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A ζ-Function Method for Weyl Fermionic Determinants

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Abstract. We present an extension of the ζ-function method adapted to handle the regularization of Dirac operator determinants when Weyl fermions are present. The method we propose makes use of an auxiliary operator which takes into account regularization ambiguities in anomalous gauge theories. As an application, we consider a two-dimensional model where these ambiguities allow for the definition of a consistent quantum theory.

Quantization of gauge theories with Weyl fermions has recently received a lot of attention, particularly after the results of Jackiw and Rajaraman [1] on the consistency of the chiral Schwinger model and of Faddeev and Shatashvili [2] on the modification of the canonical quantization method by the addition of a Wess–Zumino term.

Soon after these important developments, it was observed [3–5] within the path-integral framework, that integration over all field configurations makes the Wess–Zumino term arise naturally, establishing a link [6] between the results of [1] and [2].

An important feature of chiral theories concerns the regularization of divergent quantities. Since gauge currents possess anomalous divergences at the quantum level, there is no gauge-invariance principle to invoke when selecting a particular regularization scheme. This introduces an ambiguity which, in fact, can be exploited to render the quantum theory consistent (at least in two spacetime dimensions, as first shown in [1]. See also [7–9]).

Special regularization schemes had then been developed, by modification of the heat-kernel method, stochastic regularization, etc. [10]–[13], so as to take into account peculiarities of chiral theories. In this Letter, we present an extension of the ζ-function method adapted to handle the regularization problems posed by Weyl fermions. As will be shown, it maintains the simplicity and advantages of the original ζ-function method [14] (in particular, it works in the case of non-Hermitian and even nonnormal operators [15]) and, at the same time, it has the flexibility required in the present situation. The

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method is intended to apply to the calculation of fermion determinants and Fujikawa's Jacobians [16] in any spacetime dimensions ($d = 2, 4$, etc.) but for simplicity we present, at the end, an application to the ($d = 2$) chiral Schwinger model.

A first problem one faces with theories in which gauge fields A_μ are coupled to Weyl fermions ψ, $\bar{\psi}$ (left-handed, for definiteness) is that the Dirac operator $D_-[A]$,

$$D_-[A] = (i\partial\!\!\!/ + eA\!\!\!/)(1 - \gamma_5)/2 \tag{1}$$

maps negative chirality spinors to positive chirality spinors and, consequently, does not have a well-defined eigenvalue problem. This makes difficult the definition of the fermion determinant and, hence, limits the effective action.

The usual way to overcome this problem is to redefine the fermionic action in terms of a new operator $D[A]$,

$$D[A] = D_-[A] + i\partial\!\!\!/(1 + \gamma_5)/2 = i\partial\!\!\!/ + eA\!\!\!/(1 - \gamma_5)/2 \tag{2}$$

which defines an eigenvalue problem, since it acts on Dirac fermions rather than on Weyl fermions. Since the added (right-handed) fermionic degrees of freedom do not couple to the gauge field, one expects that only an overall normalization takes care of the change in the corresponding determinants. This definition also satisfies several consistency requirements [10].

At this point, it is important to note that $D[A]$ is not a covariant operator:

$$D[A^g] \neq g^{-1}D[A]g \tag{3}$$

with A^g the gauge transformation of A by a gauge-group element g:

$$A^g_\mu = g^{-1}A_\mu g + ig^{-1}\partial_\mu g\,. \tag{4}$$

Now, since $D[A]$ is an unbounded operator, one needs a regularization prescription in order to get a finite answer for its determinant. In the present case, there is no more reason (as there is in a gauge theory with Dirac fermions) to use a gauge-invariant regularization method since, due to (3), one generally has

$$\det D[A] \neq \det D[A^g]\,. \tag{5}$$

As we stated above, several methods have been proposed in order to take into account this peculiarity of theories with Weyl fermions [10–13]. We now present an extension of the ζ-function method, adapted to the present situation.

Given an invertible elliptic operator L with a cone of Agmon directions [17] defined on a compact manifold M without boundary, one has, within the 'classical' ζ-function method [14–15]

$$\log\det L = -\left.\frac{\mathrm{d}\zeta(L,s)}{\mathrm{d}s}\right|_{s=0}, \tag{6}$$

where $\zeta(L, s)$ is the meromorphic continuation of $\Sigma_i \lambda_i^{-s}$ with λ_i the eigenvalues of L. This continuation is given by

$$\zeta(L, s) = \mathrm{Tr} \int_M \mathrm{d}\mu K_{-s}(L, x, x) \tag{7}$$

with $K_{-s}(L, x, x)$ the continuation of the evaluation of the kernel of L^{-s} on the diagonal $x = y$ [17].

The extension we propose makes use of an auxiliary operator L_a (to be chosen on physical grounds) and it is given by the following definition

$$\det_{L_a}(L) \equiv \frac{\det(L L_a)}{\det(L_a)}, \tag{8}$$

where 'det' in the r.h.s. is taken as in Equation (6). Of course, operators L_a and LL_a have to satisfy the conditions required within the ζ-function method. Note that, for instance, if L and L_a separately satisfy these conditions and have the same principal symbol, LL_a also does. Definition (8) coincides with the usual one whenever $\det(LL_a) = \det L \det L_a$. As we shall see in an example, this is not always the case. It is true however that $\det(LL_a) = \det(L_a L)$, since the cyclic property holds for the ζ-function as given by (7) [15].

If one is interested in the computation of the anomaly associated to gauge currents in the path-integral approach, one has to evaluate the Fujikawa Jacobian [16] associated to a gauge transformation of the fermionic variables. In the present formulation, the Jacobian is given by

$$J = \frac{\det_{D_a}(D[A])}{\det_{D_a}(D[A^g])} \tag{9}$$

with $D[A]$ given by (2) and D_a a suitable auxiliary operator. It is then easy to show [16, 15] that the anomaly equation for the gauge current J_μ,

$$J\frac{a}{\mu} = \left\langle \bar{\psi}\gamma_\mu t^a \frac{(1-\gamma_5)}{2} \psi \right\rangle \quad (t^a \text{ are the gauge-group generators})$$

is

$$D^\mu J_\mu = -\left.\frac{\delta \log J}{\delta \alpha}\right|_{\alpha=0} = \mathscr{A}[A]. \tag{10}$$

It is worthwhile to stress that J, as given by (9), can be always computed in a closed form [18].

We now present a proposition which allows for the evaluation of determinants as in (9). Consider an operator $L(t)$, depending on a parameter $t \in [0, 1]$ such that

$$\frac{\mathrm{d}L(t)}{\mathrm{d}t} = fL(t) + L(t)g, \tag{11}$$

with f and g some functions and $L(t)$ satisfying the same hypothesis as in (6). We then have the proposition.

PROPOSITION

$$\frac{\det_{L_a}(L(1))}{\det_{L_a}(L(0))} = \exp \int_0^1 \mathrm{d}t \int \mathrm{d}\mu_x \, \mathrm{tr}[f(x)K_0(L(t)L_a; x, x) + g(x)K_0(L_aL(t); x, x)] \tag{12}$$

Proof. Consider the identity

$$\frac{\det_{L_a}(L(1))}{\det_{L_a}(L(0))} = \exp \int_0^1 w'(t)\,\mathrm{d}t \tag{13}$$

with

$$\begin{aligned} w'(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{tr}[L(t)L_a]^{-s}\bigg|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left[s\,\mathrm{tr}[L(t)L_a]^{-s}\,\frac{\mathrm{d}}{\mathrm{d}t}\,L(t)L_a\right]\bigg|_{s=0}, \end{aligned} \tag{14}$$

which can be proved using theorems on differentiation of the ζ-function [15, 19] which justify the naive identity 'tr log$L$' = 'log det$L$'.

From (11) we have

$$\mathrm{tr}[L(t)L_a]^{-s-1}\frac{\mathrm{d}}{\mathrm{d}t}L(t)L_a = \mathrm{tr}[\,[L(t)L_a]^{-s}f + [L_aL(t)]^{-s}g]\,, \tag{15}$$

where we have used

$$(AB)^sA = A(BA)^s \tag{16}$$

valid for any two operators A, B, such that its complex powers are well-defined. Then

$$w'(t) = \frac{\mathrm{d}}{\mathrm{d}s}\,[s\,\mathrm{tr}[L(t)L_a]^{-s}f] + \frac{\mathrm{d}}{\mathrm{d}s}\,[\mathrm{tr}[L_aL(t)]^{-s}g] \tag{17}$$

and again using (16), one gets the result (12).

It is worthwhile to note that condition (11) (sometimes known as the integrability condition) is fulfilled, for instance, by operators $L(t)$ interpolating between chiral transformed Dirac operators or, as in the present case, by gauge-transformed Dirac operators [15, 20], as will become clear in the example we will discuss below.

From the knowledge of J, given by (9), (12) one easily gets the anomaly for the gauge current J_μ from (10). Note also that J can be written in the form [18]

$$\log J = -\int_0^1 \mathrm{tr}\,\mathscr{A}[A^{g(t)}]\alpha\,\mathrm{d}t\,. \tag{18}$$

The extension of the ζ-function method we have presented is intended to apply in any dimensions (2, 4, etc.). To see how it works, we consider for simplicity a two-dimensional example, the chiral Schwinger model, with Lagrangian (in Euclidean space)

$$\mathscr{L} = \bar{\psi} D_{-}[A]\psi - \tfrac{1}{4}F_{\mu\nu}^{2}. \tag{19}$$

This model has been solved by Jackiw and Rajaraman [1] who showed that, although anomalous, it can be quantized in a consistent unitary form.

Let us evaluate the fermion determinant using the method described above.

As interpolating operator $L(t)$ we choose

$$L(t) = i\not\partial + et\not A(1 - \gamma_5)/2 \tag{20}$$

so that $L(0) = i\not\partial$ and $L(1) = D[A]$. If one writes

$$eA_\mu = -\varepsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\eta, \tag{21}$$

then it is easy to see that $L(t)$ satisfies a relation like (11) with

$$f = -(\phi - i\eta)(1 + \gamma_5)/2,$$
$$g = -(\phi - i\eta)(1 - \gamma_5)/2. \tag{22}$$

One can now employ formula (12) to evaluate $\det D[A]$. At this point, one has to select an auxiliary operator D_a satisfying the conditions required by the ζ-function method. In the present case, an auxiliary operator D_a which ensures a Lorentz invariant answer is

$$D_a = i\not\partial + e\not A(1 - \gamma_5)/2 + e\,\frac{a}{2}\,\not A(1 + \gamma_5)/2, \tag{23}$$

with a an 'a-priori' undetermined parameter.

In order to compute the kernels in the r.h.s. of Equation (12), we follow the Seeley technique [17] (see [15] for details). This requires the construction of Seeley's coefficients b_{-i} ($i = 1, \ldots, 4$) and then the use of the expression

$$K_0 = \frac{-i}{2(2)^2}\int_{|\xi|=1} \mathrm{d}\xi \int \mathrm{d}u\, b_{-4}(x, \xi, iu) \tag{24}$$

After some algebra, one gets for $K_0(L(t)D_a, x, x)$

$$K_0(L(t)D_a, x, x)$$
$$= \frac{-i}{8\pi}\left[-iet\,\varepsilon_{\mu\nu}\partial_\mu A_\nu(1 + \gamma_5)/2 - et\partial_\mu A_\mu + e\not\partial\not A\,(1 + \gamma_5)/4 + ea\not\partial\not A\,(1 + \gamma_5)/4\right] \tag{25}$$

and a similar expression for $K_0(D_aL(t), x, x)$. Then, using (12), we get for the fermion determinant

$$\log\frac{\det_{D_a}D[A]}{\det_{D_a}i\not\partial} = \frac{e^2}{8\pi}\int \mathrm{d}^2x A_\mu\left[a\delta_{\mu\nu} + (\delta_{\mu\alpha} + i\varepsilon_{\mu\alpha})\frac{\partial_\alpha\partial_\beta}{\Delta}(\delta_{\beta\nu} - i\varepsilon_{\beta\nu})\right]A_\nu \tag{26}$$

which coincides with the result presented by Jackiw and Rajaraman [1] in their solution of the chiral Schwinger model. Note that no value of a gives a gauge-invariant answer for the determinant – this is why the theory is anomalous. Also note that the a-dependence of the result implies that in this case $\det D_a D \neq \det D_a \det D$ (see Equation (8)).

Following [1], we conclude that the choice $a > 1$ guarantees a consistent unitary quantum theory which contains a massive degree of freedom, with mass

$$m^2 = \frac{e^2 a^2}{4\pi(a-1)} ,$$

and a massless excitation.

We conclude by noting that, as stressed above, the presence of the arbitrary parameter a was crucial in getting a consistent quantum theory. The arbitrariness is covered in the ζ-function method we presented by the auxiliary operator D_a. In more realistic (for example, four-dimensional) models, a careful election of D_a may also lead to consistent gauge theories with Weyl fermions and, hence, it should be interesting to investigate different choices of D_a's. We hope to report on this point in a future publication.

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