)
$$\|\nabla(u^{\mathrm{I}} - u_h)\|_{0,\Omega_0} \leqq C(h^2 \|u\|_{3,\Omega_1} + \|e\|_{0,\Omega_1}) ,$$

where u^{I} is the piecewise linear Lagrange interpolant of u. (For a survey on superconvergence results like this, see [11] or [12]). Here and thereafter C denotes a generic constant.

We denote by $\nabla_h \varepsilon$ the vector function defined by $(\nabla_h \varepsilon)|_T := \nabla\varepsilon_T$.

Theorem 2.1. *If the meshes are parallel on Ω_1 and the solution $u \in H^3(\Omega_1)$ then, for h small enough,*

$$\|\nabla e - \nabla_h \varepsilon\|_{0,\Omega_0} \leqq C(h^2 \|u\|_{3,\Omega_1} + \|e\|_{0,\Omega_1}) .$$

Proof. For $T \subset \Omega_0$, let $T^* := \bigcup\{T' \in \mathcal{T}_h : T \text{ and } T' \text{ have a common edge}\}$, as in Fig. 1. Let h be such that $T^* \subset \Omega_1$.

For $w \in H^2(T^*)$ let $q_w \in \mathcal{P}_2^0(T)$ be the solution of the problem

(2.6)
$$\int_T \nabla q_w \cdot \nabla v = -\int_T (\Delta w) v + \frac{1}{2} \sum_{l \in E_T} \int_l \left[\!\left[\frac{\partial w^{\mathrm{I}}}{\partial \boldsymbol{n}}\right]\!\right]_l v, \quad \forall v \in \mathcal{P}_2^0(T) .$$

Then, we have

(2.7)
$$\|\nabla e - \nabla_h \varepsilon\|_{0,T} \leqq \|\nabla(u - u^{\mathrm{I}}) - \nabla q_u\|_{0,T} + \|\nabla(u^{\mathrm{I}} - u_h)\|_{0,T} + \|\nabla q_u - \nabla_h \varepsilon\|_{0,T} .$$

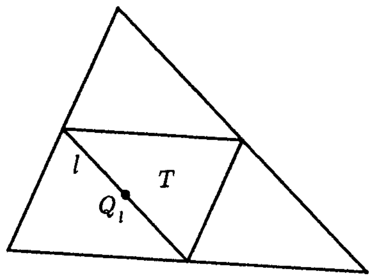

Fig. 1

Hence, in view of (2.5), it is enough to estimate the first and the last term on the right hand side of (2.7).

Let us show that if $w \in \mathscr{P}_2(T^*)$, then

$$q_w = w - w^{\mathrm{I}}, \quad \text{in } T \,. \tag{2.8}$$

Clearly $w - w^{\mathrm{I}} \in \mathscr{P}_2^0(T)$ and, since the solution of (2.6) is unique, it is enough to see that

$$\int_T \nabla(w - w^{\mathrm{I}}) \cdot \nabla v = -\int_T (\Delta w) v + \frac{1}{2} \sum_{l \in E_T} \left[\!\!\left[\frac{\partial w^{\mathrm{I}}}{\partial \boldsymbol{n}} \right]\!\!\right]_l v, \quad \forall v \in \mathscr{P}_2^0(T) \,. \tag{2.9}$$

Integration by parts shows that (2.9) holds if, $\forall l \in E_T$,

$$\int_l \frac{\partial}{\partial \boldsymbol{n}} (w - w^{\mathrm{I}}) v = \frac{1}{2} \int_l \left[\!\!\left[\frac{\partial w^{\mathrm{I}}}{\partial \boldsymbol{n}} \right]\!\!\right]_l v, \quad \forall v \in \mathscr{P}_2^0(T) \,,$$

which, by Simpson's rule, is equivalent to

$$\frac{\partial}{\partial \boldsymbol{n}} (w - w^{\mathrm{I}})(Q_l) = \frac{1}{2} \left[\!\!\left[\frac{\partial w^{\mathrm{I}}}{\partial \boldsymbol{n}} \right]\!\!\right]_l ,$$

(where Q_l is the midpoint of l as in Fig. 1). For a proof of this last equality see, for instance, [9]. Therefore, (2.8) holds.

On the other hand, using the definition of q_w and a trace theorem it is easy to see that for $w \in H^2(T^*)$

$$\| \nabla q_w \|_{0,T} \leqq C h_T |w|_{2,T^*}$$

and hence

$$\| \nabla(w - w^{\mathrm{I}}) - \nabla q_w \|_{0,T} \leqq C h_T |w|_{2,T^*} \,,$$

where h_T is the diameter of T. Consequently, because of (2.8), an application of the Bramble-Hilbert lemma gives for $w \in H^3(T^*)$,

$$\| \nabla(w - w^{\mathrm{I}}) - \nabla q_w \|_{0,T} \leqq C h_T^2 |w|_{3,T^*} \,.$$

By using $w = u$, this last inequality provides the estimate for the first term on the right hand side of (2.7). To bound the last term, observe that because of (2.6) and the definition of ε_T,

$$\int_T \nabla(q_u - \varepsilon_T) \cdot \nabla v = \frac{1}{2} \sum_{l \in E_T} \int_l \left[\!\!\left[\frac{\partial(u^{\mathrm{I}} - u_h)}{\partial \boldsymbol{n}} \right]\!\!\right]_l v, \quad \forall v \in \mathscr{P}_2^0(T) \,.$$

In particular, for $v = q_u - \varepsilon_T$, using Schwarz inequality, a trace theorem and an inverse inequality we obtain

$$\|\nabla q_u - \nabla_h \varepsilon\|_{0,T} \leqq C \|\nabla(u^I - u_h)\|_{0,T^*}.$$

Therefore, the theorem follows by using again (2.5). □

Remark 2.1. As a consequence of the theorem, the relative error

$$\frac{\|\nabla e - \nabla_h \varepsilon\|_{0,\Omega_0}}{\|\nabla e\|_{0,\Omega_0}}$$

converges to 0 when h goes to 0, whenever the error satisfies $\|\nabla e\|_{0,\Omega_0} \geqq ch$ and $\|e\|_{0,\Omega_1} = \mathcal{O}(h^{1+\epsilon})$, for some $\epsilon > 0$. The first assumption holds in all but trivial cases (see [2]). The second one holds, for instance, when Ω is a convex polygon and $u \in H^2(\Omega)$; it also holds for non convex polygons and homogeneous Dirichlet boundary conditions (see [9] and [10]). When both assumptions hold, the estimator is asymptotically exact on Ω_0.

Now, we shall analyze a simple example which shows that the estimator is not asymptotically exact for general meshes. Let us consider a particular case of problem (2.1) where Ω is a square as in Fig. 2a, Γ_d consists of the two vertical edges of Ω and Γ_n of the horizontal ones; let f be a constant and $g = 0$. The solution is a quadratic polynomial in x (and it does not depend on y). Let $\mathcal{T}_h$ be a family of criss-cross meshes; (a criss-cross mesh is a uniform mesh of squares splitted into four triangles as in Fig. 2b).

Since the solution is quadratic and the Neumann boundary conditions are zero, it is easy to see that

$$u_h(P) = \begin{cases} u(P), & \text{if } P \text{ is a vertex of a square } R, \\ u(P) + \dfrac{h^2}{24}\Delta u, & \text{if } P \text{ is the midpoint of } R\ , \end{cases}$$

where R is a square in the mesh as that in Fig. 2b (see [9]).

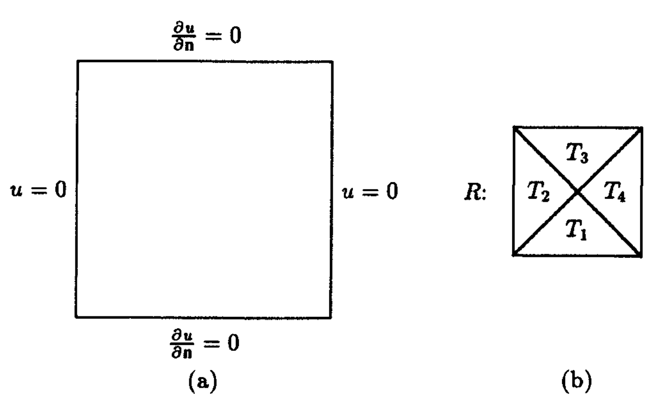

Fig. 2a,b

By using this expression for u_h it is possible to compute explicitly the true error and the estimator. The error is the same for all the squares R:

$$\|\nabla e\|_{0,R}^2 = \frac{h^4(\Delta u)^2}{18}\,.$$

And for each triangle T_i disjoint with Γ_d, the estimator is:

$$\eta_{T_i}^2 = \begin{cases} \dfrac{17h^4(\Delta u)^2}{1728}, & i = 1, 3\,, \\ \dfrac{65h^4(\Delta u)^2}{1728}, & i = 2, 4\,. \end{cases}$$

For those triangles with an edge l on the boundary Γ_d, the estimator is different; however, since the proportion of these elements goes to zero when the mesh is refined, their effect is asymptotically negligible and so, for any subdomain of Ω, the quotient between the estimated error and the exact one converges to

$$\frac{\left(\sum_{i=1}^{4} \eta_{T_i}^2\right)^{1/2}}{\|\nabla e\|_{0,R}} = \sqrt{\frac{41}{24}} \approx 1.3$$

Therefore, in spite of the uniformity of this kind of meshes, the estimator is not asymptotically exact even for very smooth solutions. This behavior is analogous to that of other well known estimators as Zienkiewicz–Zhu's [15] (see [4]) and that in [9].

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Note added in proof. The analysis of this paper is also valid for one of the estimators introduced by Verfürth in [13]. This estimator is defined by $\tilde{\eta}_T := \|\nabla \tilde{\varepsilon}_T\|_{0,T}$ where $\tilde{\varepsilon}_T$ is the solution of the local problem:

$$\tilde{\varepsilon}_T \in V_T\colon \int_T \nabla \tilde{\varepsilon}_T \cdot \nabla v = \int_T fv + \frac{1}{2} \sum_{l \in E_T} \int_l J_l v, \quad \forall v \in V_T ,$$

in the space $V_T := \mathscr{P}_2^0(T) \oplus (H_0^1(T) \cap \mathscr{P}_3(T))$ (i.e.: the space of quadratic functions vanishing at the nodes of T of Bank-Weiser's estimator, plus a cubic bubble function).

A result analogous to Theorem 2.1 is valid for this estimator (its proof needs only minor modifications) and hence the estimator is asymptotically exact under the same assumptions as Bank-Weiser's one. On the other hand, Verfürth's estimator is not asymptotically exact for general meshes, even for smooth solutions. For instance, in the particular case of problem (2.1) previously analyzed, we have:

$$\frac{\left(\sum_{i=1}^{4} \tilde{\eta}_{T_i}^2\right)^{1/2}}{\|\nabla e\|_{0,R}} = \sqrt{\frac{17}{6}} \approx 1.68 .$$

Once again the situation is the same as for Bank-Weiser's estimator.