General Relativity and Gravitation, Vol. 34, No. 4, April 2002 (© 2002)

$f(R)$ Cosmology in the First Order Formalism

D. Barraco, V. H. Hamity,[1,2] and H. Vucetich[1]

Received October 3, 2001

In the present work we consider those theories that are obtained from a Lagrangian density $\mathscr{L}_T(R) = f(R)\sqrt{-g} + \mathscr{L}_M$, that depends on the curvature scalar and a matter Lagrangian that does not depend on the connection, and apply Palatini's method to obtain the field equations. We start with a brief discussion of the field equations of the theory and apply them to a cosmological model described by the FRW metric. Then, we introduce an auxiliary metric to put the resultant equations into the form of GR with cosmological constant and coupling constant that are curvature depending. We show that we reproduce known results for the quadratic case. We find relations among the present values of the cosmological parameters q_0, H_0, $(\overset{\circ}{G}/G)_0$ and $(\overset{\circ\circ}{G}/G)_0$. Next we use a simple perturbation scheme to find the departure in angular diameter distance with respect to General Relativity. Finally, we use the observational data to estimate the order of magnitude of what is essentially the departure of $f(R)$ from linearity. The bound that we find for $f''(0)$ is so huge that permit almost any $f(R)$. This is in the nature of things: the effect of higher order terms in $f(R)$ are strongly suppressed by power of Planck's time $8\pi G_0$. In order to improve these bounds more research on mathematical aspects of these theories and experimental consequences is necessary.

KEY WORDS: Cosmology; non-linear Lagrangian; Palatini approach.

1. INTRODUCTION

The study of gravitational actions composed of the Einstein scalar curvature term plus quadratic or higher power terms of curvature and cosmological models based on these actions have been around for some time [1]. These actions lead to higher order theories which appear to enjoy better renormalizability properties than

[1] Fa.M.A.F., Universidad Nacional de Córdoba, Ciudad Universitaria, Córdoba 5000, Argentina. Observatorio Astronómico, Universidad Nacional de La Plata, La Plata, Argentina.
[2] Tel: +54 351 334051; Fax: +54 351 334054; E-mail: hamity@fis.uncor.edu.ar

0001–7701/02/0400-0533/0

General Relativity (GR) [2], and in modern cosmology have become standard since the Starobinsky model with curvature squared terms lead automatically to the desired inflationary period. More recently [3], the stability and Hamiltonian formulation of these theories have been studied.

In general, since curvature is defined in terms of second derivatives of the metric tensor, an action with n powers of curvature yields field equations of order 4 if $n > 1$ and the metric is assumed to be the only dynamical field (second-order formalism). In fact, all the good and bad properties of higher order theories are due to the increase in the order of the field equations. For instance, in the $R + R^2$ theory the Green function of the linear field equations differs from the Newtonian Green function by a Yukawa term. Then, coupling the linearized theory to a pressurized fluid distribution shows that the coefficients of the Yukawa potentials depend on the pressure and the size of the distribution. This shows that Birkhoff's theorem is not valid in these models. Therefore the theory defined by the above Lagrangian have, as we have just mentioned, some quantum and cosmological interesting property but have a great trouble with the Schwarzschild solution because it is not the one that matches to a realistic interior solution. Then, we can not say that the classical tests of general relativity are automatically satisfied, as the early investigators emphasized.

On the other hand, the Palatini approach (first-order formalism), of treating the metric tensor and the connections as independent fields, can be applied to the Hilbert Lagrangian to obtain the same field equations of GR as obtained from the second-order formalism. The Palatini approach has also been applied to more general Lagrangian densities with quadratic terms [1] to discuss a Friedmann cosmological model, or a general function $f(R)$ of the scalar curvature to study other geometrical theories of gravitation [4], its conserved quantities [5] and its spherically symmetric solutions [6]. One apparent conceptual advantage of these theories is that quantum fluctuations of the metric and the connection are independent of each other, since there are no *a priori* reasons to assume metric compatibility in the strong curvature epoch, when the quantum effects may be significant. We mention that these theories have been recently extended to include a scalar field in the Lagrangian and a connection allowing torsion [11].

Although the $f(R)$ theories in the first-order formalism yields a conformally metric theory, which implies breakdown of the Einstein equivalence principle due to breakdown of conformal symmetry, the active mass is equal to the inertial mass [5] and the active mass of a source of the Schwarzschild solution is quite different from the active mass of GR, for the same form of the energy-momentum tensor [6].

In our previous paper [7], we have shown that it is very difficult to test these models in the (post)Newtonian approximation, since the departures from Newtonian behavior are both very small and masked by other effects, since these departures from the Newtonian behavior have to be measured when the body is moving "through" a matter filled region.

Cosmological phenomena, however, are a more promising as potential tests. This is due to one of the "Dicke coincidences" [8, 9]:

$$\frac{\dot{a}^2}{a^2} \sim G\rho \tag{1}$$

which means that the expansion of the Universe is of the same order of magnitude as the Newtonian corrections of metric affine theories of gravitation (Cf. Ref. [7]):

$$V_{MA} = V_N - 2f''(0)(8\pi G)\rho \tag{2}$$

We expect thus, today cosmological effects of order H_0^2, where H_0 is the Hubble constant. (For a general discussion of the "Dicke coincidences" see Ref. [10].)

The higher powers of R in the Lagrangian may be insignificant now, but at the time when curvature becomes significantly large, quadratic contributions, for instance, will become important. An $f(R)$ theory of gravitation, in the first-order formalism, is an adequate model to include such contributions. In the present work we consider those theories that are obtained from a Lagrangian density $\mathscr{L}_T(R) = f(R)\sqrt{-g} + \mathscr{L}_M$, that depends on the curvature scalar and a matter Lagrangian that does not depend on the connection, and apply Palatini's method to obtain the field equations. We start with a review of the field equations in section 2; in section 3 we apply the field equations to a cosmological model described by the FRW metric and introduce an auxiliary metric, conformally related to the physical metric, to put the resultant equations into the form of GR with cosmological constant and coupling constant that are curvature depending. We show that we reproduce the results of Shahid-Shales [1] for the quadratic case. In section 4 we find relations between the present values of the cosmological parameters q_0, H_0, $(\overset{\circ}{G}/G)_0$ and $(\overset{\circ\circ}{G}/G)_0$ ($\overset{\circ}{G}$, means the time derivative of the time dependent gravitational constant G). In section 5 we use a simple perturbation scheme to find the departure in angular diameter distance with respect to General Relativity. In section 6, we use the observational data briefly reviewed in subsection 6.1 to estimate the order of magnitude of what is essentially the departure of $f(R)$ from linearity. Finally, in section 7, we present our conclusions and a discussion of the main results of the present paper.

2. REVIEW OF THE FIELD EQUATIONS

In this section we review the structure of the theory as presented elsewhere in previous work [1, 4, 7].

Let $\mathscr{M}$ be a manifold with metric g_{ab} and a torsionless derivative operator ∇_a, both considered as independent variables. Consider a Lagrangian density $\mathscr{L} = f(R)\sqrt{-g} + \mathscr{L}_M$, where the matter Lagrangian $\mathscr{L}_M$ does not depend on the connection.

Suppose we have a smooth one-parameter (λ) family of field configurations starting from given fields g^{ab}, ∇_a and ψ (the matter fields), with appropriate boundary conditions, and denote by δg^{ab}, $\delta\Gamma^c_{ab}$, $\delta\psi$ the corresponding variations, i.e., $\delta g^{ab} = (dg^{ab}_\lambda/d\lambda)\mid_{\lambda=0}$, etc. Then the field equations, if we vary with respect to the metric, are

$$f'(R)R_{ab} - \frac{1}{2}f(R)g_{ab} = T_{ab}. \tag{3}$$

where $f'(R) = (df/dR)$, $(\delta S_M/\delta g^{ab}) \equiv -T_{ab}\sqrt{-g}$. The variation with respect to the connection, recalling that this is fixed at the boundary, gives

$$\nabla_c(\sqrt{-g}g^{ab}f'(R)) = 0. \tag{4}$$

Now, we choose Lagrangian $f(R)$ with $f'(R)$ derivable and not null for any value of R. Then the last equation becomes

$$\nabla_c g_{ab} = b_c g_{ab} \tag{5}$$

where,

$$b_c = -[\ln f'(R)]_{,c}. \tag{6}$$

Thus, we have a Weyl conformal geometry with a Weyl field given by (6).

The vanishing of the connection in a particular frame, for example in a geodesic frame, however does not mean that the metric is flat there, because from (5) $\partial_c g_{ab} = b_c g_{ab}$. Therefore the strong equivalence principle is in general not satisfied.

From (3) we obtain

$$f'(R)R - 2f(R) = T, \tag{7}$$

which define $R(T)$, and we suppose the function $f(R)$ is such that $R(T)$ would be derivable respect to the variable T. Therefore b_c is determined by T and its derivatives except in the case $f(R) = \omega R^2$, for which $Rf' - 2f \equiv 0$, and then we must consistently have $T \equiv 0$. It is important to note that b_c has solution only if T is differentiable in $\mathscr{M}$; this condition on T, for the existence of solution, is not necessary in other theories, as GR or fourth order theories.

Therefore, the field equations (3) can be written as

$$G_{ab} + \Lambda(T)g_{ab} = \kappa(T)T_{ab}(g), \tag{8}$$

with

$$\frac{1}{2}\Lambda(T) = \left[R(T) - f(R(T))/f'(R(T))\right], \tag{9}$$

$$\kappa(T) = 1/f'(R(T)), \tag{10}$$

and both of them continuous. In equation (8) we have made explicit the dependency of T_{ab} on the metric.

The connection solution to (5) is

$$\Gamma^a_{bc} = C^a{}_{bc} - \frac{1}{2}(\delta^a_b b_c + \delta^a_c b_b - g_{bc} b^a), \tag{11}$$

where $C^a{}_{bc}$ are the Christoffel symbols (metric connection). Then we have to solve only equation (8).

The Riemann tensor can be defined in the usual way, and then the Ricci tensor and scalar curvature are

$$R_{ab} = R^0_{ab} + \frac{3}{2}D_a b_b - \frac{1}{2}D_b b_a + \frac{1}{2}g_{ab} D \cdot b + \frac{1}{2}b_a b_b - \frac{1}{2}g_{ab} b^2 \tag{12}$$

$$R = R^0 + 3D \cdot b - \frac{3}{2}b^2, \tag{13}$$

where R^0_{ab}, R^0, and D_c are the Ricci tensor, scalar curvature and covariant derivative, defined from the metric connection, respectively.

From equation (12) we obtain the skew symmetric part of the Ricci tensor in the form

$$R_{[ab]} = \partial_a b_b - \partial_b b_a, \tag{14}$$

then equation (6) gives $R_{(ab)} = R_{ab}$. Thus, the Ricci tensor is actually symmetric in this theory.

Because the matter action must be invariant under diffeomorphisms and the matter fields satisfy the matter field equations, then T_{ab} is conserved [7]

$$D^a T_{ab} = 0. \tag{15}$$

Therefore, a test particle will follow the geodesics of the metric connection. Using (6) and (7) we have

$$b_c = -\frac{f'' \nabla_c T}{f'(Rf'' - f')}. \tag{16}$$

Except for the case of GR, $f'' \equiv 0$, the Weyl field is nonzero wherever the trace of the energy-momentum tensor varies with respect to the coordinates. If T is constant, then R is also constant, $b_c = 0$ and (3) takes the form

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab}, \tag{17}$$

where Λ and κ are two functions of R. All those cases with constant trace of the energy-momentum tensor are equivalent to GR for a given cosmological constant. This is the so called [12] Universality of the Einstein equations for matter with constant T.

3. THE COSMOLOGICAL MODEL

Consider an auxiliary metric $\tilde{g}$, which is conformally related to the physical metric:

$$\tilde{g}_{ab} = \Omega^2(x) g_{ab}. \tag{18}$$

Then, the field equation (5) changes as

$$\nabla_c \tilde{g}_{ab} = (b_c + 2(\ln \Omega)_c)\tilde{g}_{ab}. \tag{19}$$

Taking into account the definition of b_c, equation (6), we can choose $\Omega(T) = C\sqrt{| f' |}$, where the constant C is such that $\Omega(0) = 1$. Then the field equation (19) for the transformed metric is,

$$\nabla_c \tilde{g}_{ab} = 0. \tag{20}$$

Therefore, the connection of our theory is the metric connection for the $\tilde{g}_{ab}$ metric. Thus, $G_{ab} = \tilde{G}^0_{ab}$ and the other field equation, equation (8), can be written in the form

$$\tilde{G}^0_{ab} + \frac{\Lambda(T)}{\Omega^2(T)}\tilde{g}_{ab} = \kappa(T) T_{ab}(\Omega^{-2}\tilde{g}), \tag{21}$$

where we have taken into account the dependence of T_{ab} on the metric. Notice that in (21) the coupling between matter and the metric will not be minimal, in general. We recall that the "physical metric" is g_{ab}; thus, once we have solved for $\tilde{g}_{ab}$ we have to apply the inverse of the transformation (18) to obtain g_{ab}.

Let us consider now a FRW spacetime manifold with a metric $\tilde{g}$ in the form:

$$d\tilde{s}^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2\right) \tag{22}$$

where $k = 0, \pm 1$. Then, the physical metric g is:

$$ds^2 = -\frac{dt^2}{\Omega^2(t)} + \frac{a^2(t)}{\Omega^2(t)}\left(\frac{dr^2}{1-kr^2} + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2\right) \tag{23}$$

By choosing a new time coordinate through $dt' = \Omega^{-1} dt$ and an expansion parameter $A(t) = a(t)\Omega^{-1}$, we obtain

$$ds^2 = -dt'^2 + A^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 \sin^2\theta d\phi^2 + r^2 d\theta^2\right) \tag{24}$$

Thus, we obtain also a FRW metric as the physical metric, as expected. We have to find the field equations, for the expansion parameter $A(t)$, in terms of the cosmic time t', in order to relate their solutions to the cosmological observables (like the Hubble constant, the deceleration parameter the age of the Universe, etc.). We need the derivatives $dA/dt', d^2A/dt'^2$. To this end we use that $d/dt' = \Omega(t)d/dt$; to simplify the notation we introduce $\dot{a} \equiv da/dt$, $\overset{\circ}{A} \equiv dA/dt'$ and similarly for higher derivatives. We obtain

$$\overset{\circ}{A} = \frac{\dot{a}\Omega - a\dot{\Omega}}{\Omega} \tag{25}$$

$$\overset{\circ\circ}{A} = \Omega\ddot{a} - \dot{a}\dot{\Omega} - a\ddot{\Omega} + a\dot{\Omega}^2/\Omega \tag{26}$$

In the physical spacetime the energy momentum tensor corresponds to a perfect fluid model:

$$T_{ab}(g) = \rho u_a u_b + p(g_{ab} + u_a u_b) \tag{27}$$

To obtain $T_{ab}(\Omega^{-2}\tilde{g})$ we notice that the energy momentum tensor will be of the form of a perfect fluid if $u_a u_b$ transform in the same way as the metric; i.e. $\tilde{u}_a\tilde{u}_b = \Omega^2 u_a u_b$. Thus, $u_a = \Omega^{-1}\tilde{u}_a$ and

$$T_{ab}(\Omega^{-2}\tilde{g}) = \tilde{\rho}\ \tilde{u}_a\tilde{u}_b + \tilde{p}\ (\tilde{g}_{ab} + \tilde{u}_a\tilde{u}_b) \tag{28}$$

where $\tilde{\rho} = \rho/\Omega^2$ and $\tilde{p} = p/\Omega^2$. The field equations (21)are:

$$\frac{3k}{a^2} + 3\ \frac{\dot{a}^2}{a^2} = \kappa(T)\frac{\rho}{\Omega^2} + \frac{\Lambda}{\Omega^2} \tag{29}$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{k}{a^2} = (\kappa(T)p - \Lambda)\ \frac{1}{\Omega^2} \tag{30}$$

Using (29)we may rewrite (30) as

$$3\ \frac{\ddot{a}}{a} = -\ \frac{1}{2\Omega^2}\ (\kappa(T)(3p + \rho) - 2\Lambda) \tag{31}$$

From the definition of Λ and (7) we obtain $\Lambda = (T + f)/f'$; then (29) becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \kappa(T)\frac{\rho}{3\ \Omega^2} + \frac{\Lambda}{3\ \Omega^2} - \frac{k}{a^2} \tag{32}$$

We are ready now to write the field equations for the physical metric in a convenient way. First, we introduce a new function through $U(R) = f'(R)$. Its inverse function is $R(U)$, and $\tilde{f}(U) = f(R(U))$. Then, using (25) and the expression that relates Ω to f', equation (32) becomes

$$\left(\frac{\overset{\circ}{A}}{A} + \frac{\overset{\circ}{U}}{2U}\right)^2 = \kappa(T)\frac{\rho}{3} + \frac{\Lambda}{3} - \frac{k}{A^2}$$

$$= \frac{\rho + 3p}{6U} + \frac{\tilde{f}(U)}{6U} - \frac{k}{A^2} \tag{33}$$

It is straightforward to check that for $f(R) = R$ we obtain one of the field equations of GR. The other equation could be the transformed of (31) or, directly, from the Bianchi identities (15), we obtain

$$3 \overset{\circ}{A} (\rho + p) + A \overset{\circ}{\rho} = 0 \tag{34}$$

which expresses conservation of energy. In particular if $p = (\gamma - 1)\rho$ where γ is a constant ($1 \leq \gamma \leq 2$)), we find

$$\rho = \rho_0 \left(\frac{A_0}{A(t')} \right)^{3\gamma} \quad \Longrightarrow \quad \frac{4}{3}\pi\rho \, A^{3\gamma} = M \tag{35}$$

where M is a constant.

4. THE COSMOLOGICAL PARAMETERS

We wish to compute the equations that relate the cosmological model to the observational cosmological parameters q_0, H_0, $(\overset{\circ}{G}/G)_0$ and $(\overset{\circ\circ}{G}/G)_0$. To this end, let t'_0 be the present value of the cosmic time ; then, $H_0 = (\overset{\circ}{A}/A)_{t'=t'_0}$ is the Hubble's constant. Since $\kappa(T) = f'^{-1} \equiv 8\,\pi G(t')$, with $G(t')$ defined as the time dependent gravitational constant, we have

$$\frac{\overset{\circ}{U}}{U} = -\frac{\overset{\circ}{G}}{G} \tag{36}$$

Since $R(t'_0) \approx 0$ in the present Universe, for $t' \approx t'_0$, we have

$$8\,\pi G(t') = 8\,\pi G_N(1 - XR(t')) \tag{37}$$

where we have made a Taylor series expansion, up to the linear term, using as independent variable the curvature scalar R. We have introduced the Newtonian gravitational constant $G_N = 1/(8\,\pi f'(0))$ and $X = f''(0)/f'(0)$. In particular

$$G_0 \equiv G(t'_0) = G_N(1 - XR(t'_0)) \tag{38}$$

Let us introduce the critical density $\rho_{CR} = 3\,H_0^2/8\,\pi G_0$, and the cosmological parameters $\Omega_0 = \rho_0/\rho_{CR}$, $\Omega_\Lambda = \Lambda_0/3H_0^2$, and $\Omega_k = k/H_0^2 A_0^2$. Note that our definition of Ω_k has the opposite sign of the usual one. Then, from (33) we obtain

$$\frac{1}{2H_0}\left(\frac{\overset{\circ}{G}}{G}\right)_0 = 1 \mp \sqrt{\Omega_0 + \Omega_\Lambda - \Omega_k} \tag{39}$$

We may use this equation to compute Ω_k in terms of the other observational parameters.

From the definition of Λ, equation (9), we obtain

$$\Lambda_0 = 2\frac{f(0)}{f'(0)}(XR(t_0') - 1) + O(R^2(t_0')) \tag{40}$$

therefore,

$$f(0) = -\frac{\Lambda_0}{8\,\pi\,G_0} \tag{41}$$

Let us compute now $(\overset{\circ}{G}/G)_0$. To this end, notice first that from (6), (16) and (36), we obtain

$$(\overset{\circ}{G}/G)_0 = -\frac{f"(t_0')\ \overset{\circ}{T}\ (t_0')}{f'(t_0')[R(t_0')\ f"(t_0') - f'(t_0')]} \tag{42}$$

From (7), we have

$$T(t_0') = -2f(0) - f'(0)R(t_0') + O(R^2(t_0')) \tag{43}$$

Let us make the further assumption that $f(0)/f'(0)$, a quantity that may be related to a nonzero cosmological constant in GR, is as most of order $R(t_0')$; then $T(t_0') = O(R(t_0'))$. Hence, since we may consider that at present time the matter is in the Newtonian regime, $\rho(t_0') = O(R(t_0'))$; i.e., $\overset{\circ}{T}\ (t_0') = -\ \overset{\circ}{\rho}\ (t_0') = -\ 3H_0\rho(t_0') = O(R(t_0'))$. Where, to write the last relation, we have used (35). Finally, we obtain

$$(\overset{\circ}{G}\ /G)_0 = 8\pi G_N X\ \overset{\circ}{T}\ (t_0') + O(R^2(t_0')) \tag{44}$$

From (35), the definitions of H_0 and Ω_0, and using $G_N = G_0 + O(R)$, we obtain,

$$(\overset{\circ}{G}\ /G)_0 = -\ 9XH_0^3\Omega_0 + O(R^2(t_0')) \tag{45}$$

We may use this equation to express X in terms of measurable quantities. Thus,

$$X = -\left(\frac{\overset{\circ}{G}}{G}\right)_0 \frac{1}{9\,H_0^3\,\Omega_0} \tag{46}$$

From (43), we obtain

$$R(t_0') = \frac{1}{f'(0)}\left(\rho_0 + \frac{4\,\Lambda_0}{8\,\pi\,G}\right) \tag{47}$$

Therefore, using this last equation in (38), we have

$$\frac{1}{f'(0)} = \frac{1 \pm \sqrt{1 - 12\,XH_0\,(4\,\Omega_\Lambda + \Omega_0)}}{\frac{6\,XH_0}{8\pi G_0}\,(4\,\Omega_\Lambda + \Omega_0)} \tag{48}$$

or, since we must recover GR in the limit $X \to 0$:

$$\frac{1}{f'(0)} = 8\pi G_0 \left[1 - 6\xi(4\Omega_\Lambda + \Omega_0)\right] + O(\xi^2) \tag{49}$$

where we have introduced the dimensionless parameter:

$$\xi = X H_0 \tag{50}$$

Once we know $f'(0)$ and X we may compute $f''(0)$.

Let us compute now the deceleration parameter q_0, as defined by

$$q_0 = -\left(\frac{\overset{\circ\circ}{A}}{\overset{\circ}{A}{}^2} A\right)_0 \tag{51}$$

From equations (25), (26) and (36), we obtain

$$\left(\frac{\overset{\circ\circ}{A}}{A}\right)_0 = \Omega^2 \left(\frac{\ddot{a}}{a}\right)_0 + \frac{H_0}{2}\left(\frac{\overset{\circ}{G}}{G}\right)_0 - (\overset{\circ\circ}{\Omega}\ \Omega)_0 \tag{52}$$

Then, using the field equation (31) into (52), and taking into account that at present time the matter may be considered in Newtonian regime, we get

$$\left(\frac{\overset{\circ\circ}{A}}{A}\right)_0 = \frac{8\pi G_0}{6}(2\Lambda_0 - \rho_0) + \frac{H_0}{2}\left(\frac{\overset{\circ}{G}}{G}\right)_0 - (\overset{\circ\circ}{\Omega}\ \Omega)_0 \tag{53}$$

Finally, since $(\overset{\circ\circ}{\Omega}\ \Omega)_0 = -(1/2)(\overset{\circ}{G}/G)^{\circ}_0$, we obtain

$$q_0 = \frac{1}{2}(\Omega_0 - 2\Omega_\Lambda) - \frac{1}{2H_0}\left(\frac{\overset{\circ}{G}}{G}\right)_0 + \frac{1}{2H_0^2}\left[\left(\frac{\overset{\circ}{G}}{G}\right)^2 - \frac{\overset{\circ\circ}{G}}{G}\right]_0 \tag{54}$$

This last equation represents, in the present formalism, a constraint among the observational parameters.

5. PERTURBATION THEORY TREATMENT

Both an estimation of the age of the Universe T_0 and the recent measurement of the first acoustic peak in the Cosmic Microwave background will provide the most strict bounds on the dimensionless parameter ξ (Eq. (50)) and on $f''(0)$. This is because the curvature effects are much enhanced in those early times, corresponding to $A_0/A = z + 1 \sim 10^3$.

The angular diameter subtended by an object of diameter D at redshift $z = A_0/A - 1$ is given by:

$$\Delta\theta = \frac{D}{A(z)\sinh\chi} \tag{55}$$

where $\chi(t)$ is the "angular distance", defined through the equations:

$$\chi = \int_{t_r'}^{t_0'} \frac{dt'}{A} = \int_{A}^{A_0} \frac{dA}{\overset{\circ}{A}\,A} = \int_{1}^{W} \frac{dw}{H(w)} \tag{56}$$

where $w = A_0/A$.

This equation, which describes the path of a photon in an open universe, is conformal invariant and valid both in the physical and auxiliary frames. In the latter one, we have the expression:

$$H(w) = H_{GR} + \delta H \tag{57}$$

where

$$\delta H = -\frac{1}{2}\frac{\overset{\circ}{U}}{U} + \frac{1}{2}H_{GR}^{-1}\left(\frac{\delta\kappa}{\kappa}\Omega_0 w^3 + \frac{\delta\Lambda}{\Lambda}\Omega_\Lambda\right) \tag{58}$$

$$\frac{\delta\kappa}{\kappa} = -6\xi(4\Omega_\Lambda + \Omega_0 w^3) \tag{59}$$

$$\frac{\delta\Lambda}{\Lambda} = 6\xi(4\Omega_\Lambda + \Omega_0 w^3) \tag{60}$$

$$\frac{\overset{\circ}{U}}{U} = -9\xi H_0 \Omega_0 w^3 \tag{61}$$

and

$$H_{GR} = \sqrt{\Omega_0 w^3 - \Omega_K w^2 + \Omega_\Lambda} \tag{62}$$

In the above equations all the parameters are the General Relativity ones.

The change in angular distance will be:

$$\delta\chi = -\int_1^w \frac{\delta H}{H_{GR}^2} dw \tag{63}$$

We shall limit ourselves to the flat case, since observation shows that $\Omega_K \sim 0$. Besides, since $\Omega_\Lambda + \Omega_0 = 1$, most of the integration range in equation (63) will be dominated by the terms containing w^3. With this approximations, we find for $w \gg 1$:

$$\delta\chi \sim -\frac{6}{5}\xi\sqrt{\Omega_0 w^5}\left(1 + O(w^{-\frac{3}{2}})\right) \tag{64}$$

We shall be interested in the change induced in the peak position. In the particular case of a flat Universe, this position is given by [13]:

$$l_1 = \pi\sqrt{\Omega_0 w_r}\,\chi_r \tag{65}$$

where w_r, χ_r refer to the surface of last scattering. The change in the first peak, induced by the terms $O(R^2)$ is, thus:

$$\frac{\delta l_1}{l_1} = \frac{\delta \chi_r}{\chi_r} \sim -\frac{3}{5}\xi \Omega_0 w_r^{\frac{5}{2}} \tag{66}$$

The evolution of the scale factor A can be treated using perturbation theory around GR in a similar form.

6. COMPARISON WITH OBSERVATION

In the following, we shall find estimates (or rather, upper bounds) on the parameters ξ, X and $f''(0)$, from the available observational data.

6.1. Observational Data

There are quite a few observational data related to the Cosmological Constant and the time variation of G: the main observational consequences of metric affine theories of gravitation.

$\mathbf{H_0}$ and the deceleration parameter $\mathbf{q_0}$: A large number of observational data obtained with different techniques is converging to a value of [14, 15, 16]:

$$H_0 = 65 \pm 5 \text{ km/s/Mpc} \simeq (6.5 \pm 5) \times 10^{-11} \text{ yr}^{-1} \tag{67}$$

The deceleration parameter q_0 has been estimated as [17]:

$$q_0 \sim -0.6 \tag{68}$$

with a large (∼20%) error.

The cosmological constant: The surprising discovery of an "acceleration" of the Universe [18, 17]; i.e. that $\frac{\ddot{a}}{a} > 0$ strongly suggests the existence of a substantial cosmological constant. Similar results can be found from the comparison of structure formation results [21, 22, 23] with the predictions of the Cosmic Background spectrum [19, 20]. The analysis of all these data suggest [24, 25]:

$$\Omega_0 \sim 0.25; \qquad \Omega_\Lambda \sim 0.75 \tag{69}$$

with large (∼20%) uncertainties. These values have been estimated using GR as the underlying theory of gravitation; however, since the departures from GR seem to be small, we shall use them in the following to estimate $f(0)$ and $f''(0)$.

Time variation of G: Many attempts to measure time variation of G have been made in latter times. From a combination of astronomical and geophysical data, an upper bound $\frac{\overset{\circ}{G}}{G} \leq 7.5 \times 10^{-12}$ yr^{-1} was found in Refs [26, 27, 28]. More modern bounds have been found based on heliosysmology [29], globular cluster evolution [30] or neutron star masses [31] yield similar bounds. The most accurate bounds come from white-dwarf evolution. The change in G induces a change in gravitational energy that is released (or absorbed) as light. Comparison of the observed and calculated luminosity distributions yields $\frac{\overset{\circ}{G}}{G} \leq 10^{-13}\ \mathrm{yr}^{-1}$ which is the smallest upper bound available [32].

To our knowledge, there is a single estimation of the second time derivative of G [33], based on paleontological data. From the results in this reference we find:

$$\frac{\overset{\circ\circ}{G}}{G} \leq 10^{-18}\ \mathrm{yr}^{-2} \tag{70}$$

The above upper bound is not very strict. A lower one can be obtained from the results of globular cluster evolution [30] or white dwarf evolution [32]. These papers approximate the functional form of $G(t) = G_0(t/t_0)^{\nu}$, from which we obtain:

$$\left(\frac{\overset{\circ}{G}}{G}\right)_0 = \frac{\nu}{t_0} \tag{71}$$

$$\left(\frac{\overset{\circ\circ}{G}}{G}\right)_0 = \frac{\nu(\nu-1)}{t_0^2} \tag{72}$$

and from [32] $\mid \nu \mid < 10^{-3}$ we obtain:

$$\left|\left(\frac{\overset{\circ}{G}}{G}\right)_0\right| < 10^{-13}\ \mathrm{yr}^{-1} \tag{73}$$

$$\left|\left(\frac{\overset{\circ\circ}{G}}{G}\right)_0\right| < 10^{-23}\ \mathrm{yr}^{-2} \tag{74}$$

We shall adopt these latter value in the following.

Position of the first acoustic peak: The recent measurements of the position of the first acoustic peak in the Cosmic Microwave Background radiation by the Boomerang [34] and Maxima [35] has provided high accuracy result for the spectra of fluctuations in the CMB. The position of the first peak can be inferred from this set of data [36]:

$$l_1 = 206 \pm 6 \tag{75}$$

This result is almost independent of the model of matter included in the theory, since it depends mainly on the geometry of the Universe.

6.2. Bounds on ξ and $f''(0)$

Let us now apply the above-mentioned values to the present theory. Thus, from equation (45) written in the form: $\overset{\circ}{G}/G = -9\xi H_0 \Omega_0$ we obtain:

$$|\ \xi\ | < 10^{-3} \tag{76}$$

and from this:

$$|\ f''(0)\ | = \left| \frac{\xi}{8\pi G_0 H_0^2} \right| < 10^{119} \tag{77}$$

This enormous bound is not surprising: the denominator in (77) is Planck's time divided by the age of the Universe.

On the other hand, the consistency condition (54) is satisfied within the very large errors in q_0.

Finally, if we accept that the corrections to GR have remained small during all the matter-dominated epoch, we can use equation (66) to put bounds on ξ and $f''(0)$, if we assume that the error in (75) is due only to ξ. Using $w_r \sim 1000$, we get:

$$|\ \xi\ | < 10^{-8} \tag{78}$$

$$|\ f''(0)\ | < 10^{113} \tag{79}$$

Equations (78) and (79) are the main results of this papers. They set the most accurate bounds on higher order gravity, based on cosmological parameters.

From Eqs. (49) and (78) we find that f'^{-1} differ from the General Relativity value less than the current observational error for the Newton constant.

7. CONCLUSIONS

In this work we have obtain the first bound, as far as we know, of the departure from General Relativity of the lower order coefficients for the general function $f(R)$, in the case of the theories which are obtained using the first order formalism. We have also obtained the cosmological equations for these theories in a very general way. The use of a conformal transformation is a trick which allows us to map these kind of theories to General Relativity as we have shown above.

The bound in Eq. (79) is so huge that permit almost any $f(R)$. This is in the nature of things: the effect of higher order terms in $f(R)$ are strongly suppressed by power of Planck's time $8\pi G_0$ as we have just remarked above. In order to improve these bounds more research on mathematical aspect of these theories and experimental consequences is necessary.

REFERENCES

1. Shahid-Saless, B. (1990). *J. Math. Phys.* **31**, 2429.
2. Stelle, K. S. (1978). *Gen. Rel. Grav.* **9**, 353.
3. Schmidt, H. J. (1994). *Phys. Rev. D* **49**, 6354; *Erratum D* **54**, 7906 (1996).
4. Hamity, V. H., and Barraco, D. E. (1993). *Gen. Rel. Grav.* **25**, 461.
5. Barraco, D., Dominguez, E., and Guibert, R. (1999). *Phys. Rev. D* **60**, 44012.
6. Barraco, D., and Hamity, V. H. (2000). *Phys. Rev. D* **62**, 44027.
7. Barraco, D. E. Guibert, R., Hamity, V. H., and Vucetich, H. (1996). *Gen. Rel. Grav.* **28**, 339.
8. Peebles, P. J. E. *Principles of Physical Cosmology*. (Princeton University Press, Princeton, New Jersey, 1993).
9. Dicke, R. H. (1961). *Nature* **192**, 440.
10. Barrow, J. D., and Tipler, F. J. *The anthropic cosmological principle* (Oxford University Press, Oxford, 1989).
11. Rubilar, G. F. (1998). *Class. Quantum. Grav.* **15**, 239.
12. Ferraris, M., Francaviglia, M., and Volovich, I. (1994). *Class. Quantum Grav.* **11**, 1505.
13. Frampton, P. H., Jack Ng, Y., and Rohm, R. (astro-ph/9806118).
14. Branch, D. (1998). *Ann. Rev. Astron. Astroph.* **36**, 17.
15. Freedman, W. L. (2000). *Phys. Rep.* **333**, 13; (astro-ph/9909076).
16. Hogan, C. J. (1998). In *Review of Particles Properties*. *Europ. Phys. J C* **3**, 1.
17. Perlmutter, S. et al. (1998). *Astroph. J* (astro-ph/9812133).
18. Riess, A. G. et al. (1998). *Astron. J.* (astro-ph/9805201).
19. Mauskopf, P. D., et al. (1999). *Astroph. J.* (astro-ph/9911444).
20. Melchorri, A., et al. (1999). *Astroph. J.* (astro-ph/9911445).
21. Carlberg, R. G., Yee, H. K. C., Ellingson, E., Abraham, R., Gravel, P., Morris, S., and Pritchet, C. J. (1996). *Astroph. J.* **462**, 32.
22. Reiss, A., Davis, M., Baker, J., and Kishner, R. P. (1997). *Astroph. J.* **488**, L1.
23. Willick, J. A., and Strauss, M. A. (1998). *Astroph. J.* **507**, 629.
24. Bridle, S. L. et al. (1999). *Mon. Not. R. Astron. Soc.* (astro-ph/9903472).
25. Tegmark, M. (1999). *Astrophys. J.* **514**, L69 (astro-ph/9809201).
26. Sisterna, P., and Vucetich, H. (1990). *Phys. Rev. D* **41**, 1024.
27. Sisterna, P., and Vucetich, H. (1991). *Phys. Rev. D* **44**, 3096.
28. Sisterna, P., and Vucetich, H. *Proceedings of the Conference on "Mach's Principle"*, J. Barbour and H. Pfister, eds., p. 375 Birkhäuser, Boston 1995.
29. Demarque, P., Krauss, L. M., Gunther, D. B., and Nydham, D. (1994). *Astroph. J.* **437**, 870.
30. Degl'Innocenti, S., Fiorentini, G., Raffelt, G. G., Ricci, B., and Weiss, A. (1996). *Astr. Astroph.* 312 345-352 (astro-ph/9509090).
31. Thorsett, S. E. (1996). *Phys. Rev. Lett.* **77**, 1432.
32. Benvenuto, O. G., Althaus, L. G., and Diego F. Torres, (1999). *Mon. Not. R. Astr. Soc.* **305**, 905.
33. Sisterna, P., and Vucetich, H. (1994). *Phys. Rev. Lett.* **72**, 454.
34. de Bernardis, P., et al. (2000). *Nature* **404**, 955.
35. Hanany, S., et al. (2000). *Astroph. J.* **545**, L5 (astro/ph/0005123v2).
36. Hu, W., Fukugita, M., Zaldarriaga, M., and Tegmark, M. (2001). *Astroph. J.* **549**, 669 (astro-ph/0006436).